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Noncommutative marked surfaces

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A R T I C L E   I N F O

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To the memory of Andrei Zelevinsky
Светлой памяти Андрея Владленовича Зелевинского посвящается

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A B S T R A C T

The aim of the paper is to attach a noncommutative cluster-like structure to each marked surface Σ. This is a noncommutative algebra $\mathcal{A}_\Sigma$ generated by "noncommutative geodesics" between marked points subject to certain triangle relations and noncommutative analogues of Ptolemy–Plücker relations. It turns out that the algebra $\mathcal{A}_\Sigma$ exhibits a noncommutative Laurent Phenomenon with respect to any triangulation of Σ, which confirms its "cluster nature". As a surprising byproduct, we obtain a new topological invariant of Σ, which is a free or a 1-relator group easily computable in terms of any triangulation of Σ. Another application is the proof of Laurentness and positivity of certain discrete noncommutative integrable systems.

Contents

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1. Introduction

The goal of the paper is to introduce and study noncommutative algebras attached to surfaces (with marked boundary points and punctures) and their triangulations. This provides an instance of the noncommutative cluster theory (which is the main theme of the forthcoming paper [3]).

Since each surface can be obtained by gluing edges of a polygon (actually, in many ways), the most important object of study are *noncommutative polygons* and their *noncommutative triangulations*.

In the commutative case, cluster structure (of type A_{n-3}) on an n-gon is based on the *Ptolemy relations*:

$$x_{ik}x_{j\ell} = x_{ij}x_{k\ell} + x_{i\ell}x_{jk} \tag{1.1}$$

for all quadrilaterals (i, j, k, ℓ) inscribed in a circle, $1 \le i, j, k, \ell \le n$, so that the chords (i, k) and (j, ℓ) are diagonals of the quadrilateral, and $x_{ij} = x_{ji}$, $i \ne j$ is the Euclidean length of the chord (ij). The Ptolemy relations (1.1) can also be interpreted as Plücker identities for $2 \times n$ matrices.

In the noncommutative version we do not assume that $x_{ij} = x_{ji}$ and we think of x_{ij} as a measurement of a directed chord from i to j. We suggest the following noncommutative generalization of the Ptolemy identity based on the theory of noncommutative quasi-Plücker coordinates developed in [17]:

$$x_{ik}x_{jk}^{-1}x_{j\ell} = x_{i\ell} + x_{ij}x_{kj}^{-1}x_{k\ell}, \tag{1.2}$$

for every quadrilateral (i, j, k, ℓ), in which (i, k) and (j, ℓ) are the diagonals.

Note that since elements x_{ij} correspond to directed arrows, the products of the form $x_{ij}x_{k\ell}^{-1}$, $x_{\ell k}^{-1}x_{ji}$ make sense only when $\ell = j$.

It turns out that in order to establish the noncommutative Laurent Phenomenon and thus to obtain a noncommutative cluster structure on the n-gon, it is crucial to impose additional *triangle relations* (also suggested by properties of quasi-Plücker coordinates):

$$x_{ij}x_{kj}^{-1}x_{ki} = x_{ik}x_{jk}^{-1}x_{ji} \tag{1.3}$$

for all distinct i, j, k (of course, (1.3) is redundant in the commutative case).

The triangle relations (1.3) are of fundamental importance because they allow to introduce *noncommutative angles* $T_i^{j,k} := x_{ji}^{-1}x_{jk}x_{ik}^{-1}$ in each triangle (i, j, k) so that $T_i^{j,k} = T_i^{k,j}$ due to (1.3). That is, the noncommutative angle at a vertex of a triangle does not depend on the order of the remaining two vertices. The "commutative" angles were introduced by Penner in [24, Section 3] (where they were called "h-lengths") and each $x_{ij} = x_{ji}$ was viewed as the λ-length of the side (i, j) of an ideal triangle (i, j, k) (see also [13, Lemma 7.9], [11, Section 12], and [14, Section 1.2], in the latter paper the term "angle" was used, apparently, for the first time) and thus noncommutative angles together with the "noncommutative λ-lengths" x_{ij} can be thought of as a totally noncommutative metric on the Lobachevsky plane. The term "angle" is justified by the following observation. The noncommutative Ptolemy relations (1.2) together with the triangle relations (1.3) are equivalent to:

$$T_j^{ik} = T_j^{i\ell} + T_j^{k\ell}$$

for every quadrilateral (i, j, k, ℓ), in which (i, k) and (j, ℓ) are the diagonals. In other words, the (both commutative and noncommutative) angles are additive, which justifies the name. Using additivity of noncommutative angles, we establish the first instance of the noncommutative Laurent Phenomenon for the n-gon with vertices $1, \ldots, n$:

$$x_{ij} = \sum_{k=i}^{j-1} x_{i,1}T_1^{k,k+1}x_{1,j}$$

for all $2 \le i < j \le n-1$, e.g., each x_{ij} is a noncommutative Laurent polynomial in $x_{1,k}$, $x_{k,1}$, $k = 2, \ldots, n-1$ and all $x_{i,i\pm 1}$. In fact, the latter elements correspond to a triangulation of the n-gon where each triangle has a vertex at 1. We generalize this to any triangulation of the n-gon in Theorem 2.10, and, as expected, the commutative "limit" of this result (with all $x_{ij} = x_{ji}$) specializes to the Schiffler formula ([25, Theorem 1.2]).

These arguments extend verbatim if we replace a polygon with a surface Σ with marked points. That is, for each such Σ one defines a $\mathbb{Z}$-algebra $\mathcal{A}_\Sigma$ generated by $x_\gamma^{\pm 1}$, where γ runs over homotopy classes of curves on Σ between marked points subject to the triangle and noncommutative Ptolemy relations. The Noncommutative Laurent Phenomenon (Theorem 3.30) asserts that for a given triangulation Δ of Σ each x_γ belongs to the subalgebra generated by all $x_{\gamma'}^{\pm 1}$, $\gamma' \in \Delta$. In any case, the assignments $\Sigma \mapsto \mathcal{A}_\Sigma$ and $\Sigma \mapsto \mathbb{T}_\Sigma$ define functors from the category of surfaces with marked points to respectively the category of algebras and the category of groups (Theorem 3.16).

A surprising byproduct of our approach is that the corresponding *triangle group* $\mathbb{T}_\Delta$ (generated by all t_γ, $\gamma \in \Delta$ subject to the triangle relations) does not depend on the triangulation of Σ and, therefore, is a topological invariant of Σ (Theorem 3.24). Moreover, each $\mathbb{T}_\Delta$ is either free or a one-relator group which looks like the fundamental group of Σ, however it is different from $\pi_1(\Sigma)$. For instance, if Σ_n is the sphere with n punctures, then $\mathbb{T}_\Delta$ is a free group in 5 generators if $n = 3$ and it is a 1-relator torsion-free group in $4n - 7$ generators if $n \geq 4$.[1] It turns out that each group $\mathbb{T}_\Delta$ has a "universal cover" $\mathbb{T}_\Sigma$ which is a group generated by t_γ, as γ runs over all isotopy classes of directed curves on Σ between marked point, subject to the triangle relations (see Sections 2.5 and 3.5 for details). This group, which we refer to as *big triangle group* is of interest as well: if Σ is the n-gon, we prove (Proposition 2.28) that $\mathbb{T}_\Sigma$ has a presentation with $\frac{(n-1)(n+2)}{2}$ generators and $(n-3)^2$ relations and expect that the multiplicative group of $\mathcal{A}_\Sigma$ is isomorphic to $\mathbb{T}_\Sigma$.

For each marked point i on Σ and each triangulation Δ we also introduce a *total (noncommutative) angle* $T_i^\Delta \in \mathcal{A}_\Sigma$ in Section 3.9 to be the sum of noncommutative angles of all adjacent triangles. Similarly to the commutative case, we establish (Theorem 3.40) that the total angles do not depend on the choice of a triangulation Δ. Thus the collection of the total angles $\{T_i\}$ can be thought of as a noncommutative version of a (hyperbolic) Riemann structure on Σ. Using them we define in Section 3.9 the algebra $\mathcal{U}_\Sigma$ to be the subalgebra of $\mathcal{A}_\Sigma$ generated by all noncommutative edges x_γ, the inverses of the boundary edges and all noncommutative angles T_i and argue that $\mathcal{U}_\Sigma$ is a totally noncommutative analogue of the *upper cluster algebra* corresponding to Σ (see e.g., [1]).

As an application of our noncommutative Laurent phenomenon, taking Σ to be a cylinder with no punctures, one marked point on the inner boundary and k marked points on the outer boundary, we prove Laurentness of the following noncommutative recursion for each $k \in 1 + 2\mathbb{Z}_{>0}$:

[1] Misha Kapovich explained to us that $\mathbb{T}_\Delta$ is related to the fundamental group of a ramified two-fold cover of Σ.

$$\begin{cases} U_{n-k}DU_n = C_n + U_{n-1}\overline{D}U_{n+1-k} & \text{if } n \text{ is even} \\ U_n\overline{D}U_{n-k} = C_n + U_{n+1-k}DU_{n-1} & \text{if } n \text{ is odd} \end{cases} \tag{1.4}$$

for all $n \geq k+1$, where $D, \overline{D}$, and C_i, $i \in \mathbb{Z}_{>0}$ belong to a noncommutative ground ring so that $C_{n+k-1} = C_{k-1}$ for $n \in \mathbb{Z}_{>0}$.

We prove (Theorem 4.5) that for odd $k > 0$ this recursion has a unique solution in the group algebra $\mathbb{Q}F_{2k+1}$ of the free group F_{2k+1} freely generated by $D, \overline{D}, C_1, \ldots, C_{k-1}$, $U_1, \ldots, U_k$, more precisely, each U_n is a sum of elements of F_{2k+1}. We also prove (Theorem 4.5) that the element H_n in the skew field of fractions of $\mathbb{Q}F_{2k+1}$, $n \geq k$, given by

$$H_n := \begin{cases} \overline{D}U_{n+1-k}U_n^{-1} + DU_{n+k-1}U_n^{-1} & \text{if } n \text{ is even} \\ U_n^{-1}U_{n+1-k}D + U_n^{-1}U_{n+k-1}\overline{D} & \text{if } n \text{ is odd} \end{cases} \tag{1.5}$$

belongs to $\mathbb{Z}F_{2k+1}$ and does not depend on n hence is a "noncommutative conserved quantity."

Setting $D = \overline{D} = C_i = 1$ for all $i > 0$, we recover the Laurentness of the noncommutative discrete dynamical system established by Di Francesco and Kedem in [19, Theorem 6.2] (conjectured by M. Kontsevich in [20, Section 3]).

We finish the introduction with establishing Laurentness of the following noncommutative recursion (which specializes to the discrete integrable system recently studied by P. Di Francesco in [9], see Section 4 for details) in the skew field $\mathcal{F}$ freely generated by $A, \overline{A}_i$, $B_i, \overline{B}_i$, $U_{i,i}$, $V_{i,i}$, $U_{i,i+1}$, $i \in \mathbb{Z}$:

$$U_{i+1,j}A_jV_{j+1,i} = B_{i+1}^{-1} + U_{i+1,j+1}\overline{A}_jV_{ji},\ V_{i+1,j}B_jU_{j+1,i} = A_{i+1}^{-1} + V_{i+1,j+1}\overline{B}_jU_{ji}\ , \tag{1.6}$$

$$U_{ij}A_jV_{j+1,i} = U_{i,j+1}\overline{A}_jV_{ij},\ V_{ij}B_jU_{j+1,i} = V_{i,j+1}\overline{B}_jU_{ij}\ . \tag{1.7}$$

We prove (Theorem 4.11) that this recursion has a (unique) solution in the group algebra $\mathbb{Q}T_\infty$ of the free group $\mathbb{T}_\infty$ freely generated by $A_i, \overline{A}_i$, $B_i, \overline{B}_i$, $U_{i,i}$, $V_{i,i}$, $U_{i,i+1}$, $i \in \mathbb{Z}$, more precisely, each U_{ij} and V_{ij} is a sum of elements of the group. We also prove (Theorem 4.11) that the elements $H_{ij}^{\pm} \in Frac(\mathbb{Z}\mathbb{T}_\infty)$, $i \in \mathbb{Z}$, given by

$$H_{ij}^{+} := U_{ji}^{-1}(U_{j,i-1}A_{i-1} + U_{j,i+1}\overline{A}_i),\ H_{ij}^{-} := V_{ji}^{-1}(V_{j,i-1}B_{i-1} + V_{j,i+1}\overline{B}_i^{-1}) \tag{1.8}$$

belong to $\mathbb{Z}\mathbb{T}_\infty$ and do not depend on j.

These examples and their treatment in Section 4 suggest the following general approach to constructing noncommutative discrete integrable systems. That is, such a system consists of a marked surface Σ, its automorphism $\tau : \Sigma :\to \Sigma$ permuting marked points, and a triangulation Δ so that the collection $\mathcal{T} = \{x_\gamma \in \mathcal{A}_\Sigma, \gamma \in \cup_{k\in\mathbb{Z}}\tau^k(\Delta)\}$ evolves in "discrete time" $k \in \mathbb{Z}$ and for each marked point p of Σ, the total noncommutative angle T_p is a (noncommutative) conserved quantity. The noncommutative

Laurent Phenomenon (Theorems 3.30 and 3.36) then guarantees that each $\mathcal{T}$ belongs to the algebra isomorphic to the group algebra of $\mathbb{T}_\Delta$.

In Appendix we collect relevant results on noncommutative localizations.

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2. Noncommutative polygons

2.1. Definition and main results

For each $n \geq 3$ consider a cyclic order $i \mapsto i^+$ on $[n] = \{1, 2, \ldots, n\}$ via

$$i^+ = \begin{cases} i+1 & \text{if } i < n \\ 1 & \text{if } i = n \end{cases}$$

(and $i \mapsto i^-$ to be the inverse of $i \mapsto i^+$). We will view $[n]$ with this cyclic order as n points on a circle (or vertices of a convex n-gon) and each pair (i, j) as a chord from i to j (or as an edge or diagonal of the n-gon).

We also say that a sequence $\mathbf{i} = (i_1, \ldots, i_\ell)$ of distinct elements in $[n]$ is *cyclic* if a cyclic permutation $\mathbf{i} \mapsto (i_k, \ldots, i_\ell, i_1, \ldots, i_{k-1})$ is strictly increasing. In particular, the sequence $(k, k+1, \ldots, n, 1, \ldots, k-1)$ is cyclic for each k.

