FINAL EXAM REVIEW, PART I

(1) Let $A = \{a_1, \dots, a_9\} \subset \{1, 2, \dots, 25\}$. Prove that there exist two different subsets $B = \{b_1, \dots, b_5\} \subset A$ and $B' = \{b'_1, \dots, b'_5\} \subset A$ such that $b_1 + \cdots + b_5 = b'_1 + \cdots + b_5$.

(2) Let k be an odd integer. Prove that there exists n such that k divides $2^n - 1$.

(3) Prove that $n^5 - n$ is divisible by 10 for all integers $n \geq 1$.

(4) Prove that $n^3 - n$ is divisible by 6 for all integers $n \geq 1$.

(5) Prove the identity:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(6) Prove the identity:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

(7) Prove that $n^2 > n + 1$ for $n \geq 2$.

(8) Prove that $11^n - 4^n$ is divisible by 7 for all n.

(9) Prove that $3^{2n+3} + 40n - 27$ is divisible by 64 for any $n \geq 1$.

(10) Prove the inequalities: $\sqrt{n} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leq 2\sqrt{n} - 1$ for all integers $n \geq 1$.

(11) Solve the Diophantine equations:

- $2013x + 741y = 3$,
- $2017x + 2013y = 1$

(12) Let $A = \{a_1, \dots, a_{10}\}$ be a subset of $\{1, 2, 3, \dots, 50\}$. Show that there exist two subsets

$$B = \{b_1, b_2, b_3, b_4\}, \quad B' = \{b'_1, b'_2, b'_3, b'_4\}, \quad B, B' \subset A,$$

such that $B \neq B'$ and $b_1 + b_2 + b_3 + b_4 = b'_1 + b'_2 + b'_3 + b'_4$.

(13) How many *odd numbers* between 100,000 and 1,000,000 have no two digits the same?

(14) How many positive divisors does $2^3 \cdot 5^4 \cdot 7^3$ have?

(15) Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $B = \{b_1, b_2, b_3, b_4\}$.

- How many functions are there from the set A to the set B?
- How many of these functions map A onto B?
- How many ways are there to put 6 objects to 4 distinct identical boxes so that no box will be left empty?
- How many ways are there to put 6 objects to 4 identical boxes so that no box will be left empty?

(16) How many integers between 1 and 10,000 (including 1 and 10,000) are not divisible by 2, 5, or 17?

(17) We are given the letters **I, I, I, I, M, P, P, S, S, S, S**.

- How many different words, i.e., strings, can be made using all of those letters?
- How many of those words have all the **I**'s together?
- How many of those words read the same backward and forward?
- If the letters are arranged in random order, what is the probability that they spell **MISSISSIPPI**?

(18) Let p be a prime.

- Prove that $\binom{p}{r}$ is a multiple of p for $0 < r < p$. Hint: $p! = \binom{p}{r} \cdot r! \cdot (p-r)!$.
- Prove that $(a+b)^p \equiv a^p + b^p \mod p$ for all integers a and b.

(19) Let $A = \{a_1, a_2, a_3, a_4, a_5, a_7, a_8, a_9\}$ and $B = \{b_1, b_2, b_3, b_4\}$.

(a) How many functions are there from the set A to the set B?

(b) How many of these functions map A onto B?

(c) How many ways are there to put 9 objects to 4 identical boxes so that no box will be left empty?

(20) Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ be a set of positive integers. Prove that there exist i, j, $i \neq j$ such that $a_i + a_j$ or $a_i - a_j$ is divisible by 10.

(21) Let n, k be positive integers.

- Let $n > k$. Prove the identity: $\sum_{r=0}^{n-1} (-1)^r \binom{n}{n-r} (n-r)^k = 0$.
- Prove the identity: $\sum_{r=0}^{n-1} (-1)^r \binom{n}{n-r} (n-r)^n = n!$.

(22) Show that for any positive integer n $\sum_{j=0}^{n} \binom{n}{j} 2^j = 3^n$.

(25) How many positive integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_7 + x_8 + x_9 = 27?$$

(23) How many ways are there to put 21 objects in 5 boxes with at least 8 objects in one box?

(24) Let $n = d_1 d_2 \cdots d_s$ be an integer, where $d_1, d_2, \ldots, d_s$ are decimal digits. Prove that n is divisible by 3 if and only if the sum $d_1 + d_2 + \cdots + d_s$ is divisible by 3.

(25) Let k, n be positive integers. Explain why there exists the greatest common divisor $\gcd(k, n)$.

(26) Write a binary decomposition of 2013, 2014 and 2015.

(27) Compute the value $\phi(2014)$, where ϕ is the Euler function.

(28) Let n be a positive integer and n is not a prime number. Prove that there exists a prime p such that p divides n.

(29) Prove that for any positive integer $n \geq 1$ the number $n^2 - 2$ is not divisible by 3.

(30) Let $b_0 = b_1 = b_2 = 1$, and $b_n = b_{n-1} + b_{n-3}$.

(a) Prove that $b_n \geq (\sqrt{2})^{n-2}$ for $n \geq 3$.

(b) Prove that $b_n \leq \left(\frac{3}{2}\right)^{n-1}$ for $n \geq 1$.

(31) For how many seven-digit integers have the sum of their digits equal to 47?

PART 2: MORE PROBLEMS ON MATHEMATICAL INDUCTION

(1) Prove that $2n^3 + 3n^2 + n$ is divisible by 6 for any $n \geq 1$.

(2) Prove that $7^n - 2^n$ is divisible by 5 for any $n \geq 1$.

(3) Prove that $3^{2n+3} + 40n - 27$ is divisible by 64 for any $n \geq 1$.

(4) Prove that $5^{2n+1} \cdot 2^{n+2} + 3^{n+2} \cdot 2^{2n+1}$ is divisible by 19 for any $n \geq 1$.

(5) Prove the following identity for any $n \geq 1$:

$$1 + 3 + 5 + \cdots + (2n-1) = \sum_{i=1}^{n} (2i-1) = n^2.$$

(6) Prove the following identity for any $n \geq 1$:

$$1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 = \sum_{i=1}^{n} (3i-2)^2 = \frac{n(6n^2-3n-1)}{2}.$$

(7) Prove the following identity for any $n \geq 1$:

$$2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2 = \sum_{i=1}^{n} (3i-1) = \frac{n(6n^2+3n-1)}{2}.$$

(8) Prove the following identity for any $n \geq 1$:

$$1 \cdot 2 + 2 \cdot 4 + \cdots + n(n+2) = \sum_{i=1}^{n} i(i+2) = \frac{n(n+1)(2n+7)}{6}.$$

(9) Prove that $3^{2n} - 1$ is divisible by 8 for any $n \geq 1$.

(10) Let $x \geq -1$. Prove that $(1+x)^n \geq 1 + nx$ for any $n \geq 1$.

(11) Prove that $n! < n^n$ for any $n \geq 2$.

(12) Prove the following identity for any $n \geq 1$:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} = \sum_{i=1}^{n-1} \frac{1}{i(i+1)} = \frac{n-1}{n}.$$

(13) Prove the following identity for any $n \geq 1$:

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}.$$

(14) Prove the following identity for any $n \geq 1$:

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{n}\right) = \prod_{i=2}^{n} \left(1 - \frac{1}{i}\right) = \frac{1}{n}.$$