Here is an attempt to clarify the issues of calculating matrix determinants and inverse matrices. I am unable to provide a full explanation, but I want to make clear how determinants and inverse matrices can be calculated, how the formula for determinants is to be understood, and what cofactors have to do with all that.

The Determinant I: Permutations

The first way to calculate the determinant is given by the formula for the determinant—if we make it intelligible. So let's do that.

$$|\mathbf{A}| = \sum (-1)^{f(j_1,j_2,\ldots,j_p)} \prod_{i=1}^{p} a_{ij_i}$$

From the formula we see that the determinant is a sum over a number of terms. These terms are themselves products of diagonal elements a_{ij_i}. These terms have either a positive sign or a negative sign, depending on $(-1)^{f(j_1,j_2,\ldots,j_p)}$. Let's be more specific. To get the product terms on the right of the formula we have to do two things: first, permutate the columns of the matrix as often as we can (changing the original columns $1, \ldots, p$ into a different order $j_1, j_2, \ldots, j_p$); second, form the product of the resulting main diagonals: $\prod_{i=1}^{p} a_{ij_i}$. From these permutations we also get the number (even or odd) that decides on the sign for our terms: The original order of columns (which counts as the first permutation) has 0 changes, so $f(j_1, j_2, \ldots, j_p) = 0$ and –1 to the power of 0 makes 1. Any permutation in which two columns are exchanged has 1 change, so $f(j_1, j_2, \ldots, j_p) = 1$ and –1 to the power of 1 makes –1; and so on. The terms (with their appropriate signs) are then simply summed up and the resulting number is the determinant. Let's look at an example.

$$\text{Matrix } \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

A matrix with two columns has only two permutations, namely, the original one and the one where the first and the second column switch. So the determinant of matrix **A** is $1 \cdot 4 - 2 \cdot 3 = -2$. The term $1 \cdot 4$ is the product of diagonal elements from the original matrix (its sign has to be positive because $(-1)^0 = 1$). The term $2 \cdot 3$ is the product of diagonal elements from the first (and only) permutation of **A** (its sign has to be negative because $(-1)^1 = -1$. (To avoid confusion, write out all the permutations of a matrix first and then form the products of the main diagonals—don't combine main diagonals with secondary diagonals unless you have a reliable algorithm to do that.)

Let's do a more difficult example now with a 3x3 matrix.

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The original column order (c1, c2, c3) gives us the product $1 \cdot 5 \cdot 9$ (no changes, sign is positive). A first permutation (c1, c3, c2) gives us the product $1 \cdot 6 \cdot 8$ (1 change, sign is negative). A second permutation of the same type (c2, c1, c3) gives us the product $2 \cdot 4 \cdot 9$ (1 change, sign is negative). A third permutation of this type (c3, c2, c1) gives us the product $3 \cdot 5 \cdot 7$ (1 change, sign is negative). Finally there are two permutations with two changes. The first (c2, c3, c1) gives us the product $2 \cdot 6 \cdot 7$ (2 changes, sign is positive). The second (c3, c1, c2) gives us the product $3 \cdot 4 \cdot 8$ (2 changes, sign is positive). So here is the determinant of **B**:

$$|\mathbf{B}| = 1 \cdot 5 \cdot 9 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 9 - 3 \cdot 5 \cdot 7 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8.$$

The Determinant II: Cofactors

There is a second way to calculate the determinant, namely, by means of cofactors. In order to calculate the determinant of a matrix **A** (with elements a_{ij}, we perform the following steps.

To begin with, we have to find the *cofactor* of each element. (There are as many cofactors in a matrix as there are elements). The cofactor of a_{ij} is defined as $(-1)^{i+j}$ times the *minor* of a_{ij}. Let's first see how we get the minor. The minor is itself a determinant, but on a reduced matrix: to get the minor of a particular element a_{ij}, you delete the ith row and the jth column (these are the row and column in which the element a_{ij} lies), and what remains is either a number or a smaller, reduced matrix. When do you get a number, when a matrix? If you are trying to find the determinant of a 2x2 matrix, the minors will be single numbers; for a 3x3 matrix, the minors will be 2x2 matrices, for 4x4 matrices, minors will be 3x3 matrices, and so on.