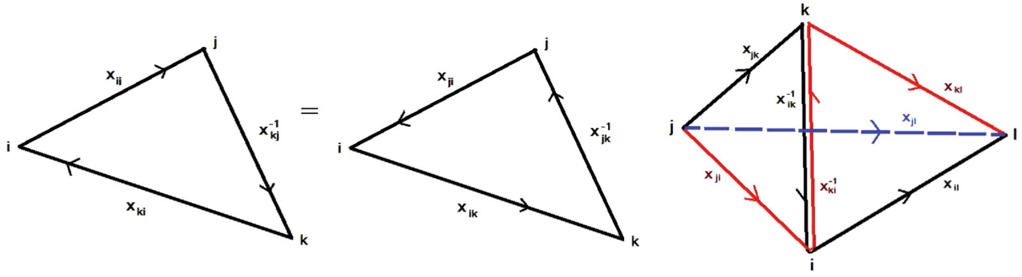
Definition 2.1. Denote by $\mathcal{A}_n$ the $\mathbb{Q}$-algebra generated by x_{ij} and x_{ij}^{-1}, $i, j \in [n]$, $i \neq j$ subject to the relations:

(i) (Triangle relations) For any distinct indices $i, j, k \in [n]$:

$$x_{ij}x_{kj}^{-1}x_{ki} = x_{ik}x_{jk}^{-1}x_{ji} \ . \tag{2.1}$$

(ii) (Exchange relations) For any cyclic (i, j, k, ℓ) in $[n]$:

$$x_{j\ell} = x_{jk}x_{ik}^{-1}x_{i\ell} + x_{ji}x_{ki}^{-1}x_{k\ell} \ . \tag{2.2}$$



Triangle and exchange relations

Remark 2.2. One can show that the exchange relations (2.2) are equivalent to noncommutative Ptolemy relations (1.2) provided the triangle relations (2.1) hold. Namely, multiplying (2.2) by $x_{ik}x_{jk}^{-1}$ on the left and using the triangle relation (2.1), we obtain (1.2):

$$x_{ik}x_{jk}^{-1}x_{j\ell} = x_{i\ell} + x_{ik}x_{jk}^{-1}x_{ji}x_{ki}^{-1}x_{k\ell} = x_{i\ell} + x_{ij}x_{kj}^{-1}x_{ki}x_{ki}^{-1}x_{k\ell} = x_{i\ell} + x_{ij}x_{kj}^{-1}x_{k\ell} \ .$$

Conversely, multiplying (1.2) on the left by $x_{jk}x_{ik}^{-1}$ and using (2.1), we recover (2.2).

At the first glance the number of relations of $\mathcal{A}_n$ greatly exceeds the number of generators, moreover, we expect that the subalgebra of $\mathcal{A}_n$ generated by all x_{ij} is a free algebra in $n^2 - n$ generators.

However, we will demonstrate below that the algebra $\mathcal{A}_n$ is “rationally” generated only by $3n - 4$ free generators.

Denote by F_m the free group on m generators, so that its group algebra $\mathbb{Q}F_m$ is the free Laurent polynomial algebra $\mathbb{Q}\langle c_1^{\pm 1}, \ldots, c_m^{\pm 1}\rangle$. Following Amitsur and Cohn (see e.g., [5] or Appendix A below) denote by $\mathcal{F}_m$ the free skew field on m generators, which is the “largest” skew field of fractions of $\mathbb{Q}F_m$. The following is our first main result, in which we freely use notation of Appendix A.

Theorem 2.3. *For each $n \geq 2$ the algebra $\mathcal{A}_n$ contains a subalgebra $\mathcal{A}'_n$ isomorphic to the free group algebra $\mathbb{Q}F_{3n-4}$ so that $\mathcal{A}_n$ is a universal localization of $\mathcal{A}'_n$ by a certain multiplicative submonoid of $A'_n \setminus \{0\}$.*

We prove the theorem in Section 2.14. In fact, it will follow from a more precise assertion (Theorem 2.8).

In view of universality of the localization (Lemma A.1), Theorem 2.3 implies the following immediate corollary.

Corollary 2.4. *The canonical monomorphism of algebras $\varphi' : \mathcal{A}'_n \hookrightarrow \mathcal{F}_{3n-4}$ uniquely extends to a homomorphism of algebras*

$$\varphi : \mathcal{A}_n \to \mathcal{F}_{3n-4} \tag{2.3}$$

In fact, we expect that (2.3) is injective, so far we can deduce this from another, "innocent looking" conjectural property of the group algebras $\mathbb{Q}F_m$ (Conjecture A.18, see also Section 2.15).

Remark 2.5. Injectivity of (2.3) would imply, in particular, that $\mathcal{A}_n$ has no zero divisors, which is a rather non-trivial assertion because of the following "counter-example" which was communicated to us by George Bergman. The universal localization $\mathbb{Q}\langle x,y\rangle[(xy)^{-1}]$ of the free algebra $\mathbb{Q}\langle x,y\rangle$ has a zero-divisor $y(xy)^{-1}x-1$.

Remark 2.6. Given $n'\geq n$ and an injective map $\mathbf{j}:[n]\hookrightarrow[n']$ for some $n'>n$, clearly, the assignments $x_{ij}\mapsto x_{\mathbf{j}(i),\mathbf{j}(j)}$ define a homomorphism of algebras $\mathbf{j}_\star:\mathcal{A}_n\to\mathcal{A}_{n'}$. One can conjecture that each $\mathbf{j}_\star$ is injective. In fact, this would directly follow from the injectivity of each (2.3).

Now we explore the "cluster" structure of $\mathcal{A}_n$. We say that a pair (i,k) *crosses* (j,ℓ) if (i,j,k,ℓ) is cyclic.

A *triangulation* Δ of $[n]$ is a maximal crossing-free subset of $[n]\times[n]\setminus\{(i,i)|i\in[n]\}$. Clearly, each triangulation of $[n]$ has cardinality $4n-6$.

For each triangulation Δ of $[n]$ define:

- The subalgebra $\mathcal{A}_\Delta$ of $\mathcal{A}_n$ generated by x_{ij}, $i,j\in[n]$, $i\neq j$ and x_{ij}^{-1}, $(i,j)\in\Delta$.
- The *triangle* group $\mathbb{T}_\Delta$ generated by all t_{ij}, $(i,j)\in\Delta$ subject to the relations:

$$t_{ij}t_{kj}^{-1}t_{ki}=t_{ik}t_{jk}^{-1}t_{ji}$$

for all $i,j,k\in[n]$ such that $(i,j),(j,k),(k,i)\in\Delta$.

The term "triangle group" is normally used for a group generated by reflections about sides of a triangle. In this paper we are using it in a somewhat similar way: a group generated by "side lengths" of noncommutative triangles.

Theorem 2.7. *Each $\mathbb{T}_\Delta$ is a free group in $3n-4$ generators.*

We prove Theorem 2.7 in Section 2.11. We generalize it in Theorem 3.24 to all surfaces.

Clearly, the assignments $t_{ij}\mapsto x_{ij}$, $(i,j)\in\Delta$ define a homomorphism of algebras:

$$\mathbf{i}_\Delta:\mathbb{Q}\mathbb{T}_\Delta\to\mathcal{A}_\Delta\ ,\tag{2.4}$$

where $\mathbb{Q}\mathbb{T}_\Delta$ is the group algebra of $\mathbb{T}_\Delta$.

Recall (see, e.g., (A.1)) that for a given algebra $\mathcal{A}$ with no zero divisors and a submonoid $S\subset\mathcal{A}\setminus\{0\}$ one has a universal localization $\mathcal{A}[S^{-1}]$ of $\mathcal{A}$ by S.

Theorem 2.8. *For each triangulation Δ of $[n]$ one has:*

(a) The homomorphism $\mathbf{i}_\Delta$ given by (2.4) is an isomorphism of algebras.

(b) $\mathcal{A}_n=\mathcal{A}_\Delta[\mathbf{S}^{-1}]$, where $\mathbf{S}$ is the multiplicative submonoid of $\mathcal{A}_\Delta$ generated by all x_{ij}.

We will prove Theorem 2.8 in Section 2.14. In fact, Theorem 2.8(a) establishes a *noncommutative cluster structure* on $\mathcal{A}_n$ and Theorem 2.8(b) – a *noncommutative Laurent Phenomenon* (see also Section 2.2).

2.2. Noncommutative Laurent Phenomenon

For each even sequence $\mathbf{i} = (i_1, \ldots, i_{2m}) \in [n]^{2m}$ such that adjacent indices are distinct define the monomial $x_\mathbf{i} \in \mathcal{A}_n$ by:

$$x_\mathbf{i} := x_{i_1,i_2} x_{i_3,i_2}^{-1} x_{i_3,i_4} \cdots x_{i_{2m-1},i_{2m-2}}^{-1} x_{i_{2m-1},i_{2m}} .$$

Definition 2.9. For a directed chord (i, j), $i, j \in [n]$, $i \neq j$ and a triangulation Δ of $[n]$, we say that a sequence $\mathbf{i} = (i_1, \ldots, i_{2m}) \in [n]^{2m}$ is (i, j, Δ)-*admissible* if:
(i) $i_1 = i$, $i_{2m} = j$ and $(i_s, i_{s+1}) \in \Delta$ for $s = 1, \ldots, 2m - 1$;
(ii) each chord (i_{2s}, i_{2s+1}), $s = 1, \ldots, m - 1$ intersects (i, j);
(iii) if $\mathbf{p} := (i_k, i_{k+1}) \cap (i, j) \neq \emptyset$ and $\mathbf{q} := (i_\ell, i_{\ell+1}) \cap (i, j) \neq \emptyset$ for some $k < \ell$, then the point $\mathbf{p}$ of (i, j) is closer in the path to i than the point $\mathbf{q}$.

We denote by $Adm_\Delta(i, j)$ the set of all (i, j, Δ)-admissible sequences $\mathbf{i}$.

Theorem 2.10 *(Noncommutative Laurent Phenomenon). Let Δ be a triangulation of $[n]$. Then*

$$x_{ij} = \sum_{\mathbf{i} \in Adm_\Delta(i,j)} x_\mathbf{i} \ , \tag{2.5}$$

for all $i, j \in [n]$, $i \neq j$.

We prove Theorem 2.10 in Section 2.13.

Remark 2.11. This is a noncommutative generalization of Schiffler's formula ([25]).

Now we illustrate Theorem 2.10 for each *starlike* triangulation

$$\Delta_i = \{(i, j), (j, i) | j \in [n] \setminus \{i\}\} \cup \{(k, k^\pm), k \in [n]\}, \quad i \in [n] \ . \tag{2.6}$$

Example 2.12. Fix $i \in [n]$. Then for each $k, \ell \in [n] \setminus \{i\}$ such that (i, k, ℓ) is cyclic, the following relation holds in $\mathcal{A}_n$:

$$x_{k\ell} = \sum_s x_{ki} x_{si}^{-1} x_{s,s^+} x_{i,s^+}^{-1} x_{i\ell}$$

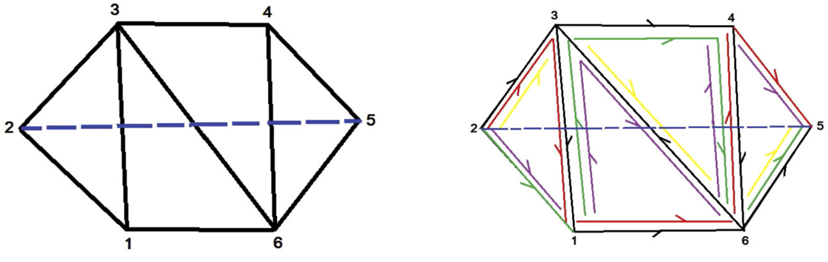
where summation is over all $s = k, k^+, \ldots, \ell^-$ in cyclic order. Hence $x_{k\ell} = \mathbf{i}_{\Delta_i}(\sum_s t_{ki} t_{si}^{-1} t_{s,s^+} t_{i,s^+}^{-1} t_{i\ell})$.

Example 2.13. (a) If $n = 5$ and $\Delta = \{(1,3),(3,1),(1,4),(4,1);(i,i^{\pm})|i \in [5]\}$, then

$$x_{25} = x_{21}x_{41}^{-1}x_{45} + x_{23}x_{13}^{-1}x_{15} + x_{21}x_{31}^{-1}x_{34}x_{14}^{-1}x_{15}.$$

(b) If $n = 6$ and $\Delta = \{(1,3),(3,1),(3,6),(6,3),(4,6),(6,4);(i,i^{\pm})|i \in [6]\}$, then

$$x_{25} = x_{23}x_{63}^{-1}x_{65} + x_{21}x_{31}^{-1}x_{36}x_{46}^{-1}x_{45} + x_{21}x_{31}^{-1}x_{34}x_{64}^{-1}x_{65} + x_{23}x_{13}^{-1}x_{16}x_{46}^{-1}x_{45}$$
$$+ x_{23}x_{13}^{-1}x_{16}x_{36}^{-1}x_{34}x_{64}^{-1}x_{65}.$$



A triangulation of a hexagon and all $(2,5)$-admissible sequences

In fact, we will streamline the formula for x_{ij} by introducing new coordinates $y_{ij}^k \in \mathcal{A}_n$ for distinct $i,j,k \in [n]$ by:

$$y_{ij}^k := x_{ki}^{-1}x_{kj} \ .$$

We refer to y_{ij}^k as *noncommutative sectors* and denote by $\mathcal{Q}_n$ the subalgebra of $\mathcal{A}_n$ generated by all y_{ij}^k (with the convention $y_{ii}^k = 1$).

Theorem 2.14. *The algebra $\mathcal{Q}_n$ is generated by all y_{ij}^k subject to the relations:*

(i) (triangle relations):

$$y_{ij}^k y_{ji}^k = 1, \ \ y_{ij}^k y_{jk}^i y_{ki}^j = 1 \tag{2.7}$$

for distinct $i,j,k \in [n]$ and

$$y_{ij}^\ell y_{jk}^\ell y_{ki}^\ell = 1 \tag{2.8}$$

for distinct $i,j,k,\ell \in [n]$.

(ii) (exchange relations) For all cyclic (i,j,k,ℓ) in $[n]$:

$$y_{i\ell}^j = y_{ij}^k y_{j\ell}^i + y_{i\ell}^k \ . \tag{2.9}$$

We prove Theorem 2.14 in Section 2.10.

For any sequence $\mathbf{j} = (j_0, j_1, \ldots, j_{2m}) \in [n]^{2m}$ such that adjacent indices are distinct define a monomial $y_{\mathbf{j}} \in \mathcal{Q}_n$ by:

$$y_{\mathbf{j}} := y_{j_0 j_2}^{j_1} y_{j_2 j_4}^{j_3} \cdots y_{j_{2m-2} j_{2m}}^{j_{2m-1}} .$$

The following is a "polynomial equivalent" in $\mathcal{Q}_n$ of Theorem 2.10.

Theorem 2.15 *(Noncommutative polynomial phenomenon). Let Δ be a triangulation of $[n]$. Then for any triple (i, j, k) of distinct indices such that $(i, k) \in \Delta$ one has:*

$$y_{kj}^{i} = \sum_{\mathbf{i} \in Adm_\Delta(i,j)} y_{(k,\mathbf{i})} \tag{2.10}$$

where $(k, \mathbf{i})$ stands for the sequence $\mathbf{i}$ preceded by k.

We prove Theorem 2.15 in Section 2.13.

Example 2.16. Following Example 2.13,

(a) If $n = 5$ and $\Delta = \{(1, 3), (3, 1), (1, 4), (4, 1); (i, i^\pm)|i \in [5]\}$, then

$$y_{15}^{2} = y_{15}^{4} + y_{13}^{2} y_{35}^{1} + y_{14}^{3} y_{45}^{1} .$$

(b) If $n = 6$ and $\Delta = \{(1, 3), (3, 1), (3, 6), (6, 3), (4, 6), (6, 4); (i, i^\pm)|i \in [5]\}$, then

$$y_{15}^{2} = y_{16}^{3} y_{65}^{4} + y_{13}^{2} y_{35}^{6} + y_{14}^{3} y_{46}^{5} + y_{13}^{2} y_{36}^{1} y_{65}^{4} + y_{13}^{2} y_{36}^{1} y_{64}^{3} y_{45}^{6} .$$

Similarly to Section 2.1, for each triangulation Δ of $[n]$ define:

- The subalgebra $\mathcal{Q}_\Delta$ of $\mathcal{Q}_n$ generated by all y_{ij}^{k}, $i, j, k \in [n]$ such that $(i, k), (k, j) \in \Delta$.
- The subgroup $\mathbb{U}_\Delta$ of $\mathbb{T}_\Delta$ generated by

$$u_{ij}^{k} := t_{ki}^{-1} t_{kj} \tag{2.11}$$

for $i, j, k \in [n]$ such that $(i, k), (kj) \in \Delta$.

