FIRST MIDTERM REVIEW

(1) How many multiples of 6 are there between -7 and 2014?

(2) Twenty people are to be seated at three circular tables, one of which seats 5, the second one seats 7 and the third one seats 8 people. How many different seating arrangements are possible?

(3) How many distinct five-digit integers can one make from the digits 1, 2, 2, 3, 3, 5, 7, 7, 8, 9?

(4) Prove that $\dfrac{(\ell k)!}{(\ell!)^k}$ is an integer. [1]

(5) How many seven-digit integers are there such that

- no digits are repeated and
- which are divisible by 4?

(6) How many arrangements of the letters in **NEWTOWNMOUNTKENNEDY** do not have consecutive **N**'s? [2]

(7) Let $\Sigma = \{0,1,2,3\}$ be an alphabet. We consider strings (words) over Σ of length 12, such as

$$x_1 x_2 \dots x_{12}, \quad x_1, \dots, x_{12} \in \Sigma.$$

Then we define a weight $w(x_1 x_2 \dots x_{12}) = x_1 + \cdots + x_{12}$. How many strings of length 12 have weight 4?

(8) Let S be a set of integers between 1 and 1,000,000, i.e.

$$S = \{1, 2, \dots, 1,000,000\}.$$

- How many integers from S are multiples of 7 and also multiples of 37?
- How many integers from S are multiples of 7 or of 37 or both?
- How many integers from S are not divisible by either 7 or 37?
- How many integers from S are divisible by 7 or 37, but not both?

(9) Show that for any positive integer $n > 0$

$$\sum_{j=0}^{n} \binom{n}{j} 2^j = 3^n \ .$$

(10) What is the coefficient of $a^5 b^3 c^2$ in the expansion of $(a+b+c)^{10}$?

(11) Prove that

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

(12) For how many seven-digit integers have the sum of their digits equal to 9? 14?

[1] Hint: try a combinatorial argument

[2] NEWTOWNMOUNTKENNEDY is a village in County Wicklow, Ireland

(13) How many positive integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 23?$$

(14) How many ways are there to put 14 objects in 3 boxes with at least 8 objects in one box?

(15) On the xy-plane, we can travel using the moves

$$R : (x, y) \mapsto (x + 1, y), \quad U : (x, y) \mapsto (x, y + 1).$$

Every path from $(0, 0)$ to (k, n) could be written as a sequence of k R's and of n U's. We are allowed to take only such paths that the number of U's will never exceed the number of R's along the path taken. How many such paths are there?

(16) Prove the following equivalence: $(p \wedge q) \iff \neg(p \to \neg q)$.

(17) Prove the following implications: [3]

- $p \to [q \to (p \wedge q)]$
- $[(p \to q) \wedge \neg q] \to \neg p$

(18) Prove the following satement:
For any positive integer $n \geq 1$ the number $n^2 - 2$ is not divisible by 3.

(19) Let $\mathbf{F}_0$ stand for a contradiction. Prove that the statement $(\neg p \to \mathbf{F}_0) \to p$ is a tautology.

(20) Prove that the statement $[(p \wedge q) \vee (\neg p \wedge r)] \to (q \vee r)$ is a tautology.

(21) Prove that the statement $[(p \wedge q) \wedge [p \to (q \to r)] \to r$ is a tautology.

(22) Prove that $\sqrt{5}$ is irrational number.

(23) The following statements are tautologies

(a) **T** **F** $\neg(p \wedge \neg p)$
(b) **T** **F** $p \to (p \vee q)$
(c) **T** **F** $p \to (p \wedge q)$
(d) **T** **F** $(\neg p \vee q) \to (q \to p)$

(24) Prove that there is an infinite number of primes.

(25) Let $\{x_n\}$ be a sequence of real numbers. Write what does it mean that $\lim_{n\to\infty} x_n = a$.

(26) Negate the statement $\forall x \, \exists y \, (p(x, y) \to q(x, y))$.

(27) Prove that the implication

$$[\exists x \, (p(x) \wedge q(x))] \to [(\exists x \, p(x)) \wedge (\exists x \, q(x))]$$

is a tautology. Show that the converse implication

$$[(\exists x \, p(x)) \wedge (\exists x \, q(x))] \to [\exists x \, (p(x) \wedge q(x))]$$

is not a tautology. Give an example.

[3] i.e., to show that those implications are tautologies