SECOND MIDTERM EXAM REVIEW

(1) Prove that $n^5 - n$ is divisible by 5 for all integers $n \geq 1$.

(2) Prove that $2^{2n+1} + 1$ is divisible by 3 for all integers $n \geq 1$.

(3) Prove that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all integers $n \geq 1$.

(4) Let n be a positive odd integer which is not divisible by 5. Prove that there exists an integer $\ell > 0$ such that the last digit of n^ℓ is equal to 1.

(5) Prove the inequalities: $2^n < \binom{2n}{n} < 4^n$ for all integers $n \geq 2$.

(6) Prove that $n^3 + 5n$ is divisible by 6 for all integers $n \geq 1$.

(7) Prove the identity:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(8) Prove the identity:

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

(9) Prove that $n^2 > n + 1$ for $n \geq 2$.

(10) Prove the identity:

$$r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

for all integers $n \geq 1$.

(11) How many seven-digit integers have the sum of their digits equal to 46?

(12) Prove that $8^{n+2} + 9^{2n+1}$ is divisible by 73 for all integers $n \geq 1$.

(13) Prove the inequalities: $\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1$ for all integers $n \geq 1$.

(14) Solve the Diophantine equations:

- $2000x + 643y = 1$,
- $1647x + 788y = 1$,
- $1647x + 788y = 24$.

(15) How many *odd numbers* between 100,000 and 1,000,000 have no two digits the same?

(16) Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3\}$.

- How many functions are there from the set A to the set B?
- How many of these functions map A onto B?
- How many ways are there to put 5 objects to 3 identical boxes so that no box will be left empty?

(17) How many integers between 1 and 10,000 (including 1 and 10,000) are not divisible by 2, 5, or 17?

(18) Let $A = \{a_1, a_2, a_3, a_4, a_5, a_7, a_8, a_9\}$ and $B = \{b_1, b_2, b_3, b_4\}$.

(a) How many functions are there from the set A to the set B?

(b) How many of these functions map A onto B?

(c) How many ways are there to put 9 objects to 4 identical boxes so that no box will be left empty?

(19) Let n, k be positive integers.

- Let $n > k$. Prove the identity: $\displaystyle\sum_{r=0}^{n-1} (-1)^r \binom{n}{n-r} (n-r)^k = 0$.
- Prove the identity: $\displaystyle\sum_{r=0}^{n-1} (-1)^r \binom{n}{n-r} (n-r)^n = n!$.

(20) How many positive integral solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_7 + x_8 + x_9 = 27 \ ?$$

(21) How many ways are there to put 21 objects in 5 boxes with at least 8 objects in one of the boxes?

(22) Let $n = d_1 d_2 \cdots d_s$ be an integer, where $d_1, d_2, \ldots, d_s$ are decimal digits. Prove that n is divisible by 3 if and only if the sum $d_1 + d_2 + \cdots + d_s$ is divisible by 3.

(23) Write a binary decomposition of 2017, 2018 and 2019.

(24) Prove that for any positive integer $n \geq 1$ the number $n^2 - 2$ is not divisible by 3.

(25) Let F_n be the n-the Fibonacci number. Prove that

$$\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}$$

for any positive integer n.

(26) Find the last two digits of the integer 54^{54}.

(27) Use the Euclidian Algorithm to find $d = \gcd(231, 1920)$ and the integers s and t such that $d = s \cdot 231 + t \cdot 1820$.

(28) Prove that $\gcd(7n+4, 5n+3) = 1$ for every positive interger n.

(29) Compute the Euler function $\phi(n)$ for $n = 51$, $n = 452$, $n = 12,300$.

(30) For which positive integers n is $\phi(n)$ power of two? (Here $\phi(n)$ is the Euler function.)