Clearly, the restriction of the homomorphism $\mathbf{i}_\Delta$ given by (2.4) to $\mathbb{Q}\mathbb{U}_\Delta \subset \mathbb{Q}\mathbb{T}_\Delta$ is a surjective homomorphism of algebras:

$$\mathbf{i}'_\Delta : \mathbb{Q}\mathbb{U}_\Delta \twoheadrightarrow \mathcal{Q}_\Delta . \tag{2.12}$$

Since $\mathcal{Q}_\Delta$ is a subalgebra of $\mathcal{A}_\Delta$, the following is an immediate corollary of Theorem 2.8.

Corollary 2.17. *For each triangulation Δ one has:*

(a) The homomorphism $\mathbf{i}'_\Delta$ given by (2.12) is an isomorphism.

(b) $\mathcal{Q}_n = \mathcal{Q}_\Delta[\mathbf{S}'^{-1}]$ for some multiplicative submonoid $\mathbf{S}' \subset \mathcal{Q}_\Delta \setminus \{0\}$.

(c) $\mathbf{i}'_\Delta$ extends to a monomorphism of algebras $\mathbb{Q}\mathcal{Q}_n \hookrightarrow Frac(\mathcal{Q}_\Delta) = \mathcal{F}_{2n-4}$.

2.3. Regular elements in noncommutative polygons

We start with a more economical presentation of $\mathcal{A}_n$. Denote by $\mathcal{U}_n$ the subalgebra of $\mathcal{A}_n$ generated by all x_{ij}, $i \neq j$ and $x_{i,i^\pm}^{-1}$. The following result is obvious.

Lemma 2.18. *The algebra $\mathcal{U}_n$ satisfies the following relations*

(a) (reduced triangle relations) for all $i, j \in [n]$, $i \notin \{j^-, j\}$:

$$x_{i,j^-} x_{j,j^-}^{-1} x_{ji} = x_{ij} x_{j^-,j}^{-1} x_{j^-,i} \ , \tag{2.13}$$

(b) (reduced exchange relations) for all cyclic (i, j, k) in $[n]$ such that $i^- \notin \{j, k\}$:

$$x_{ij} x_{j^-,j}^{-1} x_{j^-,k} = x_{ik} + x_{i,j^-} x_{j,j^-}^{-1} x_{jk}, \ \ x_{k,j^-} x_{j,j^-}^{-1} x_{ji} = x_{ki} + x_{kj} x_{j^-,j}^{-1} x_{j^-,i} \ . \tag{2.14}$$

Remark 2.19. We expect that these relations are defining for the algebra $\mathcal{U}_n$.

Noncommutative Laurent phenomenon (2.10) guarantees that $\mathcal{U}_n$ belongs to each subalgebra $\mathcal{A}_\Delta$. The following conjecture implies, in particular, that $\mathcal{U}_n$ is a totally noncommutative analogue of the upper cluster algebra of type A_{n-3}.

Conjecture 2.20. *For each $n \geq 2$ one has:*

$$\mathcal{U}_n = \bigcap_\Delta \mathcal{A}_\Delta \ , \tag{2.15}$$

where the intersection is over all triangulations Δ of $[n]$.

We say that an element $x \in \mathcal{A}_n$ is *regular* if it belongs to each subalgebra $\mathcal{A}_\Delta$ as Δ runs over all triangulations Δ of $[n]$. Thus, Conjecture 2.20 asserts that each regular element of $\mathcal{A}_n$ belongs to $\mathcal{U}_n$, i.e., is a noncommutative polynomial in x_{ij} and $x_{i,i^\pm}^{-1}$.

2.4. Noncommutative angles

Now we take advantage of the "invariant" algebra $\mathcal{Q}_n$ and will view the ambient algebra $\mathcal{A}_n$ as some "Galois extension" of $\mathcal{Q}_n$ (in fact, Proposition 2.34 below guarantees that $\mathcal{A}_n$ is freely generated by x_{i,i^-}, $i \in [n]$ and $\mathcal{Q}_n$).

However, we want a more symmetric and "geometric" presentation of $\mathcal{A}_n$ over $\mathcal{Q}_n$. The following result provides such a presentation of $\mathcal{A}_n$, $n \geq 3$.

Proposition 2.21. *The algebra $\mathcal{A}_n$ is generated by $\mathcal{Q}_n$ and $(T_i^{jk})^{\pm 1}$ for all distinct $i, j, k \in [n]$ subject to:*

(i) (triangle relations) $T_i^{jk} = T_i^{kj}$ for all distinct (i, j, k) in $[n]$.

(ii) (modified exchange relations) $T_i^{j\ell} = T_i^{jk} + T_i^{k\ell}$ for any cyclic (i, j, k, ℓ) in $[n]$.

(iii) (consistency relations) $y_{ji}^k T_i^{jk} = y_{ji}^\ell T_i^{j\ell}$ for all distinct $i, j, k, \ell \in [n]$.

Proof. Denote by $\mathcal{A}'_n$ the algebra whose presentation is given in the proposition. It is easy to see that the assignments $y_{ij}^k \mapsto x_{ki}^{-1}x_{kj}, T_i^{jk} \mapsto x_{ji}^{-1}x_{jk}x_{ik}^{-1}$ for distinct $i,j,k \in [n]$ define a homomorphism of algebras $\mathcal{A}'_n \to \mathcal{A}_n$.

On the other hand, the consistency relations imply that the element $(T_i^{jk})^{-1}y_{ij}^k$ does not depend on k.

The following is immediate.

Lemma 2.22. *The assignments $x_{ij} \mapsto (T_i^{jk})^{-1}y_{ij}^k$ for distinct $i,j \in [n]$ define a homomorphism of algebras $f : \mathcal{A}_n \to \mathcal{A}'_n$.*

In particular, $f(x_{ji}^{-1}x_{jk}x_{ik}^{-1}) = y_{ij}^k T_j^{ik}(T_j^{ki})^{-1}y_{jk}^i y_{ki}^j T_i^{jk} = y_{ij}^k y_{jk}^i y_{ki}^j T_i^{jk} = T_i^{jk}$ by (2.7).

These homomorphisms are, clearly, inverse to each other and hence are isomorphisms.

The proposition is proved. □

We refer to $T_i^{jk} := x_{ji}^{-1}x_{jk}x_{ik}^{-1}$ for all distinct $i,j,k \in [n]$ as *noncommutative angles* by a number of reasons. First, because of the triangle relations in Proposition 2.21 (so that we can attach T_i^{jk} to the angle in the triangle (i,j,k) at the vertex i) and, second, because of the modified exchange relations (ii) of Proposition 2.21 can be viewed as an “addition law” of angles in a quadrilateral. In fact, such an addition law holds in more general situation.

Corollary 2.23. *For any cyclic $(i_0,i_1,i_2,\ldots,i_\ell)$ one has: $T_{i_0}^{i_1,i_k} = T_{i_0}^{i_1,i_2} + T_{i_0}^{i_2,i_3} + \cdots + T_{i_0}^{i_{\ell-1},i_\ell}$. In particular, $T_1^{2,n} = T_1^{23} + T_1^{34} + \cdots + T_1^{n-1,n}$.*

Moreover, this view is supported by the following observation. For each triangulation Δ of n and each $i \in [n]$ define the *total angle* T_i^Δ around the vertex i to be the sum of all noncommutative angles in Δ at the vertex i. For instance, we have in Example 2.16:

$$T_1^\Delta = T_1^{23} + T_1^{34} + T_1^{45},\ T_2^\Delta = T_2^{13},\ T_3^\Delta = T_3^{12} + T_3^{14},\ T_4^\Delta = T_4^{13} + T_4^{15},\ T_5^\Delta = T_5^{14}\ .$$

Corollary 2.24. *$T_i^\Delta = T_i^{i^-,i^+}$ for any triangulation Δ of $[n]$ and any $i \in [n]$. In particular, T_i^Δ does not depend on a choice of Δ.*

Remark 2.25. Based on Corollary 2.24, we can view $T_i := T_i^{i^-,i^+}$ as the *total angle* of the noncommutative n-gon at the vertex i. The sum of all total angles $T := T_1 + T_2 + \cdots + T_n$ also does not depend on a choice of triangulations and, in particular, can be specialized to any constant value (e.g., to $\pi \cdot (n-2)$).

Remark 2.26. The independence of T_i of a choice of Δ means that T_i is invariant under noncommutative mutations. We will encounter the noncommutative angles again in Section 3.

2.5. Big triangle group of noncommutative polygons

For each $n \geq 2$ let $\mathbb{T}_n$ be a group generated by t_{ij}, $i, j \in [n]$, $i \neq j$ subject to the triangle relations

$$t_{ij}t_{kj}^{-1}t_{ki} = t_{ik}t_{jk}^{-1}t_{ji}$$

for all distinct $i, j, k \in [n]$; and refer to this group as the *big triangle group* of the n-gon.

The following is obvious.

Lemma 2.27. *For any $n \geq 3$ one has:*

(a) the assignments $t_{ij} \mapsto \begin{cases} t_{1j} & \textit{if } i = n \\ t_{i1} & \textit{if } j = n \\ t_{ij} & \textit{otherwise} \end{cases}$ *for $i, j \in [n]$, $i \neq j$ (with the convention $t_{11} = 1$) define an epimorphism of groups $\pi_n^+ : \mathbb{T}_n \twoheadrightarrow \mathbb{T}_{n-1}$.*

(b) The assignments $t_{ij} \mapsto t_{ij}$ for $i, j \in [n-1]$, $i \neq j$ define an injective homomorphism of groups $\mathbb{T}_{n-1} \hookrightarrow \mathbb{T}_n$ which splits π_n^+.

The following result gives a presentation of $\mathbb{T}_n$.

Proposition 2.28. *For each $n \geq 3$ the group $\mathbb{T}_n$ is generated by t_{ij}, $1 \leq i < j \leq n$ and t_{i1}, $i = 2, \ldots, n$, subject to:*

$$t_{i1}t_{j1}^{-1}t_{jk}t_{1k}^{-1}t_{1j}t_{ij}^{-1}t_{ik} = t_{ik}t_{1k}^{-1}t_{1j}t_{ij}^{-1}t_{i1}t_{j1}^{-1}t_{jk}$$

for all $2 \leq i < j < k \leq n$.

Proof. By a slight abuse of notation, set $T_i^{jk} = t_{ji}^{-1}t_{jk}t_{ik}^{-1}$. Clearly, if $n = 3$, then $\mathbb{T}_3$ is free in $t_{12}, t_{13}, t_{23}, t_{21}, t_{31}$. Furthermore, let $n \geq 4$. Then we can group the defining relations for $\mathbb{T}_n$ into the following quadruples for $2 \leq i < j < k \leq n$:

$$T_1^{ij} = T_1^{ji},\ T_1^{ik} = T_1^{ki},\ T_1^{jk} = T_1^{kj},\ T_j^{ik} = T_j^{ki}\ . \tag{2.16}$$

It is easy to see that each such quadruple (2.16) is equivalent to the following quadruple of relations (here $(i', j') \in \{(i, j), (i, k), (j, k)\}$): $t_{j',i'} = t_{j',1}t_{i',1}^{-1}t_{i',j'}t_{1,j'}^{-1}t_{1,i'}$, $t_{i1}t_{j1}^{-1}t_{jk}t_{1k}^{-1}t_{1j}t_{ij}^{-1}t_{ik} = t_{ik}t_{1k}^{-1}t_{1j}t_{ij}^{-1}t_{i1}t_{j1}^{-1}t_{jk}$. Thus, eliminating the redundant generators $t_{j',i'}$, we finish the proof of the proposition. □

The following is obvious.

Lemma 2.29. *For each n one has:*

(a) The assignments $t_{ij} \mapsto x_{ij}$ define a ring epimorphism $\pi_n : \mathbb{Z}\mathbb{T}_n \twoheadrightarrow \mathcal{A}_n$.

(b) For each triangulation Δ of $[n]$ the assignments $t_{ij} \mapsto t_{ij}$ define a group homomorphism $\hat{\mathbf{j}}_\Delta : \mathbb{T}_\Delta \to \mathbb{T}_n$.

(c) The symmetric group S_n acts on $\mathbb{T}_n$ by automorphisms: $\sigma(t_{ij}) := t_{\sigma(i),\sigma(j)}$ for $\sigma \in S_n$, $i,j \in [n]$, $i \neq j$.

Conjecture 2.30. *The restriction of π_n to $\mathbb{T}_n$ is an isomorphism of monoids $\mathbb{T}_n \widetilde{\to} \mathcal{A}_n^\times$.*

Theorem 2.31. *For any triangulation Δ of $[n]$ there exists an epimorphism $\pi_\Delta : \mathbb{T}_n \twoheadrightarrow \mathbb{T}_\Delta$ such that*

$$\hat{\mathbf{j}}_\Delta \circ \pi_\Delta = Id_{\mathbb{T}_\Delta} \ .$$

In particular, $\hat{\mathbf{j}}_\Delta$ is an injective homomorphism $\mathbb{T}_\Delta \hookrightarrow \mathbb{T}_n$.

We prove Theorem 2.31 in Section 2.12.

The following is obvious.

Corollary 2.32. *For any triangulations Δ, Δ' of n the composition $\tau_{\Delta,\Delta'} := \pi_{\Delta'} \circ \hat{\mathbf{j}}_\Delta$ is an isomorphism $\mathbb{T}_\Delta \to \mathbb{T}_{\Delta'}$ such that $\tau_{\Delta,\Delta} = Id_{\mathbb{T}_\Delta}$ and $\tau_{\Delta,\Delta''} = \tau_{\Delta',\Delta''} \circ \tau_{\Delta,\Delta'}$ for any triangulation Δ'' of $[n]$.*

2.6. Representation of $\mathcal{A}_n$ and $\mathcal{Q}_n$ in noncommutative $2 \times n$ matrices

In what follows, we identify the free skew field generated by all a_{1i}, a_{2i}, $i \in [n]$ with $\mathcal{F}_{2n}$ and view it as totally noncommutative rational functions on the space $Mat_{2\times n}$ of $2 \times n$ matrices.

Following [17] and [2] (see also [15], [16][18]) define 2×2-*positive quasiminors* by

$$\begin{vmatrix} a_{1i} & \boxed{a_{1j}} \\ a_{2i} & a_{2j} \end{vmatrix}_+ = \mathrm{sgn}(i-j)(a_{1j} - a_{1i}a_{2i}^{-1}a_{2j}), \quad \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & \boxed{a_{2j}} \end{vmatrix}_+ = \mathrm{sgn}(j-i)(a_{2j} - a_{2i}a_{1i}^{-1}a_{1j})$$

for $i,j \in [n]$ and *positive* quasi-Plücker coordinates Q_{ij}^k for distinct $i,j,k \in [n]$ by:

$$Q_{ij}^k = \begin{vmatrix} a_{1k} & \boxed{a_{1i}} \\ a_{2k} & a_{2i} \end{vmatrix}_+^{-1} \cdot \begin{vmatrix} a_{1k} & \boxed{a_{1j}} \\ a_{2k} & a_{2j} \end{vmatrix}_+ = \begin{vmatrix} a_{1k} & a_{1i} \\ a_{2k} & \boxed{a_{2i}} \end{vmatrix}_+^{-1} \cdot \begin{vmatrix} a_{1k} & a_{1j} \\ a_{2k} & \boxed{a_{2j}} \end{vmatrix}_+ \tag{2.17}$$

(the latter identity is proved in [17, Section 4.3] and in [18, Proposition 4.2.1]).