Once we have the minor for our first element, we have to figure out whether it has a positive sign or a negative sign—this is what the term $(-1)^{i+j}$ provides. You either get +1 (if the sum of i and j was even) or –1 (if the sum of i and j was odd). You just sum up the row number and the column number of the element whose minor you just calculated (e.g., $1 + 2 = 3$ for element a_{12}) and take –1 to the power of that number, here $(-1)^3 = -1$. Now we have the cofactor, namely, the minor with either a positive sign or a negative sign. We proceed with this method of finding cofactors for each and every element in the original matrix. Let's look at a simple 2x2 matrix, with elements 1, 2, 3, and 4.

$$\text{Matrix } \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \text{Matrix of cofactors} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

We are looking for the cofactor of element a_{11} first. If we delete the first row and the first column of **A** (the row and column of a_{11}), we are left with a number, namely 4 (the element a_{22}). So a_{22} is the minor of a_{11}. Now we have to find its sign. The sum of i and j is 2, so the sign is positive, because $(-1)^2 = +1$. Cofactors are denoted A_{ij}, corresponding to the original elements a_{ij}, so the cofactor of a_{11} is $A_{11} = 4$. Next, we find the minor of a_{12}. If we delete the first row and the second column of **A** (the row and column of a_{12}), we are left with a number—this time 3 (the element a_{21}). The sum of i and j is 3, so the sign is negative, because $(-1)^3 = -1$. Thus, the cofactor $A_{12} = -3$. Continuing like this, we find that the cofactors of elements in 2x2 matrices are other elements.

Now we are one step short of calculating the determinant of **A**. We only need the elements from the first row of **A** and their corresponding cofactors. We multiply each element from the first row of the original matrix $(a_{11}, ...a_{1p})$ by its cofactor $(A_{11}, ..., A_{1p})$ and sum over the pairwise products: $\sum_{j=1}^{p} a_{1j}A_{1j}$. In the above example, we have $4 \cdot 1 - 3 \cdot 2 = -2$, which is the determinant of **A**. (We could do the same with *any* row of **A**—we would always arrive at the same number for the determinant. For example, taking the second row of **A**, $3 \cdot -2 + 4 \cdot 1 = -2$. So the determinant is defined as $\sum_{j=1}^{p} a_{ij}A_{ij}$, for any i.)

Now let's put the two ways of calculating determinants together, namely, the one based on cofactors and the one based on products of permutated diagonals. They should give the same results, so let's test that for matrix **B** whose determinant we already calculated using the permutation method. Here is the stepwise calculation using cofactors:

element a_{ij}	$i+j$	cofactor A_{ij}	$a_{ij} \cdot A_{ij}$
1	2	$5 \cdot 9 - 6 \cdot 8$	$1 \cdot 5 \cdot 9 - 1 \cdot 6 \cdot 8$
2	3	$-(4 \cdot 9 - 7 \cdot 6)$	$-2 \cdot 4 \cdot 9 + 3 \cdot 7 \cdot 6$
3	4	$4 \cdot 8 - 5 \cdot 7$	$3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7$

If you look at the terms in the far right column you will see that they are exactly the same as the ones we got from the permutation method. This has to be the case because $|\mathbf{A}| = \sum_{j=1}^{p} a_{ij} A_{ij}$. I cannot tell you why exactly the two formulas are equivalent, but at least you may appreciate the convergence of the two methods.

The Inverse Matrix

The permutation method might be a little more handy, but the cofactor method is closer to the definition of an inverse matrix. In fact, the elements of $\mathbf{A}^{-1}$, which is the inverse of $\mathbf{A}$, are on the basis of the cofactors. Let's look at the formula.

$$\text{elements of } \mathbf{A}^{-1}\text{: } a^{ij} = \frac{A_{ji}}{|\mathbf{A}|}$$

Any inverse element a^{ij} is the transposed cofactor of the corresponding original element a_{ij} divided by the determinant. Transposing means taking the cofactor A_{ij} and switching its row number with its column number so it becomes A_{ji}. The elements a^{ij} together form the inverse matrix $\mathbf{A}^{-1}$; the transposed cofactors A_{ji} together form the so-called adjoint matrix $\mathbf{A}^{+}$. Thus the inverse matrix is defined as follows:

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}^{+}$$

By implication, for any 2x2 matrix $\mathbf{M}$ (with elements a, b, c, d), $\mathbf{M}^{-1}$ is defined as

$$\mathbf{M}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{|\mathbf{M}|} & \frac{-b}{|\mathbf{M}|} \\ \frac{-c}{|\mathbf{M}|} & \frac{a}{|\mathbf{M}|} \end{bmatrix},$$

$$|\mathbf{M}| = ad - bc.$$

Outlook

To compute the inverse of 3x3 matrices, we first need to form the 9 cofactors (one for each element), which are themselves determinants of 2x2 submatrices. Next, we transpose the matrix of these 9 cofactors to get the adjoint matrix. Then we calculate the determinant (using the already computed cofactors of the first row) and divide each transposed cofactor by the determinant. The result is the 3x3 inverse matrix.

You can already see that determinants of higher-dimensional matrices are built upon the determinants and cofactors of lower-dimensional matrices. For example, the determinant of a 3x3 matrix requires 6 determinants of 2x2 submatrices; the determinant of a 4x4 matrix requires 12 determinants of 3x3 matrices, which makes 72 2x2 submatrices, and so on. Perhaps you value your computer now a bit more.