Summary on Lecture 2, October 1, 2014

Notation: $a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i$.

Let $A = \{a_1, \dots, a_n\}$. How many subsets of size k are there in A? In other words, how many selections of k elements with no reference to their order are there? The asnwer:

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{P(n,k)}{k!}.$$

Conventions: $0! = 1$, $\binom{n}{0} = \frac{n!}{n!\,0!} = 1$.

Examples: There are 52 cards and then there are

$$\binom{52}{5} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!} = 2,598,960$$

different "hands".

(1) **Full house.** Recall that "full house" means that a "hand" has 3 cards of one value, and 2 of another. Say, if we have 3 Jacks, and 2 8's, we say that this is a full house of the type $(J, 8)$. Then there are $13 \cdot 12$ different types of "full houses". Then we can choose 3 out of 4 suits for J, and 2 out of 4 for 8. Here the number of "full houses":

$$13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2} = 13 \cdot 12 \cdot 4 \cdot 6 = 3,744$$

(2) **Two pairs.** Recall that "two pairs" means that a "hand" has 2 cards of one value, 2 of the second, and the remaining card of the third value. Say, if we have 2 queens and 2 4's, we denote such type of "two pairs" as the set $\{Q, 4\}$. Clearly there are $\binom{13}{2}$ types of "two pairs". Here the number of "two pairs":

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44 = 123,552$$

(2) **Straights.** Recall that "royal flush" is a hand with 10, J, Q, K, A of the same suit. Clearly there are 4 royal flushes. Then a "straight flush" is "straight", say, 8,9,10,J,Q, of the same suit. A hand with A,2,3,4,5 is also a straight. Then a hand is "straight" if it is straight, but it is not "royal flush" or "straight flush". For each straight we can keep a record of the top card. This gives 10 types of straights. Then there are 4 choices for each card. Thus there are $10 \cdot 4^5 = 10,240$ "straights" including 'royal flushes" and "straight flushes". Since there are 36 "straight flushes", the number of "straights" is $10,240 - (4 + 36) = 10,200$.

(3) Count the number of poker hands of the following kinds:

(a) "four of the kind";
(b) "flush" but not "royal flush";
(c) "three of the kind";
(d) "one pair".

(4) Let $\Sigma = \{0, 1, 2\}$ be an alphabet. For each word (string) $\bar{x} = x_1 \dots x_r \in \Sigma^r$, we define a weight $w(\bar{x}) = x_1 + \cdots + x_r$. How many words of length $2n$ have even weight? Hint: consider the cases $2n = 4, 6, 8$. The answer:

$$\sum_{i=0}^{n} \binom{2n}{2i} 2^{2n-2i}.$$

We notice: $\binom{n}{k} = \binom{n}{n-k} = \dfrac{n!}{(n-k)! \cdot k!}$.

Theorem 1. (Binomial theorem)

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = \sum_{r=0}^{n} \binom{n}{n-r} x^r y^{n-r}.$$

Examples:

$$\sum_{r=0}^{n} \binom{n}{r} = (1+1)^n = 2^n$$

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = (1-1)^n = 0$$

Pascal's triangle: Prove that $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. This gives us the Pascal's triange:

1	$(x+y)^0$
1 1	$(x+y)^1$
1 2 1	$(x+y)^2$
1 3 3 1	$(x+y)^3$
1 4 6 4 1	$(x+y)^4$
1 5 10 5 1	$(x+y)^5$
1 6 15 15 6 1	$(x+y)^6$
1 7 21 30 21 7 1	$(x+y)^7$
1 8 28 51 51 28 8 1	$(x+y)^8$
1 9 36 79 102 79 36 9 1	$(x+y)^9$

Notation: Let $n = n_1 + \cdots + n_s$. Then we denote $\binom{n}{n_1 \dots n_s} = \frac{n!}{n_1! \cdots n_s!}$.

Theorem 2. (Multinomial theorem)

$$(x_1 + \cdots + x_s)^n = \sum_{n_1 + \cdots + n_s} \binom{n}{n_1 \dots n_s} x_1^{n_1} \cdots x_s^{n_s}.$$

Examples:

$$(x_1 + x_2 + x_3)^7 = \sum_{n_1+n_2+n_3=7} \binom{7}{n_1 \dots n_s} x_1^{n_1} x_2^{n_2} x_3^{n_3}$$

Say, the monomial $x_1^2 x_2^3 x_3^2$ has the coefficient

$$\binom{7}{2\ 3\ 2} = \frac{7!}{2!3!2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 2 \cdot 3 \cdot 2} = 7 \cdot 6 \cdot 5 = 210$$

Placing objects to boxes.

Theorem. There are $\binom{r+n-1}{n-1} = \binom{r+n-1}{r}$ ways to place r identical objects to n distinguishable boxes.

Proof. First, we let $r = 6$, $n = 4$. We'll represent objects by six 0's, and we add three l's to serve as dividers among the four boxes. We claim that there is a one-to-one correspondence between the strings consisting of six 0's and three l's and the ways to place the five 0's into four boxes. For example:

000110001 $\mapsto$ 000| |000| $\mapsto$

000		000	

100100010 $\mapsto$ |00| 000 |0 $\mapsto$

	00	000	0

Thus, in general case every string of r 0' and $(n-1)$ 1's corresponds to a placement of r identical objects to n distinguishable boxes. Clearly, every placement of r identical objects to n distinguishable boxes gives such a string. Then it is easy to count how many strings like that do we have. Indeed, the length of the string is $(r+n-1)$, then we have to choose places for 1' (we have $(n-1)$ of 1's), or, equivalently, for 0's (we have r 0's). We get the answer: $\binom{r+n-1}{n-1} = \binom{r+n-1}{r}$.

Selection with repetition. Now we would like to "reverse" the process of placement r identical objects to n distinguishable boxes. To explain what's going on, we let $r = 6$, $n = 4$ again. Moreover, we label 4 boxes with the letters P, N, D, Q which stand for "penny", "nickel", "dime" and "quarter". Now we would like to count, how many ways are there to select 6 coins with repetition out of those four boxes? What we can do here is to first select 6 coins, say, 2 N's, 3 D's and 1 Q, then we can place them back to their original boxes. In other words, it is exactly the same number as the number of placements 6 identical objects to 4 distinguishable boxes. In general, we have the same answer as above: $\binom{r+n-1}{n-1} = \binom{r+n-1}{r}$.

Exercises:

(1) Determine number of integral solutions $x_i \geq 0$, $i = 1, \ldots, n$ of the equation $x_1 + \cdots + x_n = r$.

(2) Determine number of integral solutions $x_i \geq 1$, $i = 1, \ldots, n$ of the equation $x_1 + \cdots + x_n = r$.

(3) Determine number of integral solutions $x_i \geq 0$, $i = 1, \ldots, n$ of the inequality $x_1 + \cdots + x_n \leq r$.

(4) Determine number of integral solutions $x_i \geq 0$, $i = 1, \ldots, n$ of the inequality $x_1 + \cdots + x_n < r$.

(5) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 9?

(6) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 10?

(7) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 14?

(8) Determine how many integers between 1 and 1,000,000 have the sum of their digits equal to 21?