Theorem 2.33. *For each $n \geq 2$ one has:*

(a) The assignments $x_{ij} \mapsto \begin{vmatrix} a_{1i} & \boxed{a_{1j}} \\ a_{2i} & a_{2j} \end{vmatrix}_+$, $x_{ij} \mapsto \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & \boxed{a_{2j}} \end{vmatrix}_+$ for all distinct $i,j \in [n]$ define respectively the homomorphisms of algebras

$$\varphi_n^+ : \mathcal{A}_n \to \mathcal{F}_{2n}, \ \varphi_n^- : \mathcal{A}_n \to \mathcal{F}_{2n} \ . \tag{2.18}$$

(b) The restrictions of φ_n^+ and φ_n^- to $\mathcal{Q}_n$ are equal to an algebra homomorphism $\varphi_n : \mathcal{Q}_n \to \mathcal{F}_{2n}$ such that $\varphi_n(y_{ij}^k) = Q_{ij}^k$ for distinct $i, j, k \in [n]$.

Proof. Our proof is based on Proposition 2.34 below. It follows from [17, Section 4.4] that positive quasi-Plücker coordinates satisfy (2.7), (2.8), and (2.9) This implies that the assignments

$$y_{ij}^k \mapsto Q_{ij}^k \tag{2.19}$$

for distinct $i, j \in [n]$ define a homomorphism of algebras $\varphi_n : \mathcal{Q}_n \to \mathcal{F}_{2n}$.

Furthermore, for any $\mathbb{Q}$-algebras $\mathcal{A}$ and $\mathcal{B}$ denote by $\mathcal{A} * \mathcal{B}$ their free product, i.e., the universal algebra generated by $\mathcal{A}$ and $\mathcal{B}$ as subalgebras (with no relations between them). The most fundamental property of the free product is that any algebra homomorphisms $f_1 : \mathcal{A} \to \mathcal{C}$, $f_2 : \mathcal{B} \to \mathcal{C}$ canonically lift to an algebra homomorphism $f_1 * f_2 : \mathcal{A} * \mathcal{B} \to \mathcal{C}$.

Denote by F_m the free group generated by $c_i^{\pm 1}$, $i = 1, \ldots, m$.

By definition, the group algebra $\mathbb{Q}F_m$, is free Laurent polynomial algebra $\mathbb{Q} < c_1^{\pm 1}, \ldots, c_m^{\pm 1} >$.

Proposition 2.34. *For each $n \geq 2$ the assignments $x_{ij} \mapsto c_i * y_{i^-,j}^i$, $i, j \in [n]$, $i \neq j$ define an isomorphism of algebras*

$$f : \mathcal{A}_n \widetilde{\to} (\mathbb{Q}F_n) * \mathcal{Q}_n \ . \tag{2.20}$$

Proof. Let us prove that the homomorphism (2.20) is well-defined. We need the following obvious fact.

Lemma 2.35. *Let $\mathcal{B}$ be a $\mathbb{Q}$-algebra and let $c_1, \ldots, c_n$ be invertible elements of $\mathcal{B}$. Then the assignments*

$$x_{ij} \mapsto c_i * x_{ij} \tag{2.21}$$

for $i, j \in [n]$, $i \neq j$ define a homomorphism of algebras $\mathcal{A}_n \to \mathcal{B} * \mathcal{A}_n$.

By the above Lemma $\mathcal{B} := \mathbb{Q}F_n$ generated by $c_i^{\pm 1}$, $i \in [n]$, the assignments (2.21) define a homomorphism of algebras

$$\mathcal{A}_n \to (\mathbb{Q}F_n) * \mathcal{A}_n \ . \tag{2.22}$$

Furthermore, the assignments $c_i \mapsto c_i * x_{i,i^-}^{-1}$, $i \in [n]$ define an algebra homomorphism $f_1 : \mathbb{Q}F_n \to (\mathbb{Q}F_n) * \mathcal{A}_n$ and the identity map $\mathcal{A}_n \to \mathcal{A}_n$ defines a homomorphism of algebras $f_2 : \mathcal{A}_n \to (\mathbb{Q}F_n) * \mathcal{A}_n$. This gives an algebra homomorphism $f_1 * f_2 :$

$(\mathbb{Q}F_n)*\mathcal{A}_n \to (\mathbb{Q}F_n)*\mathcal{A}_n$ determined by $c_i \mapsto c_i * x_{i,i^-}^{-1}$, $x_{ij} \mapsto x_{ij}$. Then the composition of the homomorphism (2.22) with $f_1 * f_2$ is a homomorphism of algebras

$$\mathcal{A}_n \to (\mathbb{Q}F_n) * \mathcal{A}_n$$

given by $x_{ij} \mapsto c_i * x_{ij} \mapsto c_i * x_{i,i^-}^{-1} x_{ij} = c_i * y_{i^-,j}^i$ for all $i, j \in [n]$, $i \neq j$. Since the image of the latter homomorphism belongs to $(\mathbb{Q}F_n) * \mathcal{Q}_n$, we see that the algebra homomorphism $f : \mathcal{A}_n \to (\mathbb{Q}F_n) * \mathcal{Q}_n$ given by (2.20) is well-defined.

It remains to show that f is invertible. Indeed, denote by $f_1' : \mathbb{Q}F_n \to \mathcal{A}_n$ the homomorphism of algebras given by $c_i \mapsto x_{i,i^-}$, $i \in [n]$ and denote by f_2' the natural inclusion $\mathcal{Q}_n \hookrightarrow \mathcal{A}_n$. This defines a homomorphism of algebras $g = f_1' * f_2' : (\mathbb{Q}F_n) * \mathcal{Q}_n \to \mathcal{A}_n$ which is determined by $c_i \mapsto x_{i,i^-}$, $y_{ij} \mapsto y_{ij}$. This immediately implies that $(g \circ f)(x_{ij}) = g(c_i * y_{i^-,j}^i) = x_{i,i^-} y_{i^-,j}^i = x_{ij}$ for all $i \neq j$. Therefore, $g \circ f = Id$. Similarly,

$$\begin{aligned}(f \circ g)(c_i) = f(x_{i,i^-}) = c_i * y_{i^-,i^-}^i = c_i * 1 = c_i,\ (f \circ g)(y_{i^-,j}^i) = f(y_{i^-,j}^i) \\ = f(x_{i,i^-}^{-1} x_{ij}) = f(x_{i,i^-})^{-1} f(x_{ij}) = (c_i * x_{i,i^-})^{-1} c_i * x_{ij} = x_{i,i^-}^{-1} x_{ij} = y_{ij}\ .\end{aligned}$$

Therefore, $f \circ g = Id$ as well.

Proposition 2.34 is proved. □

Now we can finish the proof of Theorem 2.33. Define algebra homomorphisms $\psi_n^+, \psi_n^- : \mathbb{Q}F_n \to \mathcal{F}_{2n}$ by

$$\psi_n^+(c_i) = \begin{vmatrix} a_{1i} & \boxed{a_{1,i^-}} \\ a_{2i} & a_{2,i^-} \end{vmatrix}_+ , \psi_n^-(c_i) = \begin{vmatrix} a_{1i} & a_{1,i^-} \\ a_{2i} & \boxed{a_{2,i^-}} \end{vmatrix}_+$$

for $i \in [n]$. Universality of free products gives natural homomorphisms of algebras

$$\psi_n^+ * \varphi_n, \psi_n^- * \varphi_n : (\mathbb{Q}F_n) * \mathcal{Q}_n \to \mathcal{F}_{2n}\ ,$$

where φ_n is given by (2.19). Composing these homomorphisms with the isomorphism (2.20), we obtain respectively algebra homomorphisms $\varphi_n^+, \varphi_n^- : \mathcal{A}_n \to \mathcal{F}_{2n}$, whose restriction to $\mathcal{Q}_n$ equals φ_n.

Finally, note that

$$\varphi_n^+(x_{ij}) = \psi_n^+ * \varphi_n(c_i * y_{i^-,j}^i) = \begin{vmatrix} a_{1i} & \boxed{a_{1,i^-}} \\ a_{2i} & a_{2,i^-} \end{vmatrix}_+ \cdot Q_{i^-,j}^i = \begin{vmatrix} a_{1i} & \boxed{a_{1j}} \\ a_{2i} & a_{2j} \end{vmatrix}_+ ,$$

$$\varphi_n^-(x_{ij}) = \psi_n^- * \varphi_n(c_i * y_{i^-,j}^i) = \begin{vmatrix} a_{1i} & a_{1,i^-} \\ a_{2i} & \boxed{a_{2,i^-}} \end{vmatrix}_+ \cdot Q_{i^-,j}^i = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & \boxed{a_{2j}} \end{vmatrix}_+$$

for all distinct $i, j \in [n]$.

Theorem 2.33 is proved. □

Remark 2.36. Proposition 2.34 is a noncommutative algebraic analogue of the following assertion: if a group G acts freely on a set X, then there a bijection $X \widetilde{\to} G \times (X/G)$.

Remark 2.37. Unlike φ_n (see Theorem 2.45), the homomorphisms φ_n^+ and φ_n^- are not injective. In particular, one can show that $x_{ij}x_{kj}^{-1} + 1 - x_{ik}x_{jk}^{-1} \in Ker\ \varphi_n^-$ for all distinct $i, j, k \in [n]$.

For any groups G and H denote by $G * H$ their free product. It is well-known (see, e.g., [5]) that $\mathbb{Q}(G * H) = (\mathbb{Q}G) * (\mathbb{Q}H)$.

Proposition 2.38. *For each triangulation Δ of $[n]$ the assignments $t_{ij} \to c_i * u^i_{i^-,j}$ for all $(i, j) \in \Delta$ (in the notation of (2.11)) define an isomorphism of (free) groups*

$$\mathbb{T}_\Delta \widetilde{\to} F_n * \mathbb{U}_\Delta \ . \tag{2.23}$$

Proof. We essentially copy the proof of Proposition 2.34. Indeed, the following fact is obvious.

Lemma 2.39. *Let G be any group and let $c_1, \ldots, c_n \in G$. Then for any triangulation Δ of $[n]$ the assignments*

$$t_{ij} \mapsto c_i * t_{ij} \tag{2.24}$$

for $(i, j) \in \Delta$, define a homomorphism of groups $\mathbb{T}_\Delta \to G * \mathbb{T}_\Delta$.

Clearly, the assignments $c_i \mapsto c_i * t^{-1}_{i,i^-}$ for $i \in [n]$ define a group homomorphism $F_n \to F_n * \mathbb{T}_\Delta$. Composing this with (2.24) taken with $G = F_n =< c_1, \ldots, c_n >$ and applying the multiplication homomorphism $\mathbb{T}_\Delta * \mathbb{T}_\Delta \to \mathbb{T}_\Delta$, we obtain a group homomorphism: $\mathbb{T}_\Delta \to F_n * \mathbb{T}_\Delta * \mathbb{T}_\Delta \to F_n * \mathbb{T}_\Delta$ given by $t_{ij} \mapsto c_i * u^i_{i^-,j}$ for all $i, j \in \Delta$. Clearly, the image of this homomorphism contains all c_i and u^k_{ij}, $(i, j), (jk) \in \Delta$ and is contained in $F_n * \mathbb{U}_\Delta$, hence this gives a group homomorphism (2.23). It is also clear that the homomorphism $F_n * \mathbb{U}_\Delta \to \mathbb{T}_\Delta$ given by $c_i \mapsto t_{i,i^-}$, $u^k_{ij} \mapsto u^k_{ij}$ is inverse of (2.23).

The proposition is proved. □

Taking into account that $F_n * F_m \cong F_{m+n}$, we obtain an obvious corollary from Theorem 2.7.

Corollary 2.40. *For each triangulation Δ of $[n]$ the group $\mathbb{U}_\Delta$ is isomorphic to F_{2n-4}, the free group in $2n - 4$ generators.*

Furthermore, denote by $\mathcal{F}'_{2n-4}$ the skew sub-field of $\mathcal{F}_{2n}$ generated by $\varphi_n(\mathcal{Q}_n)$, i.e., by all Q^k_{ij}.

Proposition 2.41. *$\mathcal{F}'_{2n-4}$ is isomorphic to $\mathcal{F}_{2n-4}$.*

Proof. Denote:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \end{pmatrix}, B = \begin{pmatrix} a_{13} & \cdots & a_{1n} \\ a_{23} & \cdots & a_{2n} \end{pmatrix}, C = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{2.25}$$

so that $A = [C \,|\, B]$.

The following lemma easily follows from Theorem 4.4.4 and Proposition 4.5.2 in [18].

Lemma 2.42. $C^{-1}B = \begin{pmatrix} q_{13}^2 & \cdots & q_{1n}^2 \\ q_{23}^1 & \cdots & q_{2n}^1 \end{pmatrix}$, *where* $q_{ij}^k = q_{ij}^k(A) = \mathrm{sgn}(k-j)\mathrm{sgn}(k-i)Q_{ij}^k$ *for distinct* $i, j, k \in [n]$ *are quasi-Plücker coordinates (in the notation (2.17)).*

It was proved in [17, Section 4] that $q_{ij}^k(A) = q_{ij}^k(DA)$ for all distinct $i, j, k \in [n]$ and any invertible 2×2 matrix D over $\mathcal{F}_{2n}$. In particular, taking $D = C^{-1}$, we see that $q_{ij}^k = q_{ij}^k([C \,|\, B]) = q_{ij}^k([I_2 \,|\, C^{-1}B])$, therefore, each q_{ij}^k belongs to the skew subfield of $\mathcal{F}_{2n}$ generated by the matrix coefficients of C (here I_2 is the 2×2 identity matrix). This proves that $\mathcal{F}'_{2n-4}$ is a sub-field of $\mathcal{F}_{2n}$ generated by the entries of $C^{-1}B$, i.e., by all q_{1j}^2, q_{2j}^1, $j = 3, \ldots, n$.

It remains to show that matrix coefficients of $C^{-1}B$ (freely) generate a free subfield of $\mathcal{F}_{2n}$. We need the following obvious fact.

Lemma 2.43. *Let $\mathcal{F}$ be a skew field, $C \in GL_m(\mathcal{F})$ and $B \in Mat_{m,n-m}(\mathcal{F})$ such that matrix coefficients of the partitioned matrix $A = [C \,|\, B]$ generate $\mathcal{F}$. Then the matrix coefficients of $[C \,|\, C^{-1}B]$ also generate $\mathcal{F}$.*

Now we take $m = 2$ and B, C as in (2.25), $\mathcal{F} = \mathcal{F}_{2n}$, the free skew field freely generated by matrix coefficients of $A = [C \,|\, B]$. Then $C \in GL_2(\mathcal{F}_{2n})$ and $B \in Mat_{2,n-2}(\mathcal{F}_{2n})$. Then, by Lemma 2.43, the matrix coefficients $A' = [C \,|\, C^{-1}B]$ also generate $\mathcal{F}_{2n}$. Since A' is $2 \times n$, then Proposition A.8 implies that the matrix coefficients of A' are free generators of $\mathcal{F}_{2n}$. In particular, the matrix coefficients of the $2 \times (n-2)$ matrix $C^{-1}B$ are free generators of the free skew sub-field of $\mathcal{F}_{2n}$. That is, $\mathcal{F}'_{2n-4}$ is freely generated by the matrix coefficients q_{1j}^2, q_{2j}^1, $j = 3, \ldots, n$ of $C^{-1}B$.

