Summary on Lecture 9, November 3, 2014

- **More examples on the principle of inclusion and exclusion:**

(3) Let $A=\{a_1,a_2,a_3,a_4,a_5,a_6\}$, $B=\{b_1,b_2,b_3,b_4,b_5\}$. Then $N=|\mathcal{F}(A,B)|=5^6$,

$$\begin{array}{lcl} N & = & 5^6 \\ N(c_i) & = & 4^6 \\ N(c_{i_1}c_{i_2}) & = & 3^6 \\ N(c_{i_1}c_{i_2}c_{i_3}) & = & 2^6 \\ N(c_{i_1}c_{i_2}c_{i_3}c_{i_4}) & = & 1^6 \end{array}$$

We obtain the answer:

$$\begin{aligned} |\mathcal{F}^{\text{onto}}(A,B)| &= 5^6-\binom{5}{1}4^6+\binom{5}{2}3^6-\binom{5}{3}2^6+\binom{5}{4}1^6 \\ &= 15,625-5\cdot 4,096+10\cdot 729-10\cdot 64+5\cdot 1 \\ &= 15,625-20,480+7,290-640+5=1,800 \end{aligned}$$

(4) **Euler function.** For given positive integer n, consider the set of numbers m such that $1\le m<n$ and $\gcd(m,n)=1$. Leonhard Euler defined the function:

$$\phi(n)=\|\{\ m\mid 1\le m<n,\ \text{and}\ \gcd(m,n)=1\ \}|.$$

Here is the values of $\phi(n)$ for some n:

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\phi(n)$	1	2	2	4	2	6	4	9	4	10	4	12	6	8	8	16

There is a simple formula to compute $\phi(n)$. Recall that for every integer n there exist primes $p_1,\dots,p_s$ and positive $e_1,\dots,e_s$ such that $n=p_1^{e_1}\cdots p_s^{e_s}$. Here is the formula:

$$\phi(n)=n\prod_{i=1}^{s}\left(1-\frac{1}{p_i}\right)$$

Example: Let $n=p_1^{e_1}p_2^{e_2}p_3^{e_3}p_4^{e_4}$, and $S=\{1,\dots,n\}$. We notice that for $m<n$ with $\ge (m,n)>1$, m has to be divisible by one of the primes p_i. We say that "m satifies c_i" iff $p_i|m$. Let

$$S_i=\{\ m\in S\mid p_i|m\ \},\ i=1,2,3,4.$$

Then $N=|S|=n$, $N(c_i)=|S_i|=\frac{n}{p_i}$. Then $N(c_ic_j)=\frac{n}{p_ip_j}$, $N(c_ic_jc_k)=\frac{n}{p_ip_jp_k}$, $N(c_1c_2c_3c_4)=\frac{n}{p_1p_2p_3p_4}$. Then

$$\begin{aligned} N(\bar c_1\bar c_2\bar c_3\bar c_4) &= n-\left(\tfrac{n}{p_1}+\tfrac{n}{p_2}+\tfrac{n}{p_3}+\tfrac{n}{p_4}\right)+\left(\tfrac{n}{p_1p_2}+\tfrac{n}{p_1p_3}+\tfrac{n}{p_1p_4}+\tfrac{n}{p_2p_3}+\tfrac{n}{p_2p_4}+\tfrac{n}{p_2p_4}\right) \\ &\quad -\left(\tfrac{n}{p_1p_2p_3}+\tfrac{n}{p_1p_2p_4}+\tfrac{n}{p_1p_3p_4}+\tfrac{n}{p_2p_3p_4}\right)+\tfrac{n}{p_1p_2p_3p_4} \end{aligned}$$

It is easy to check:

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) \quad = \quad \frac{n(p_1-1)(p_2-1)(p_3-1)(p_4-1)}{p_1p_2p_3p_4} = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)\left(1-\frac{1}{p_4}\right).$$

Examples: (1) Let p be a prime. Then $\phi(p) = p-1$, and $\phi(p^k) = p^{k-1}(p-1)$.
(2) Since $2014 = 2\cdot 19\cdot 53$, we obtain:

$$\phi(2014) = 2\cdot 19\cdot 53\left(1-\tfrac{1}{2}\right)\left(1-\tfrac{1}{19}\right)\left(1-\tfrac{1}{53}\right) = 18\cdot 52 = 936.$$

- **Recursive definitions.** There are many mathematical objects which we can define only *recursively*. We start with well-known example:

(1) **Fibonacci numbers** F_n. We define:
(B) $F_0 = 0,\ F_1 = 1$,
(R) $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
Here are the first few values of F_n:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

We prove that $\sum_{i=1}^{n} F_i^2 = F_nF_{n+1}$ by induction. Indeed, it's true if $n=1$.

