Summary on Lecture 14, November 26, 2014

• **The Pigeon Hole Principle.** The usual Pigeon-Hole Principle asserts that, if m objects are placed in n boxes or pigeon-holes and if $m > n$, then some box will receive more than one object.

Examples.

(1) Let $A = \{a_1, \ldots, a_{1008}\} \subset S = \{1, 2, \ldots, 2014\}$. Then there exist $a_i, a_j \in A$ such that a_i divides a_j. Indeed, any number $b \in S$ could be written as $b = 2^k c$, where c is an odd number. Thus we have that $c \in T = \{1, 3, \ldots, 2013\}$, where $|T| = 1007$. Since there are 1008 integers in A, there must be two integers $a_i = 2^{k_i} c_i$ and $a_j = 2^{k_j} c_j$, where $c_i = c_j$. Then if $k_i \geq k_j$, then a_j divides a_i. If $k_j \geq k_i$, then a_i divides a_j.

(2) Let k be a positive odd integer. Then there exists n such that k divides $2^n - 1$. Indeed, we consider $k + 1$ numbers
$$2^1 - 1,\ 2^2 - 1,\ \ldots,\ 2^k - 1,\ 2^{k-1} - 1.$$
Then for each $i = 1, \ldots, k + 1$, we divide $2^i - 1$ by k and find a remainder r_i:
$$2^i - 1 = k \cdot q_i + r_i, \quad 0 \leq r_i \leq k - 1.$$
Sinnce we have $k+1$ remainders r_i, and there are only k possible values, we conclude that $r_i = r_j$ for some $i > j$. Then the difference
$$(2^i - 1) - (2^j - 1) = 2^i - 2^j = 2^j (2^{i-j} - 1)$$
has to be divisible by k. Since k is odd, we conclude that $(2^{i-j} - 1)$ is divisible by k.

(3) Let A be a 10-element subset of $\{1, 2, 3, \cdots, 50\}$. Then A has two different 4-element subsets, the sums of whose elements are equal.

Indeed, let $B = \{b_1, b_2, b_3, b_4\} \subset A$. Then we notice that the minimal possible value of the sum $b_1 + b_2 + b_3 + b_4$ is $1 + 2 + 3 + 4 = 10$, and the maximal possible value is $47 + 48 + 49 + 50 = 194$. Then we count the size of the set
$$\mathcal{B} = \{\ B = \{b_1, b_2, b_3, b_4\} \mid B \subset A\ \}.$$
Clearly, for given B, we just have to choose 4 elements out of 10. We obtain that
$$|\mathcal{B}| = \binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 3 \cdot 7 = 210.$$
For each $B = \{b_1, b_2, b_3, b_4\} \in \mathcal{B}$, we compute the value $\Sigma(B) = b_1 + b_2 + b_3 + b_4$. We have noticed that the values of Σ are in the range $\{10, 11, \ldots, 194\}$. Since $|\mathcal{B}| = 210 > 194$, there must be at least two different $B, B' \in \mathcal{B}$ with $\Sigma(B) = \Sigma(B')$.

(4) Consider nine nonnegative real numbers $a_l, a_2, a_3, \ldots, a_9$ with sum 90. First we show that there must be three of the numbers having sum at least 30. This is easy because
$$90 = (a_1 + a_2 + a_3) + (a_4 + a_5 + a_6) + (a_7 + a_8 + a_9),$$
so at least one of the sums in parentheses must be at least 30.

Next, we show that there must be four of the numbers having sum at least 40. Consider the table:

a_1	a_2	a_3	$\cdots$	a_8	a_9	90
a_2	a_3	a_4	$\cdots$	a_9	a_1	90
a_3	a_4	a_5	$\cdots$	a_1	a_2	90
a_4	a_5	a_6	$\cdots$	a_2	a_3	90
Σ_1	Σ_2	Σ_3	$\cdots$	Σ_8	Σ_9	360

Here we shifted the sequence of a_i's, and Σ_j's are the sums of the elements in j-th column, and the last column gives the sum of all a_i's, which is 90. Then we must have that

$$\Sigma_1 + \Sigma_2 + \cdots + \Sigma_9 = 390.$$

This cannot happen if $\Sigma_i < 40$ for each $i = 1, \ldots, 9$. Thus there exists i so that $\Sigma_i \geq 40$.

(5) **Exercise.** Let $a_l + a_2 + a_3 + \cdots + a_9 = 90$ as above, and $k = 5, 6, 7, 8$. Prove that there exist k numbers $a_{i_1}, \ldots, a_{i_k}$ such that $a_{i_1} + \cdots + a_{i_k} \geq 10k$.