The proposition is proved. □

Remark 2.44. Proposition 2.41 and its proof generalize verbatim to $m \times n$ matrices.

Theorem 2.45. *For each triangulation Δ of $[n]$ the homomorphism*

$$\varphi_n \circ \mathbf{i}'_\Delta : \mathbb{QU}_\Delta \to \mathcal{F}'_{2n-4} \tag{2.26}$$

is injective.

Proof. Taking any free generating set $u_1, \ldots, u_{2n-4}$ of the free group $\mathbb{U}_\Delta \cong F_{2n-4}$, we see that $t_i := \varphi_n(\mathbf{i}'_\Delta(u_i))$, $i = 1, \ldots, 2n-4$ generate $\mathcal{F}'_{2n-4}$ due to the following fact.

Lemma 2.46. *For each triangulation Δ of $[n]$ the image $\varphi_n(\mathcal{Q}_\Delta)$ generates the skew field $\mathcal{F}'_{2n-4}$.*

Proof. Denote by $\mathcal{F}''_{2n-4}$ the skew subfield of $\mathcal{F}_{2n}$ generated by image $\varphi_n(\mathcal{Q}_\Delta)$. Since image $\mathcal{Q}_\Delta \subset \mathbb{Q}_n$, we have an obvious inclusion $\mathcal{F}''_{2n-4} \subseteq \mathcal{F}'_{2n-4}$. $\square$

Therefore using Proposition A.8 with $\ell = 2n-4$, we see that $t_1, \ldots, t_{2n-4}$ are free generators of $\mathcal{F}'_{2n-4}$ and hence the homomorphism (2.26) is injective.

Theorem 2.45 is proved. $\square$

2.7. Some symmetries of noncommutative polygons

First, we establish a new presentation of $\mathcal{A}_n$ in generators $\tilde{x}_{ij}^{\pm 1} := \text{sgn}(j-i)x_{ij}^{\pm 1}$ and $\tilde{T}_i^{jk} = \tilde{x}_{ji}^{-1}\tilde{x}_{jk}\tilde{x}_{ik}^{-1} = \text{sgn}(i-j)\text{sgn}(k-j)\text{sgn}(k-i)x_{ji}^{-1}x_{jk}x_{ik}^{-1}$ for distinct $i,j,k \in [n]$ (see also Section 2.4). Also define $\tilde{y}_{ij}^k = \tilde{x}_{ki}^{-1}\tilde{x}_{kj} = \text{sgn}(i-k)\text{sgn}(j-k)y_{ij}^k$ for distinct $i,j,k \in [n]$.

We need the following useful fact.

Lemma 2.47. *For each $n \geq 2$ one has:*

(a) The algebra $\mathcal{A}_n$ is generated by $\tilde{x}_{ij}$ for distinct $i,j \in [n]$ subject to the relations:

$$\tilde{T}_i^{jk} = -\tilde{T}_i^{kj} \tag{2.27}$$

for any distinct $i,j,k \in [n]$:

$$\tilde{T}_i^{jk} + \tilde{T}_i^{k\ell} + \tilde{T}_i^{\ell j} = 0 \tag{2.28}$$

for any distinct $i,j,k,\ell \in [n]$.

(b) The algebra $\mathcal{Q}_n$ is generated by all $\tilde{y}_{ij}^k$ subject to the relations:

$$\tilde{y}_{ij}^k\tilde{y}_{ji}^k = 1, \;\; \tilde{y}_{ij}^k\tilde{y}_{jk}^i\tilde{y}_{ki}^j = -1 \tag{2.29}$$

for distinct $i,j,k \in [n]$,

$$\tilde{y}_{ij}^\ell\tilde{y}_{jk}^\ell\tilde{y}_{ki}^\ell = 1, \;\; \tilde{y}_{ik}^j\tilde{y}_{ki}^\ell + \tilde{y}_{i\ell}^j\tilde{y}_{\ell i}^k = 1 \tag{2.30}$$

for distinct $i,j,k,\ell \in [n]$.

Proof. Prove (a). Denote by $\mathcal{A}''_n$ the algebra freely generated by all $\tilde{x}_{ij}^{\pm 1}$, $i \neq j$. That is, $\mathcal{A}''_n$ is the group algebra of a free group in $n^2 - n$ generators. Define $\tilde{r}_{ijk} = \tilde{T}_i^{kj}(\tilde{T}_i^{jk})^{-1}$, for all distinct $i,j,k \in [n]$. Clearly,

$$\tilde{r}_{ijk} = \tilde{x}_{ki}^{-1}\tilde{x}_{kj}\tilde{x}_{ij}^{-1}\tilde{x}_{ik}\tilde{x}_{jk}^{-1}\tilde{x}_{ji} = -x_{ki}^{-1}x_{kj}x_{ij}^{-1}x_{ik}x_{jk}^{-1}x_{ji} = \tilde{y}_{ij}^k\tilde{y}_{jk}^i\tilde{y}_{ki}^j = -y_{ij}^k y_{jk}^i y_{ki}^j$$

for all distinct $i, j, k \in [n]$. Denote by $\mathcal{I}'$ the ideal in $\mathcal{A}_n''$ generated by all $\tilde{r}_{ijk} + 1$. Then the quotient $\mathcal{A}_n' := \mathcal{A}_n''/\mathcal{I}'$ is an algebra generated by x_{ij}, $i, j \in [n]$, $i \neq j$ subject to the triangle relations (2.1).

Furthermore, for any distinct $i, j, k, \ell \in [n]$ define $\tilde{r}_{i;j,k,\ell} \in \mathcal{A}_n'$ by $\tilde{r}_{i;j,k,\ell} = \tilde{T}_i^{jk} + \tilde{T}_i^{k\ell} + \tilde{T}_i^{\ell j}$.

Clearly, $\tilde{r}_{i;j,k,\ell} = -\tilde{r}_{i;k,j,\ell} = -\tilde{r}_{i;j,\ell,k}$ for all i, j, k, ℓ, i.e., $r_{i;j,k,\ell}$ is skew-symmetric in j, k, ℓ because of (2.27). Note also that

$$\begin{aligned}\tilde{r}_{i;j,k,\ell} &= \tilde{x}_{ji}^{-1}(\tilde{x}_{jk}\tilde{x}_{ik}^{-1}\tilde{x}_{i\ell} + \tilde{x}_{ji}\tilde{x}_{ki}^{-1}\tilde{x}_{k\ell} - \tilde{x}_{j\ell})\tilde{x}_{i\ell}^{-1} = (\tilde{y}_{ik}^{j}\tilde{y}_{k\ell}^{i} + \tilde{y}_{i\ell}^{k} - \tilde{y}_{i\ell}^{j})\tilde{x}_{i\ell}^{-1}\\ &= (-\tilde{y}_{ik}^{j}\tilde{y}_{ki}^{\ell} + 1 - \tilde{y}_{i\ell}^{j}\tilde{y}_{\ell i}^{k})\tilde{x}_{i\ell}^{-1}\end{aligned}$$

for all distinct $i, j, k, \ell \in [n]$. Moreover, if (i, j, k, ℓ) is cyclic, i.e., (i, k) crosses (j, ℓ), this gives:

$$\tilde{r}_{i;j,k,\ell} = \pm x_{ji}^{-1}(x_{jk}x_{ik}^{-1}x_{i\ell} + x_{ji}x_{ki}^{-1}x_{k\ell} - x_{j\ell})x_{i\ell}^{-1} \ .$$

Therefore, if we denote by $\mathcal{I}$ the ideal in $\mathcal{A}_n'$ generated by all $\tilde{r}_{i;j,k,\ell}$, then, obviously, $\mathcal{A}_n'/\mathcal{I} \cong \mathcal{A}_n$.

This proves (a).

Part (b) also follows because the relations (2.29) and (2.30) are equivalent to (2.7), (2.8), and (2.9). The lemma is proved. $\square$

In the notation of Lemma 2.47 define the action of the symmetric group S_n on the set $\tilde{X} = \{\tilde{x}_{ij} | i, j \in [n], i \neq j\}$ by the formula

$$w(\tilde{x}_{ij}) = \tilde{x}_{w(i),w(j)}$$

for all $w \in S_n$, $i, j \in [n]$, $i \neq j$.

Proposition 2.48. *For each $n \geq 2$ one has:*

(a) The above action uniquely extends to an action of S_n on $\mathcal{A}_n$ by algebra automorphisms.

(b) The action commutes with homomorphisms $\varphi_n^+, \varphi_n^- : \mathcal{A}_n \to \mathcal{F}_{2n}$ given by (2.18), where the action of S_n on $\mathcal{F}_{2n}$ is given by $w(a_{s,i}) = a_{s,w(i)}$ for $s = 1, 2$, $i \in [n]$, $w \in S_n$.

(c) The subalgebra $\mathcal{Q}_n$ is invariant under the S_n-action, i.e., $w(\tilde{y}_{ij}^k) = \tilde{y}_{w(i),w(j)}^{w(k)}$ for all $i, j, k \in [n]$, $w \in S_n$.

Proof. Prove (a). The following fact is obvious.

Lemma 2.49. *The S_n action on $\tilde{X}$ uniquely extends to that on $\mathcal{A}_n'' = \mathbb{Q}\langle \tilde{X} \rangle$ by algebra automorphisms.*

Thus, it suffices to prove that the S_n-action on $\mathcal{A}''_n$ preserves the ideal of triangle relations (2.27) and exchange relations (2.28).

Let us prove that the ideal $\mathcal{I}'$ of $\mathcal{A}''$ generated by all $\tilde{r}_{ijk}$ is invariant under the S_n-action. Indeed, for distinct $i, j, k \in [n]$ and $w \in S_n$ one has

$$w(\tilde{r}_{ijk}) = w(\tilde{x}_{ij})w(\tilde{x}_{kj})^{-1}w(\tilde{x}_{ki})w(\tilde{x}_{ji})^{-1}w(\tilde{x}_{jk})w(\tilde{x}_{ik})^{-1} = \tilde{r}_{w(i),w(j),w(k)} \ .$$

This proves that $S_n(\mathcal{I}') = \mathcal{I}'$ hence S_n acts on $\mathcal{A}'_n$ by algebra automorphisms.

It remains to prove that the ideal of exchange relations (2.28) in $\mathcal{A}'_n$ is invariant under the S_n-action. Now we show that the ideal $\mathcal{I}$ of $\mathcal{A}'_n = \mathcal{A}''_n/\mathcal{I}'_n$ generated by all $\tilde{r}_{i;j,k,\ell}$ is invariant under the S_n-action. Indeed,

$$\begin{aligned} w(\tilde{r}_{i;j,k,\ell}) &= w(\tilde{T}_i^{jk}) + w(\tilde{T}_i^{k\ell}) + w(\tilde{T}_i^{\ell j}) = \tilde{T}_{w(i)}^{w(j),w(k)} + \tilde{T}_{w(i)}^{w(k),w(\ell)} + \tilde{T}_{w(i)}^{w(\ell),w(j)} \\ &= \tilde{r}_{w(i);w(j),w(k),w(\ell)} \end{aligned}$$

for all distinct $i, j, k, \ell \in [n]$. This proves that $S_n(\mathcal{I}) = \mathcal{I}$.

Part (a) is proved.

Part (b) follows from the fact that the homomorphisms $\varphi_n^+, \varphi_n^- : \mathcal{A}_n \to \mathcal{F}_{2n}$ from Theorem 2.33 commute with the S_n-action.

Part (c) is obvious.

The proposition is proved. □

The Lie algebra $gl_n(\mathbb{Q})$ (viewed as $Mat_{n\times n}$) naturally acts on the space $Mat_{2\times n}$ by right multiplications, i.e., $E_{ij}(a_{s,t}) = \delta_{t,j}a_{s,i}$ for $s \in \{1,2\}$, $i, j, t \in [n]$), where $E_{ij} \in gl_n(\mathbb{Q})$ are the matrix units.

This action uniquely extends to $\mathcal{F}_{2n}$ by the Leibniz rule: $E(fg) = E(f)g + fE(g)$, $E(h^{-1}) = -h^{-1}E(h)h^{-1}$ for any $E \in gl_n(\mathbb{Q})$, $f, g \in \mathcal{F}_{2n}$, $h \in \mathcal{F}_{2n} \setminus \{0\}$.

Proposition 2.50. *For each $n \geq 2$ there exists a unique action of $gl_n(\mathbb{Q})$ on $\mathcal{Q}_n$ by derivations such that the homomorphism $\varphi_n : \mathcal{Q}_n \to \mathcal{F}_{2n}$ from Theorem 2.33(b) is $gl_n(\mathbb{Q})$-equivariant. The action is given by:*

$$E_{i',j'}(\tilde{y}_{i,j}^k) = \begin{cases} 0 & \text{if } j' \notin \{i,j,k\} \\ \tilde{y}_{i,i'}^k & \text{if } j' = j \\ -\tilde{y}_{i,i'}^k \tilde{y}_{ij}^k & \text{if } j' = i \\ -\tilde{y}_{i,i'}^k \tilde{y}_{kj}^i & \text{if } j' = k \end{cases} \tag{2.31}$$

for any distinct indices $i, j, k \in [n]$.