Assume $\sum_{i=1}^{k} F_i^2 = F_kF_{k+1}$. Then

$$\begin{aligned}\sum_{i=1}^{k+1} F_i^2 &= \sum_{i=1}^{k} F_i^2 + F_{k+1}^2 \\ &= F_kF_{k+1} + F_{k+1}^2 = F_{k+1}(F_k + F_{k+1}) = F_{k+1}F_{k+2}.\end{aligned}$$

(2) We define a sequence of numbers a_n as:
(B) $a_0 = 1,\ a_1 = 1,\ a_2 = 1$, and
(R) $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.
Here are the first few values of a_n:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a_n	1	1	1	1	2	3	5	8	13	21	34	55	89	144	233	377

We notice that $a_n = F_{n-1}$ for $n \geq 3$. We would like to prove that $a_{n+2} \geq (\sqrt{2})^n$ for all $n \geq 0$. Indeed, it's true if $n = 0, 1$. Assume $a_{k+2} \geq (\sqrt{2})^k$ for all $k = 0, 1, \ldots, n$. We should prove that $a_{n+3} \geq (\sqrt{2})^{n+1}$. We have:

$$\begin{aligned} a_{n+3} = a_{n+2} + a_{n+1} &\geq (\sqrt{2})^n + (\sqrt{2})^{n-1} \\ &= (\sqrt{2})^{n-1}(\sqrt{2}+1) \geq (\sqrt{2})^{n-1}\cdot 2 = (\sqrt{2})^{n+1}.\end{aligned}$$

Here we use that $\sqrt{2}+1 \geq 2$ and $2 = (\sqrt{2})^2$.

(3) We can define recursively the binomial coefficients $\binom{n}{r}$:

(B) $\binom{n}{0} = 1$, $\binom{n}{0} = 0$ if $r < 0$ and $r > n$.

(R) $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$.

(4) We define factorial $\mathtt{FAC}(n)$:
(B) $\mathtt{FAC}(0) = 1$
(R) $\mathtt{FAC}(n) = \mathtt{FAC}(n-1) \cdot n$ for $n \geq 1$.

(5) We define the Harmonic numbers H_n:
(B) $H_1 = 1$
(R) $H_n = H_{n-1} + \frac{1}{n}$ for $n \geq 2$.

(6) We define the sequence $\mathtt{SEC}(n)$:
(B) $\mathtt{SEC}(0) = 1$
(R) $\mathtt{SEC}(n+1) = \frac{n+1}{\mathtt{SEC}(0)}$.

Exercise. Use induction to prove that the sequence $\mathtt{SEC}(n)$ is well-defined.

(7) We define the sequence $T(n)$ as follows:
(B) $T(1) = 1$
(R) $T(n) = 2 \cdot T(\lfloor \frac{n-1}{2} \rfloor)$ for $n \geq 2$.
We compute a couple of values of $T(n)$:

$$\begin{aligned} T(73) &= 2 \cdot T(36) = 2^2 \cdot T(18) = 2^3 \cdot T(9) = 2^4 \cdot T(4) = 2^5 \cdot T(2) = 2^6 \\ T(2014) &= 2 \cdot T(1007) = 2^2 \cdot T(503) = 2^3 \cdot T(251) = 2^4 \cdot T(125) = 2^5 \cdot T(62) \\ &= 2^6 \cdot T(31) = 2^7 \cdot T(15) 2^8 \cdot T(7) = 2^9 \cdot T(3) = 2^{10} \end{aligned}$$

Exercise. Use induction to prove that $T(n) = \max\{\ k \mid 2^k \leq n\ \}$.

Exercise. Define a sequence $S(n)$ such that $S(n) = \min\{\ k \mid n \leq 2^k\ \}$.

Exercise. Let p be a prime. Define recursively a sequence $T_p(n)$ such that

$$T(n) = \max\{\ k \mid p^k \leq n\ \}.$$

Exercise. Let p be a prime. Define recursively a sequence $S_p(n)$ such that

$$S_p; (n) = \min\{\ k \mid n \leq p^k\ \}.$$

Exercise. Define recursively what does it mean "well-formed formula", see Ex. 17, p. 220.