• **Pigeon-Hole Principle and rational numbers.** Let $k < n$ be two positive integers. We have a decimal expansion:

$$\frac{k}{n} = 0.d_1 d_2 d_3 \cdots d_j \cdots \ .$$

Sometimes this decimal expansion is finite, however, there are many examples when the expansion is infinite:

$$\begin{aligned} \tfrac{1}{3} &= 0.\underline{3}33333333\cdots \ , \\ \tfrac{7}{11} &= 0.\underline{63}636363\cdots \ , \\ \tfrac{29}{54} &= 0.5\underline{370}370370\cdots \ . \end{aligned}$$

We notice that in the above examples the groups of underlined digits are repeated, and, possibly, going like that forever. We describe the above decimal decomposition of $\frac{7}{11}$ and compare with the decimal decomposition of the fraction $\frac{k}{n}$. We have:

$$\begin{array}{rclcrcl} 10 \cdot 7 &=& 6 \cdot 11 + 4 & \text{or} & 10k &=& d_1 n + r_1 \\ 10 \cdot 4 &=& 3 \cdot 11 + 7 & \text{or} & 10r_1 &=& d_2 n + r_2 \\ 10 \cdot 7 &=& 6 \cdot 11 + 4 & \text{or} & 10r_2 &=& d_3 n + r_3 \\ 10 \cdot 4 &=& 3 \cdot 11 + 7 & \text{or} & 10r_3 &=& d_4 n + r_4 \\ 10 \cdot 7 &=& 6 \cdot 11 + 4 & \text{or} & 10r_4 &=& d_5 n + r_5 \\ \cdots & & \cdots & & \cdots & & \cdots \end{array}$$

For the fraction $\frac{7}{11}$, we have that the remainders r_1 and r_3 are the same. Since a division by 11 gives a unique answer, we have that

$$d_2 = d_4 \quad \text{and} \quad r_2 = r_4.$$

Similarly, $d_3 = d_5$ and $r_3 = r_5$ and so on. We obtain:

$$4 = r_1 = r_3 = r_5 = \cdots = r_{2i-1} = \cdots, \quad 7 = r_2 = r_4 = r_6 = \cdots = r_{2i} = \cdots,$$

$$3 = d_2 = d_4 = d_6 = \cdots = d_{2i} = \cdots, \quad 6 = d_3 = d_5 = d_7 = \cdots = d_{2i+1} = \cdots.$$

Theorem. *Let $0 < k < n$ be two integers and*

$$\frac{k}{n} = 0.d_1d_2d_3\cdots d_j\cdots \tag{1}$$

be a corresponding decimal expansion. Then there exist integers $s, t > 0$ such that the blocks of t digits

$$\underline{d_sd_{s+1}\cdots d_{s+t-1}}, \quad \underline{d_{s+t}d_{s+t+1}\cdots d_{s+2t-1}}, \quad \cdots \quad \underline{d_{s+\ell t}d_{s+\ell t+1}\cdots d_{s+(\ell+1)t-1}}, \cdots \tag{2}$$

in the expansion (1) are identical for all $\ell = 1, 2, \cdots$.

Proof. We consider the process of obtaining a decimal decomposition (1):

$$\begin{array}{rcll}
10k &=& d_1n + r_1 & 0 \le r_1 \le n-1 \\
10r_1 &=& d_2n + r_2 & 0 \le r_2 \le n-1 \\
10r_2 &=& d_3n + r_3 & 0 \le r_3 \le n-1 \\
10r_3 &=& d_4n + r_4 & 0 \le r_4 \le n-1 \\
10r_4 &=& d_5n + r_5 & 0 \le r_5 \le n-1 \\
\cdots & & \cdots & \cdots \\
10r_n &=& d_{n+1}n + r_{n+1} & 0 \le r_{n+1} \le n-1 \\
\cdots & & \cdots & \cdots
\end{array}$$

We consider the remainders $r_1, r_2, \ldots, r_{n+1}$. Since

$$r_1, r_2, \ldots, r_{n+1} \in \{0, \ldots, n-1\},$$

there are two indices i, j, $1 \le i < j \le n+1$, such that $r_i = r_j$. Then we have:

$$\begin{array}{rcl}
10r_i &=& d_{i+1}n + r_{i+1} \\
10r_j &=& d_{j+1}n + r_{j+1}
\end{array}$$

Since a division by n gives a unique result, $r_{i+1} = r_{j+1}$ and $d_{i+1} = d_{j+1}$. Then we have

$$\begin{array}{rcl}
10r_{i+1} &=& d_{i+2}n + r_{i+2} \\
10r_{j+1} &=& d_{j+2}n + r_{j+2}
\end{array}$$

which gives $r_{i+2} = r_{j+2}$ and $d_{i+2} = d_{j+2}$. We can choose $s = i+1$ and $t = j - i$ to see that the first two blocks in (2) are indeed identical. Then the same argument shows that all other blocks in (2) are identical as well. $\square$

We obtain that the desimal expansion of a rational number $\frac{k}{n}$ is a *periodic fraction*:

$$\frac{k}{n} = 0.d_1d_2d_3\cdots d_{s-1}\underline{d_sd_{s+1}\cdots d_{s+t-1}}\ \underline{d_sd_{s+1}\cdots d_{s+t-1}}\cdots\underline{d_sd_{s+1}\cdots d_{s+t-1}}\cdots$$

for some s and t.