Proof. Indeed, in view of Theorem 2.45, it suffices to prove (2.31) for $q_{ij}^k = \varphi_n(\tilde{y}_{ij}^k)$. Indeed, if we abbreviate $\underline{x}_{ij} = \begin{vmatrix} a_{1i} & \boxed{a_{1j}} \\ a_{2i} & a_{2j} \end{vmatrix}$ for distinct $i, j \in [n]$, then

$$E_{i'j'}(\underline{x}_{ij}) = E_{i',j'}(a_{1j} - a_{1i}a_{2i}^{-1}a_{2j}) = \begin{cases} 0 & \text{if } j' \notin \{i,j\} \\ a_{1,i'} - a_{1i}a_{2i}^{-1}a_{2,i'} & \text{if } j' = j \\ -E_{i',i}(a_{1i}a_{2i}^{-1})a_{2j} & \text{if } j' = i \end{cases}$$

$$= \begin{cases} 0 & \text{if } j' \notin \{i,j\} \\ \underline{x}_{i,i'} & \text{if } j' = j \\ \underline{x}_{i,i'}\underline{x}_{ji}^{-1}\underline{x}_{ij} & \text{if } j' = i \end{cases}$$

because $-E_{i',i}(a_{1i}a_{2i}^{-1}) = -a_{1,i'}a_{2i}^{-1} + a_{1i}a_{2i}^{-1}a_{2,i'}a_{2i}^{-1} = -\underline{x}_{i,i'}a_{2i}^{-1}$ and $a_{2i}^{-1}a_{2j} = -\underline{x}_{ji}^{-1}\underline{x}_{ij}$ for $i \neq j$. Therefore,

$$E_{i'j'}(q_{ij}^k) = E_{i'j'}(\underline{x}_{ki}^{-1}\underline{x}_{kj}) = E_{i'j'}(\underline{x}_{ki}^{-1})\underline{x}_{kj} + \underline{x}_{ki}^{-1}E_{i'j'}(\underline{x}_{kj})$$

$$= \begin{cases} \underline{x}_{ki}^{-1}E_{i'j}(\underline{x}_{kj}) & \text{if } j' = j \\ E_{i'i}(\underline{x}_{ki}^{-1})\underline{x}_{kj} & \text{if } j' = i \\ E_{i',k}(\underline{x}_{ki}^{-1})\underline{x}_{kj} + \underline{x}_{ki}^{-1}\underline{E}_{i'k}(x_{kj}) & \text{if } j' = k \\ 0 & \text{otherwise} \end{cases}.$$

Note that

$$\begin{aligned} E_{i',k}(\underline{x}_{ki}^{-1})\underline{x}_{kj} + \underline{x}_{ki}^{-1}E_{i'k}(\underline{x}_{kj}) &= -\underline{x}_{ki}^{-1}(\underline{x}_{k,i'}\underline{x}_{ik}^{-1}\underline{x}_{ki})\underline{x}_{ki}^{-1}\underline{x}_{kj} + \underline{x}_{ki}^{-1}(\underline{x}_{k,i'}\underline{x}_{jk}^{-1}\underline{x}_{kj}) \\ &= \underline{x}_{ki}^{-1}\underline{x}_{k,i'}(-\underline{x}_{ik}^{-1} + \underline{x}_{jk}^{-1})\underline{x}_{kj} \\ &= -\underline{x}_{ki}^{-1}\underline{x}_{k,i'}\underline{x}_{ik}^{-1}(\underline{x}_{ik} - \underline{x}_{jk})\underline{x}_{jk}^{-1}\underline{x}_{kj} \\ &= \underline{x}_{ki}^{-1}\underline{x}_{k,i'}\underline{x}_{ik}^{-1}\underline{x}_{ij} \end{aligned}$$

because

$$\begin{aligned} (\underline{x}_{ik} - \underline{x}_{jk})\underline{x}_{jk}^{-1}\underline{x}_{kj} &= ((a_{1k} - a_{1i}a_{2i}^{-1}a_{2k}) - (a_{1k} - a_{1j}a_{2j}^{-1}a_{2k}))(-a_{2k}^{-1}a_{2j}) \\ &= -(-a_{1i}a_{2i}^{-1}a_{2j} + a_{1j}) = -\underline{x}_{ij} \end{aligned}$$

Therefore,

$$E_{i'j'}(q_{ij}^k) = \begin{cases} 0 & \text{if } j' \notin \{i,j,k\} \\ \underline{x}_{ki}^{-1}\underline{x}_{k,i'} & \text{if } j' = j \\ -\underline{x}_{ki}^{-1}\underline{x}_{k,i'}\underline{x}_{ki}^{-1}\underline{x}_{kj} & \text{if } j' = i \\ -\underline{x}_{ki}^{-1}\underline{x}_{k,i'}\underline{x}_{ik}^{-1}\underline{x}_{ij} & \text{if } j' = k \end{cases} = \begin{cases} 0 & \text{if } j' \notin \{i,j,k\} \\ q_{i,i'}^k & \text{if } j' = j \\ -q_{i,i'}^k q_{ij}^k & \text{if } j' = i \\ -q_{i,i'}^k q_{kj}^i & \text{if } j' = k \end{cases}.$$

The proposition is proved. □

For $i, j \in [n]$ define the elements $\tilde{y}_{ij} \in \mathcal{F}_n$ by:

$$\tilde{y}_{ij} = \tilde{y}^i_{i^-,j} = \tilde{x}^{-1}_{i,i^-}\tilde{x}_{ij}$$

(with the convention that $y_{ii} = 0$). Clearly, $\tilde{y}_{i,i^-} = 1$ and $\tilde{y}_{i,i^+} = \tilde{x}^{-1}_{i,i^-}\tilde{x}_{i,i^+}$.

Denote by $\overline{\mathcal{A}}'_n$ the subalgebra of $\mathcal{Q}_n$ generated by all $\tilde{y}_{ij}$ and $\tilde{y}^{-1}_{i,i^+}$. The following is an immediate corollary of Proposition 2.50.

Corollary 2.51. *For each $i, j, i', j' \in [n]$ one has:*

$$E_{i',j'}(\tilde{y}_{ij}) = \begin{cases} 0 & \text{if } j' \notin \{i^-, i, k\} \\ \tilde{y}_{i,i'} & \text{if } j' = j \\ -\tilde{y}_{i,i'}\tilde{y}_{ij} & \text{if } j' = i^- \\ \tilde{y}_{i,i'}(\tilde{y}_{i^-,i})^{-1}\,\tilde{y}_{i^-,j} & \text{if } j' = i \end{cases}.$$

In particular, $\overline{\mathcal{A}}'_n$ is invariant under the $gl_n(\mathbb{Q})$-action.

Remark 2.52. Note, however, that the subalgebra $\mathcal{U}_n$ of $\mathcal{A}_n$ defined in Section 2.3 is not $gl_n(\mathbb{Q})$-invariant.

2.8. Extended noncommutative n-gons

In this section we define a larger algebra $\widetilde{\mathcal{A}}_n$ which is an extension of $\mathcal{Q}_n$ and can be viewed as a carrier of *double* noncommutative triangulations of the n-gon.

Definition 2.53. Let $\mathcal{A}^{\pm}_n$ be the algebra generated by x^{ε}_{ij} and $(x^{\varepsilon}_{ij})^{-1}$, $i, j \in [n]$, $i \neq j$, $\varepsilon \in \{-, +\}$ subject to the relations:

(i) (triangle relations) For any triple (i, j, k) of distinct indices in $[n]$:

$$x^+_{ij}(x^-_{kj})^{-1}x^+_{ki} = x^-_{ik}(x^+_{jk})^{-1}x^-_{ji}\ . \tag{2.32}$$

(ii) (exchange relations) For all cyclic (i, j, k, ℓ) in $[n]$:

$$x^-_{j\ell} = x^+_{jk}(x^+_{ik})^{-1}x^-_{i\ell} + x^-_{ji}(x^+_{ki})^{-1}x^+_{k\ell},\ x^+_{j\ell} = x^+_{jk}(x^-_{ik})^{-1}x^-_{i\ell} + x^-_{ji}(x^-_{ki})^{-1}x^+_{k\ell}\ . \tag{2.33}$$

The following result is obvious.

Lemma 2.54. *The assignments $x^{\pm}_{ij} \mapsto x_{ij}$ define an epimorphism of algebras $\widetilde{\mathcal{A}}_n \twoheadrightarrow \mathcal{A}_n$.*

In what follows, we adopt a convention for all distinct $i, j, k \in [n]$:

$$x_{ij}^k := \begin{cases} x_{ij}^+ & \text{if the triangle } (i,j,k) \text{ is to the } \textbf{right} \text{ of the chord } (i,j) \text{ when one goes from } i \text{ to } j \\ x_{ij}^- & \text{if the triangle } (i,j,k) \text{ is to the } \textbf{left} \text{ of the chord } (i,j) \text{ when one goes from } i \text{ to } j \end{cases}.$$

Equivalently, $x_{ij}^k = x_{ij}^\ell$ whenever (i,k) crosses (j,ℓ).

The following result is a generalization of Proposition 2.21.

Proposition 2.55. *The algebra $\widetilde{\mathcal{A}}_n$ is generated by $\mathcal{Q}_n$ and $(T_i^{jk})^{\pm 1}$ for all distinct triples (i,j,k) subject to:*

(i) (triangle relations) $T_i^{jk} = T_i^{kj}$ for all distinct $i, j, k \in [n]$.

(ii) (modified exchange relations) $T_i^{j\ell} = T_i^{jk} + T_i^{k\ell}$ for all cyclic (i,j,k,ℓ) in $[n]$.

(iii) (consistency relations) $y_{ji}^k T_i^{jk} = y_{ji}^\ell T_i^{j\ell}$ for all cyclic (i,j,k,ℓ) in $[n]$.

Proof. We proceed similarly to the proof of Proposition 2.21. Denote by $\tilde{\mathcal{A}}_n'$ the algebra whose presentation is given in the proposition. It is easy to see that the assignments $y_{ij}^k \mapsto (x_{ki}^j)^{-1} x_{jk}^i, T_i^{jk} \mapsto (x_{ji}^k)^{-1} x_{jk}^i (x_{ik}^j)^{-1}$ for distinct $i, j, k \in [n]$ define a homomorphism of algebras $\tilde{\mathcal{A}}_n' \to \tilde{\mathcal{A}}_n$.

On the other hand, the consistency relations imply that the $(T_i^{jk})^{-1} y_{ij}^k = (T_i^{j\ell})^{-1} y_{ij}^\ell$ if (ik) crosses $(j\ell)$.

The following is immediate.

Lemma 2.56. *The assignments $x_{ij}^k \mapsto (T_i^{jk})^{-1} y_{ij}^k$ for distinct $i, j, k \in [n]$ define a homomorphism of algebras $\tilde{f} : \tilde{\mathcal{A}}_n \to \tilde{\mathcal{A}}_n'$.*

In particular, $\tilde{f}((x_{ji}^k)^{-1} x_{jk}^i (x_{ik}^i)^{-1}) = y_{ij}^k T_j^{ik} (T_j^{ki})^{-1} y_{jk}^i y_{ki}^j T_i^{jk} = y_{ij}^k y_{jk}^i y_{ki}^j T_i^{jk} = T_i^{jk}$ by (2.7).

These homomorphisms are, clearly, inverse to each other and hence are isomorphisms.

The proposition is proved. □

Similarly to Section 2.1, for each triangulation Δ of $[n]$ define:

• The subalgebra $\tilde{\mathcal{A}}_\Delta$ of $\tilde{\mathcal{A}}_n$ generated by $x_{k\ell}^\pm$ for all distinct $k, \ell \in [n]$ and by $(x_{ij}^\pm)^{-1}$ for all $(i,j) \in \Delta$.

• The group $\tilde{\mathbb{T}}_\Delta$ generated by all $t_{ij}^\pm$ subject to the triangle relations (2.32).

Clearly, $\tilde{\mathbb{T}}_\Delta$ is a free group on $5(n-2)$ generators. It is also clear that the assignments $t_{ij}^\pm \mapsto x_{ij}^\pm$, $(i,j) \in \Delta$ define a homomorphism of algebras $\tilde{\mathbf{i}}_\Delta : \mathbb{Z}\tilde{\mathbb{T}}_\Delta \to \tilde{\mathcal{A}}_\Delta$.

The following is immediate.

Proposition 2.57 *(Laurent Phenomenon for extended noncommutative polygons). For each triangulation Δ of $[n]$ the homomorphism $\tilde{\mathbf{i}}_\Delta$ is an epimorphism.*

Note, however, that $\tilde{\mathbf{i}}_\Delta$ is not an isomorphism, unlike its counterpart $\mathbf{i}_\Delta$ given by (2.4).

Proposition 2.58. *For each triangulation Δ of $[n]$ the kernel of $\tilde{\mathbf{i}}_\Delta$ contains the elements*

$$(\partial_{i_4,i_1}-\partial_{i_3,i_1})T_{i_1}^{i_4,i_5}+T_{i_2}^{i_1,i_3}(\partial_{i_1,i_4}^{-1}-\partial_{i_1,i_3}^{-1})+(t_{i_3,i_1}^-)^{-1}t_{i_3,i_4}^+(t_{i_1,i_4}^+)^{-1} \\ -(t_{i_3,i_1}^+)^{-1}t_{i_3,i_4}^+(t_{i_1,i_4}^-)^{-1} \tag{2.34}$$

for each 5-tuple (i_1,i_2,i_3,i_4,i_5) in the cyclic order such that $(i_k,i_\ell)\in\Delta$ for all distinct $(k,\ell)\in[5]\times[5]$ except for $(k,\ell)=(2,4),(4,2),(2,5),(5,2)$, where we abbreviated $\partial_{ij}=(t_{ij}^+)^{-1}t_{ji}^-$.

Proof. Without loss of generality, we assume that $i_k=k$ for $k=1,2,3,4,5$. Then

$$x_{25}^-=x_{21}^-(x_{41}^+)^{-1}x_{45}^++x_{24}^+(x_{14}^+)^{-1}x_{15}^-$$

hence $x_{25}^-=x_{21}^-(x_{41}^+)^{-1}x_{45}^++x_{23}^+(x_{13}^-)^{-1}x_{14}^-(x_{14}^+)^{-1}x_{15}^-+x_{21}^-(x_{31}^-)^{-1}x_{34}^+(x_{14}^+)^{-1}x_{15}^-$. On the other hand,

$$x_{25}^-=x_{21}^-(x_{31}^+)^{-1}x_{35}^++x_{23}^+(x_{13}^+)^{-1}x_{15}^-$$

and

$$x_{35}^+=x_{31}^-(x_{41}^-)^{-1}x_{45}^++x_{34}^+(x_{14}^-)^{-1}x_{15}^-$$

hence $x_{25}^-=x_{21}^-(x_{31}^+)^{-1}x_{31}^-(x_{41}^-)^{-1}x_{45}^++x_{23}^+(x_{13}^+)^{-1}x_{15}^-+x_{21}^-(x_{31}^+)^{-1}x_{34}^+(x_{14}^-)^{-1}x_{15}^-$.

Comparing the expressions for x_{25}^-, we obtain a relation in $\tilde{\mathcal{A}}_\Delta$ which gives the appropriate element in the kernel of $\tilde{\mathbf{i}}_\Delta$. The proposition is proved. □

Remark 2.59. It is natural to conjecture that the kernel of $\tilde{\mathbf{i}}_\Delta$ is generated (as a two-sided ideal in $\mathbb{Z}\tilde{\mathbb{T}}_\Delta$) by the elements (2.34).

2.9. Further generalizations and specializations

Definition 2.60. Let $\widehat{\mathcal{A}}_n$ be the algebra generated by all $x_{ij}^k,(x_{ij}^k)^{-1}$, where i,j,k are distinct indices in $[1,n]$ subject to the relations:

(i) (triangle relations) $T_i^{jk}=T_i^{kj}$ for all distinct i,j,k, where $T_i^{jk}=(x_{ji}^k)^{-1}x_{jk}^i(x_{ik}^j)^{-1}$.

(ii) (exchange relations) $T_i^{j\ell}=T_i^{jk}+T_i^{k\ell}$ whenever (i,k) crosses (j,ℓ).

The following result is obvious.

Lemma 2.61. *(a) The assignments $x_{ij}^k\mapsto x_{ij}$ define an epimorphism of algebras $\widehat{\mathcal{A}}_n\to\mathcal{A}_n$.*
(b) The assignments $x_{ij}^k\mapsto x_{ij}^k$ define an epimorphism of algebras $\widehat{\mathcal{A}}_n\to\widetilde{\mathcal{A}}_n$ (as in Section 2.8).

We refer to each T_i^{jk} as the generalized noncommutative angle and view it as a certain measure of the angle at the vertex i in the triangle (ijk). For any triangulation Δ of the n-gon and $i \in [n]$, define the *total angle* T_i^{Δ} to be the sum of all noncommutative angles of all triangles of Δ at the vertex i.

Theorem 2.62. *For any triangulations Δ and Δ' of the n-gon, we have $T_{\Delta} = T_{\Delta'}$.*

Furthermore, let $\mathcal{A}'_n$ be the algebra generated by x_{ij}, $c_i^{jk} = c_i^{kj}$, $d_i^{jk} = d_i^{kj}$ and their inverses subject to the relations:

(i) (triangle relations) $T_i^{jk} = T_i^{kj}$ for all distinct $i, j, k \in [n]$, where

$$T_i^{jk} = x_{ji}^{-1} x_{jk} x_{ik}^{-1} \ ;$$

(ii) (exchange relations) $(d_i^{j\ell})^{-1} T_i^{j\ell} (c_i^{j\ell})^{-1} = (d_i^{jk})^{-1} T_i^{jk} (c_i^{jk})^{-1} + (d_i^{k\ell})^{-1} T_i^{k\ell} (c_i^{k\ell})^{-1}$ for any cyclic (i, j, k, ℓ) in $[n]$.

Proposition 2.63. *The assignments $x_{ij}^k \mapsto c_i^{jk} x_{ij} d_j^{ik}$ define a homomorphism of algebras:*

$$\hat{\varphi} : \hat{\mathcal{A}}_n \hookrightarrow \mathcal{A}'_n \ . \tag{2.35}$$

Proof. Denote by $\hat{\mathcal{A}}'_n$ the algebra freely generated by all x_{ij}^k. Then, clearly, the assignments $x_{ij}^k \mapsto c_i^{jk} x_{ij} d_j^{ik}$ define an algebra homomorphism

$$\hat{\mathcal{A}}'_n \to \mathcal{A}'_n \ .$$

Denote $T_i'^{\,jk} := (x_{ji}^k)^{-1} x_{jk}^i (x_{ik}^j)^{-1}$. We need the following fact.

Lemma 2.64. $\hat{\varphi}'(T_i'^{\,jk}) = (d_i^{jk})^{-1} T_i^{jk} (c_i^{jk})^{-1}$.

Proof. Indeed,

$$\begin{aligned}\hat{\varphi}'(T_i'^{\,jk}) &= \hat{\varphi}'((x_{ji}^k)^{-1} x_{jk}^i (x_{ik}^j)^{-1}) = (c_j^{ik} x_{ji} d_i^{jk})^{-1} c_j^{ik} x_{jk} d_k^{ij} (c_i^{jk} x_{ik} d_k^{ij})^{-1} \\ &= (d_i^{jk})^{-1} x_{ji} x_{jk} x_{ik} (c_i^{jk})^{-1} = (d_i^{jk})^{-1} T_i^{jk} (c_i^{jk})^{-1} \ .\end{aligned}$$

The lemma is proved. $\square$

We can finish now the proof of the proposition. The lemma implies that $\hat{\varphi}'(T_i'^{\,jk}) = \hat{\varphi}'(T_i'^{\,kj})$ and:

$$\hat{\varphi}'(T_i'^{\,j\ell} - T_i'^{\,jk} - T_i'^{\,k\ell}) = (d_i^{j\ell})^{-1} T_i^{j\ell} (c_i^{j\ell})^{-1} - (d_i^{jk})^{-1} T_i^{jk} (c_i^{jk})^{-1} - (d_i^{k\ell})^{-1} T_i^{k\ell} (c_i^{k\ell})^{-1} = 0 \ .$$

This proves the proposition. $\square$

Proposition 2.65. *For each collection of integers $a = \{a_i^{jk} = a_i^{kj} | i, j, k \in [n]$ are distinct$\}$, the assignments $x_{ij}^k \mapsto (T_i^{jk})^{a_i^{jk}} x_{ij} (T_j^{ik})^{-a_i^{jk}}$ define an algebra homomorphism $\varphi_a : \hat{\mathcal{A}}_n \to \mathcal{A}_n$.*

Proof. Clearly, $\varphi_a = \psi \circ \hat{\varphi}$, where $\hat{\varphi}$ is given by (2.35) and $\psi : \mathcal{A}'_n \to \mathcal{A}_n$ is an epimorphism given by

$$x_{ij} \mapsto x_{ij},\ c_i^{jk} \mapsto (T_i^{jk})^a,\ d_i^{jk} \mapsto (T_i^{jk})^{-a}\ . \qquad \square$$

Remark 2.66. Note that if $a_i^{jk} = 1$, then $\varphi_a(x_{ij}^k) = x_{ki}^{-1} x_{kj} x_{jk} x_{ik}^{-1} x_{ij}$.

2.10. Free factorizations of $\mathcal{A}_n$ and proof of Theorem 2.14

First, we verify that the relations (2.7), (2.8), and (2.9) hold. The left hand side of the first relation (2.7) is:

$$y_{ij}^k y_{ji}^k = (x_{ki}^{-1} x_{kj})(x_{kj}^{-1} x_{ki}) = 1\ .$$

Furthermore, the left hand side of the second relation (2.7) is:

$$y_{ij}^k y_{jk}^i y_{ki}^j = (x_{ki}^{-1} x_{kj})(x_{ij}^{-1} x_{ik})(x_{jk}^{-1} x_{ji}) = (x_{ki}^{-1} x_{kj} x_{ij}^{-1})(x_{ik} x_{jk}^{-1} x_{ji}) = 1$$

for all distinct $i, j, k \in [n]$ by the triangle relations (2.1). Similarly, the left hand side of (2.8) is:

$$y_{ij}^\ell y_{jk}^\ell y_{ki}^\ell = (x_{\ell i}^{-1} x_{\ell j})(x_{\ell j}^{-1} x_{\ell k})(x_{\ell k}^{-1} x_{\ell i}) = 1$$

for all distinct quadruples (i, j, k, ℓ).

Finally, the difference between the right and left hand sides of (2.9) is:

$$\begin{aligned} y_{ij}^k y_{j\ell}^i + y_{i\ell}^k - y_{i\ell}^j &= (x_{ki}^{-1} x_{kj})(x_{ij}^{-1} x_{i\ell}) + x_{ki}^{-1} x_{k\ell} - x_{ji}^{-1} x_{j\ell} \\ &= (x_{ji}^{-1} x_{jk} x_{kj}^{-1}) x_{i\ell} + x_{ki}^{-1} x_{k\ell} - x_{ji}^{-1} x_{j\ell} \\ &= x_{ji}^{-1}(x_{jk} x_{kj}^{-1} x_{i\ell} + x_{ji} x_{ki}^{-1} x_{k\ell} - x_{j\ell}) = 0 \end{aligned}$$

for all cyclic (i, j, k, ℓ) by the exchange relations (2.2).

Now let us show that the relations (2.7), (2.8), (2.9) are defining. Indeed, Proposition 2.34 implies an epimorphism of algebras $\mathcal{A}_n \twoheadrightarrow \mathcal{Q}_n$ given by $x_{ij} \mapsto y_{i^-,j}^i$. In other words, $\mathcal{Q}_n$ is isomorphic to the quotient of $\mathcal{A}_n$ by the ideal generated by elements $x_{i,i^-} - 1$, $i \in [n]$.

Therefore, we obtain the following obvious result.

Lemma 2.67. *The algebra $\mathcal{Q}_n$ is generated by all $y_{ij} := y^i_{i^-,j}$ and y_{ij}^{-1}, $i, j \in [n]$, $i \neq j$, subject to $y_{i,i^-} = 1$, $i \in [i]$ and the relations (2.1), (2.2), i.e.,*

$$y_{ij} y_{kj}^{-1} y_{ki} = y_{ik} y_{jk}^{-1} y_{ji} \tag{2.36}$$

for any distinct indices $i, j, k \in [n]$;

$$y_{j\ell} = y_{jk} y_{ik}^{-1} y_{i\ell} + y_{ji} y_{ki}^{-1} y_{k\ell} \tag{2.37}$$

for all cyclic (l, k, j, i) in $[n]$.

Since $y^k_{ij} = y_{ki}^{-1} y_{kj}$, the relations (2.7) directly follow from (2.36) and the relations (2.9) directly follow from (2.37) (this is obvious if we "reverse engineer" the fist part of the proof and replace all x_{ij} by y_{ij} there).

Therefore, Theorem 2.14 is proved. $\square$

The following obvious corollary from the proof of Theorem 2.14 will be instrumental in Section 3.

Corollary 2.68. *For each triangulation Δ of $[n]$ the $\mathbb{U}_\Delta$ is generated by u^k_{ij}, $(i, k), (jk) \in \Delta$ subject to the relations (2.7) and (2.8), i.e., for all distinct $i, j, k, \ell \in [n]$ such that $(i, j), (jk) \in \Delta$ one has:*

$$u^k_{ii} = 1, \;\; u^k_{ij} u^k_{ji} = u^k_{ij} u^i_{jk} u^j_{ki}, \;\; u^\ell_{ij} u^\ell_{jk} u^\ell_{ki} = 1 \; .$$

2.11. Freeness of $\mathbb{T}_\Delta$ and proof of Theorem 2.7

Let Δ be a triangulation of $[n]$. Fix a *directed* triangulation $\underline{\Delta} \subset \Delta$ so for each $(i, j) \in \Delta$ with $j \notin \{i^+, i^-\}$ exactly one out of (i, j) and (j, i) belongs to $\underline{\Delta}$ and $\underline{\Delta}$ contains all $(i, i^\pm)$, $i \in [n]$. By definition, any such $\underline{\Delta}$ has cardinality $3n - 3$.

Proposition 2.69. *Given $i_0 \in [n]$. Then for any triangulation Δ and any $\underline{\Delta}$ as above, the group $\mathbb{T}_\Delta$ is freely generated by t_{ij}, $(i, j) \in \underline{\Delta} \setminus \{(i_0, i_0^+)\}$.*

Proof. We proceed by induction on n. The assertion is obvious for $n \leq 3$. Suppose that $n \geq 4$. Then it is easy to see that there exist distinct $j_0, j'_0 \in [n]$ such that $(j_0^-, j_0^+), ({j'}_0^-, {j'}_0^+) \in \Delta$.

Without loss of generality we may assume that $j'_0 = n$ and $j_0 \neq i$ (hence $j_0 \notin \{i, n-1, n, 1\}$). Then $\hat{\Delta} = \Delta \setminus \{(1, n), (n, 1), (n-1, n), (n, n-1)\}$ is a triangulation of $[n-1]$ and $\hat{\underline{\Delta}} = \underline{\Delta} \setminus \{(1, n), (n, 1), (n-1, n), (n, n-1)\}$ is the corresponding directed triangulation.

The following result is obvious.

Lemma 2.70. *(a) The assignments* $t_{ij} \mapsto t_{ij}$, $(i,j) \in \hat{\Delta}$, $t_{n-1,n} \mapsto 1$, $t_{n,n-1} \mapsto 1$, $t_{1,n} \mapsto 1$, $t_{n,1} \mapsto t_{n-1,1}^{-1} t_{1,n-1}$ *define an epimorphism of groups* $\varphi : \mathbb{T}_\Delta \twoheadrightarrow \mathbb{T}_{\hat{\Delta}}$.

(b) $\iota \circ \varphi = Id_{\mathbb{T}_{\hat{\Delta}}}$, *where* $\iota : \mathbb{T}_{\hat{\Delta}} \to \mathbb{T}_\Delta$ *is a homomorphism given by* $\iota(t_{ij}) = t_{ij}$ *for* $(i,j) \in \hat{\Delta}$.

(c) The homomorphism $\iota : \mathbb{T}_{\hat{\Delta}} \to \mathbb{T}_\Delta$ *is injective.*

Denote by Δ_0 the triangulation of the triangle with the vertices $1, n-1, n$. Clearly, $\mathbb{T}_\Delta$ is generated by $\mathbb{T}_{\hat{\Delta}}$ (via the embedding ι) and $\mathbb{T}_{\Delta_0}$, more precisely,

$$\mathbb{T}_\Delta = \mathbb{T}_{\hat{\Delta}} * \mathbb{T}_{\Delta_0} / \langle (t_{n-1,1} * 1)(1 * t_{n-1,1})^{-1}, (t_{1,n-1} * 1)(1 * t_{1,n-1})^{-1} \rangle \ .$$

This and the inductive hypothesis (asserting that $\mathbb{T}_{\hat{\Delta}}$ is freely generated by t_{ij}, $(i,j) \in \underline{\hat{\Delta}} \setminus \{(j_0, j_0^+)\}$) imply (by eliminating $t_{n-1,1}$ and $t_{1,n-1}$ and setting $t_{k\ell} := 1 * t_{k,\ell}$ for $(k,\ell) = (1,n), (n,1), (1,n-1), (n-1,1)$) that $\mathbb{T}_\Delta$ is freely generated by all t_{ij}, $(i,j) \in \underline{\Delta} \setminus \{(j_0, j_0^+)\}$. $\square$

The theorem is proved. $\square$

2.12. Retraction of $\mathbb{T}_n$ *onto* $\mathbb{T}_\Delta$ *and proof of Theorem 2.31*

It suffices to construct an element $\tau_{ij} \in \mathbb{T}_\Delta$ for each pair $(i,j) \in [n] \times [n]$, $i \neq j$ such that $\tau_{ij} = t_{ij}$ whenever $(i,j) \in \Delta$ and for any distinct $i,j,k \in n$ one has the triangle relation:

$$\hat{T}_i^{jk} = \hat{T}_i^{kj} \tag{2.38}$$

where $\hat{T}_i^{j,k} := \tau_{ji}^{-1} \tau_{jk} \tau_{ik}^{-1}$.

We construct such τ_{ij} by induction on n. Retain notation from the proof of Theorem 2.7 and assume, without loss of generality, that $(n-1, n+1) \in \Delta$. If $n \notin \{i,j\}$, then, by deleting the vertex n and using the natural inclusion $\mathbb{T}_{\hat{\Delta}} \subset \mathbb{T}_\Delta$ given by Lemma 2.70(c), we set τ_{ij} to be that one which belongs to $\mathbb{T}_{\hat{\Delta}}$. Finally, we set $\tau_{1,n} := t_{1,n}$, $\tau_{n,1} := t_{n,1}$ and:

$$\tau_{i,n} := \tau_{i,n-1} \tau_{1,n-1}^{-1} \tau_{1,n}, \ \ \tau_{n,i} := \tau_{n,1} \tau_{n-1,1}^{-1} \tau_{n-1,i}$$

for $1 < i < n$.

Now verify that so constructed elements satisfy (2.38). Indeed, if $i,j,k \in [n-1]$, we have nothing to prove because (2.38) holds by the inductive hypothesis. Otherwise, it suffices to consider the case when $k = n$ and verify:

$$T_i^{n,j} = T_i^{j,n} \tag{2.39}$$

for all $i, j \in [n-1]$, $i \neq j$. Indeed,

$$\hat{T}_i^{n,j} = \tau_{ni}^{-1}\tau_{nj}\tau_{ij}^{-1} = \tau_{n-1,i}^{-1}\tau_{n-1,j}\tau_{ij}^{-1} = \hat{T}_i^{n-1,j},\ \hat{T}_i^{j,n} = \tau_{ji}^{-1}\tau_{jn}\tau_{in}^{-1}$$
$$= \tau_{ji}^{-1}\tau_{j,n-1}\tau_{i,n-1}^{-1} = \hat{T}_i^{j,n-1}$$

which, together with the inductive hypothesis, proves (2.39).

Therefore, the assignments $t_{ij} \mapsto \tau_{ij}$ for all $i \neq j$ define a group epimorphisms $\mathbb{T}_n \to \mathbb{T}_\Delta$.

Theorem 2.31 is proved. □

2.13. Noncommutative Laurent Phenomenon and proof of Theorems 2.10 and 2.15

Clearly, Theorem 2.10 is a direct corollary of Theorem 2.15, so we will only prove the latter one. We proceed by induction on n. In fact, due to the relations (2.8) in the form $y_{kj}^i = y_{k,i^+}^i y_{i^+,j}^i$ (hence $y_{(k,\mathbf{i})} = y_{k,i^+}^i y_{(i^+,\mathbf{i})}$), it suffices to prove (2.10) only with $k = i^+$ (however, we will use the inductive hypothesis without this restriction).

Indeed, if $n \leq 3$, the assertion is immediate. Now suppose that $n \geq 4$. In what follows we retain some notation of Section 2.11, that is, we fix a triangulation Δ and suppose that $(n-1, 1) \in \Delta$ and $(j_0, j_0^+) \in \Delta$ for some $j_0 \notin \{i, 1, n-1, n\}$. If $1 \notin \{i, j\}$, then the assertion (2.10) for Δ coincides with that for $\hat{\Delta} = \Delta \setminus \{(1,n), (n,1), (n-1,n), (n,n-1)\}$ and we have nothing to prove. Now suppose that $n \in \{i, j\}$. Without loss of generality we may assume that $i = n$ (the case $j = n$ is obtained by reversing all chords in $[n]$). Then, we will use the inductive hypothesis (2.10) for $\hat{\Delta}$ in the form:

$$y_{1,j}^{n-1} = \sum_{\mathbf{i}'} y_{(1,\mathbf{i}')},\ y_{n-1,j}^1 = \sum_{\mathbf{i}''} y_{(n-1,\mathbf{i}'')}\ ,$$

where the first (resp. the second) summation is over all $(n-1, j, \hat{\Delta})$ (resp. $(1, j, \hat{\Delta})$)-admissible sequences.

Using these and the relation (2.9) in the form $y_{1,j}^n = y_{1,j}^{n-1} + y_{1,n-1}^n y_{n-1,j}^1$, we obtain:

$$y_{1,j}^n = \sum_{\mathbf{i}'} y_{(1,\mathbf{i}')} + \sum_{\mathbf{i}''} y_{1,n-1}^n y_{(n-1,\mathbf{i}'')} = \sum_{\mathbf{i}'} y_{(1,n,1,\mathbf{i}')} + \sum_{\mathbf{i}''} y_{(1,n,n-1,\mathbf{i}'')}\ .$$

Clearly, this gives (2.10) because each (n, j, Δ)-admissible sequence is either of the form $(n, 1, \mathbf{i}')$, where $\mathbf{i}'$ is $(n, j, \hat{\Delta})$-admissible or is of the form $(n, n-1, \mathbf{i}'')$, where $\mathbf{i}''$ is $(1, j, \hat{\Delta})$-admissible (and vice versa).

Theorem 2.15 is proved. □

Therefore, Theorem 2.10 is proved. □

2.14. Noncommutative cluster variables and proof of Theorems 2.3 and 2.8

For each triangulation Δ of $[n]$ and $(p,q)\in[n]\times[n]$, $p\ne q$ define an element $t_{pq}^\Delta\in\mathbb{QT}_\Delta$ (in the notation of Theorem 2.10) by

$$t_{pq}^\Delta=\sum_{\mathbf{i}\in Adm_\Delta(p,q)} t_{\mathbf{i}}\ , \tag{2.40}$$

where $t_{\mathbf{i}}\in\mathbb{T}_\Delta$ is given by: $t_{\mathbf{i}}:=t_{i_1,i_2}t_{i_3,i_2}^{-1}t_{i_3,i_4}\cdots t_{i_{2m-1},i_{2m-2}}^{-1}t_{i_{2m-1},i_{2m}}$ for any $\mathbf{i}\in[n]^{2m}$ (with the convention $t_{ii}=1$ for $i\in[n]$).

We need the following result.

Theorem 2.71. *For any triangulations Δ and Δ' of $[n]$ the assignments $t_{ij}^{\Delta'}\mapsto t_{ij}^\Delta$ for $(i,j)\in[n]\times[n]$, $i\ne j$ define an isomorphism of algebras*

$$\psi_{\Delta,\Delta'}:\mathbb{QT}_{\Delta'}[S_{\Delta'}^{-1}]\widetilde{\to}\mathbb{QT}_\Delta[S_\Delta^{-1}]\ , \tag{2.41}$$

where S_Δ (resp. S'_Δ) is a submonoid in $\mathbb{QT}_\Delta$ generated by all t_{ij}^Δ. These isomorphisms satisfy:

$$\psi_{\Delta,\Delta'}=\psi_{\Delta,\Delta''}\circ\psi_{\Delta'',\Delta'} \tag{2.42}$$

for any triangulations Δ,Δ',Δ'' of $[n]$.

Proof. First, prove the assertion for *adjacent* triangulations Δ,Δ' of Σ, i.e., such that $\Delta\setminus\Delta'=\{(i,k),(k,i)\}$, $\Delta'\setminus\Delta=\{(j,\ell),(\ell,j)\}$, where (i,j,k,ℓ) is a cyclic quadruple.

By definition,

$$t_{j\ell}^\Delta=t_{jk}t_{ik}^{-1}t_{i\ell}+t_{ji}t_{ki}^{-1}t_{k\ell},\ t_{\ell j}^\Delta=t_{\ell i}t_{ki}^{-1}t_{kj}+t_{\ell k}t_{ik}^{-1}t_{ij}\ . \tag{2.43}$$

We need the following result.

Lemma 2.72. *For any adjacent triangulations Δ,Δ' of $[n]$ with $\Delta\setminus\Delta'=\{(i,k),(k,i)\}$, $\Delta\setminus\Delta'=\{(j,\ell),(\ell,j)\}$ there is a unique homomorphism of algebras $\varphi_{\Delta',\Delta}:\mathbb{QT}_{\Delta'}\to\mathbb{QT}_\Delta[(t_{j\ell}^\Delta)^{-1}]$ such that*

$$\varphi_{\Delta,\Delta'}(t_{i',j'})=\begin{cases}t_{i',j'} & \textit{if } \{i',j'\}\ne\{j,\ell\}\\ t_{j\ell}^\Delta & \textit{if } (i',j')=(j,\ell)\\ t_{\ell j}^\Delta & \textit{if } (i',j')=(\ell,j)\end{cases}$$

for all $(i',j')\in\Delta'$.

Proof. Indeed, it suffices only to prove that $\varphi_{\Delta,\Delta'}$ respects the triangle relations

$$T_{i'}^{j',k'} = T_{i'}^{k',j'}$$

for all triangles (i', j', k') in Δ'. Clearly, if (i', j', k') belongs to $\Delta \cap \Delta'$, then we have nothing to prove. It suffices only to consider the case when $(j', k') = (j, \ell)$, i.e., we have to prove that

$$\varphi_{\Delta,\Delta'}(T_{i'}^{j\ell}) = \varphi_{\Delta,\Delta'}(T_{i'}^{\ell j})$$

for $i' \in \{i, k\}$. Taking into account that both $(i'j)$ and $(i'\ell)$ belong to $\Delta \cap \Delta'$, we have only to prove that in $\mathbb{QT}_\Delta$ one has:

$$t_{ji'}^{-1} t_{j\ell}^{\Delta} t_{i'\ell}^{-1} = t_{\ell i'}^{-1} t_{\ell j}^{\Delta} t_{i'j}^{-1} \ .$$

In view of (2.43), this is equivalent to:

$$t_{ji'}^{-1}(t_{jk}t_{ik}^{-1}t_{i\ell} + t_{ji}t_{ki}^{-1}t_{k\ell})t_{i'\ell}^{-1} = t_{\ell i'}^{-1}(t_{\ell i}t_{ki}^{-1}t_{kj} + t_{\ell k}t_{ik}^{-1}t_{ij})t_{i'j}^{-1} \ . \tag{2.44}$$

If $i' = i$, then both sides of (2.44) are, clearly, equal to $T_i^{jk} + T_i^{k\ell}$, and if $i' = k$, then both sides of (2.44) are equal to $T_k^{ij} + T_k^{i\ell}$.

This proves that $\varphi_{\Delta,\Delta'}$ is well-defined homomorphism of algebras. □

Furthermore, we prove that in the assumptions of Lemma 2.72 one has

$$\varphi_{\Delta,\Delta'}(t_{pq}^{\Delta'}) = t_{pq}^{\Delta} \tag{2.45}$$

for all $(p, q) \in [n] \times [n]$, $p \neq q$.

Define a partial order $\prec$ on $[n]^\bullet$ by the covering insertion relations $\mathbf{i} \prec \mathbf{i}'$ if

$$\mathbf{i} = (\ldots, i_k, i_{k+1}, i_{k+2}, \ldots), \mathbf{i}' = (\ldots, i_k, i_{k+1}, a, i_{k+1}, i_{k+2}, \ldots) \tag{2.46}$$

for any $a \in [n]$.

We need the following obvious fact.

Lemma 2.73. *For each $\mathbf{i} \in [n]^\bullet$ there is a unique element $[\mathbf{i}]$ such that:*

- $[\mathbf{i}] \preceq \mathbf{i}$.
- $[\mathbf{i}]$ *is minimal in the partial order* $\prec$.

Clearly, if $\mathbf{i}, \mathbf{i}' \in [n]^{2\bullet}$ and $\mathbf{i} \prec \mathbf{i}'$, then $t_{\mathbf{i}} = t_{\mathbf{i}'}$.

Furthermore, fix a distinct quadruple $P := (i, j, k, \ell)$ in $[n]$ and denote by $\underline{P}$ the underlying set $\{i, j, k, \ell\}$.

For any $\mathbf{i} = (i_1, \ldots, i_r) \in [n]^r$, $r \geq 2$ define the *index set* $Ind_{\mathbf{i}}(P) \subset [r-1]$ by:

$$Ind_{\mathbf{i}}(P) = \{s \in [r-1] : \{i_s, i_{s+1}\} \in \{\{i,k\}, \{j,\ell\}\}$$

(with the convention that $i_k = 0$ if $k \leq 0$ and $i_k = \infty$ if $k > r$) and the *index* $ind_{\mathbf{i}}(P) \in \mathbb{Z}_{\geq 0}$ by

$$ind_{\mathbf{i}}(P) = \min Ind_{\mathbf{i}}(P)$$

with the convention that $\min \emptyset := 0$.

Denote by I_P the set of all sequences $\mathbf{i}$ such that $|Ind_{\mathbf{i}}(P)| = 1$.

Clearly, $I_{P'} = I_P$ for any permutation $P' = (i', j', k', \ell')$ of $P = (i, j, k, \ell)$ such that $\{i', k'\} \in \{\{i,k\}, \{j,\ell\}\}$.

Proposition 2.74. *For each $\mathbf{i} \in I_P$ one has $[\mathbf{i}] \in I_P$ and $ind_{[\mathbf{i}]}(P) \equiv ind_{\mathbf{i}}(P) \mod 2$.*

Proof. We need the following fact.

Lemma 2.75. *Let $\mathbf{i}, \mathbf{i}' \in [n]^\bullet$ be such that $\mathbf{i} \prec \mathbf{i}'$ and $\mathbf{i}' \in I_P$. Then $\mathbf{i} \in I_P$.*

Proof. It suffices to prove the assertion only for $\mathbf{i}$ and $\mathbf{i}' = \mathbf{j}_{ab}^t(\mathbf{i})$ as in (2.46). Let $s' = ind_{\mathbf{i}'}(P)$. Since $|Ind_{\mathbf{i}'}(P)| = 1$, and $i'_{s'-1} \neq i'_{s'+1}$, $i'_{s'} \neq i_{s'+2}$, but $i'_{t+1} = i'_{t+3}$, then $s' \notin \{t+1, t+2\}$. In particular, $\{i_t, a\} \notin \{\{i,k\}, \{j,\ell\}\}$. This immediately implies that $|Ind_{\mathbf{i}}(P)| = 1$ and

$$Ind_{\mathbf{i}}(P) = \begin{cases} \{s'\} & \text{if } s' \leq t \\ \{s'-2\} & \text{if } s' \geq t+3 \end{cases} . \tag{2.47}$$

The lemma is proved. $\square$

Thus, for any $\mathbf{i} \in I_P$ we see that $\{\mathbf{i}'' \in [n]^\bullet : \mathbf{i}'' \prec \mathbf{i}\} \subset I_P$, in particular, $[\mathbf{i}] \in I_P$.

The proposition is proved. $\square$

For $a, b \in [n]$ and $1 \leq s < r$ define the map $\mathbf{j}_{ab}^s : [n]^r \to [n]^{r+2}$ by $(\ldots, i_s, i_{s+1}, \ldots) \mapsto (\ldots, i_s, a, b, i_{s+1}, \ldots)$. Define a map $J_P : I_P \times \{-1, 1\} \to [n]^\bullet \times \{-1, 1\}$ by

$$J_P(\mathbf{i}, \varepsilon) = (\mathbf{j}_{i'k'}^s(\mathbf{i}), (-1)^{(s-1)\chi_{\{i,j\}}(i_s)}) \tag{2.48}$$

where $s := ind_{\mathbf{i}}(P)$ and $\chi_{\{b,c\}}(a)$ is the characteristic function, i.e., it is 1 if $a \in \{b, c\}$ and 0 otherwise, and the pair (i', k') is determined by $\{i', k'\} = \underline{P} \setminus \{i_s, i_{s+1}\}$ and:

- If s is odd then $\{i'\} = \underline{P}_\varepsilon \setminus \{i_s, i_{s+1}\}$, where we abbreviated $\underline{P}_\varepsilon := \begin{cases} \{i,j\} & \text{if } \varepsilon = -1 \\ \{k,\ell\} & \text{if } \varepsilon = 1 \end{cases}$.

• If s is even then $\{i'\} = \begin{cases} \{i_{s-1}\} & \text{if } i_{s-1} \in \underline{P} \setminus \{i_s, i_{s+1}\} \\ \underline{P} \setminus \{i_s, i_{s+1}, i_{s+2}\} & \text{if } i_{s+2} \in \underline{P} \setminus \{i_{s-1}, i_s, i_{s+1}\} \\ \{i,j\} \setminus \{i_s, i_{s+1}\} & \text{otherwise} \end{cases}.$

Let $\hat{I}_P$ be the set of all $(\mathbf{i}, \varepsilon) \in I_P \times \{-1, 1\}$ such that

• if $s = ind_{\mathbf{i}}(P)$ is even, then $\varepsilon = 1$;

• if $s = ind_{\mathbf{i}}(P)$ is odd then:

(i) If $\{i_{s-1}\} = \underline{P}_\varepsilon \setminus \{i_s, i_{s+1}\}$, $\{i_{s+2}\} = \underline{P}_{-\varepsilon} \setminus \{i_s, i_{s+1}\}$, $i_{s-2} \neq i_s$, $i_{s+3} \neq i_{s+1}$, then $i_s \in \{i, j\}$.

(ii) If $\{i_{s-1}\} = \underline{P}_\varepsilon \setminus \{i_s, i_{s+1}\}$, $\{i_{s+2}\} \neq \underline{P}_{-\varepsilon} \setminus \{i_s, i_{s+1}\}$, then $i_{s-2} \neq i_{s+1}$.

Proposition 2.76. *$J_P(\hat{I}_P) \subset \hat{I}_P$, that is, J_P is a map $J_P : \hat{I}_P \to \hat{I}_P$.*

Proof. We need the following fact.

Lemma 2.77. *Let $\mathbf{i} \in I_P$ and let $s = ind_{\mathbf{i}}(P)$. Then*

$$Ind_{\mathbf{j}^s_{i'k'}(\mathbf{i})}(P) = \{ind_{\mathbf{i}}(P) + 1\} \tag{2.49}$$

for any $i', k' \in [n]$ such that $\{i', k'\} = \underline{P} \setminus \{i_s, i_{s+1}\}$.

Proof. Let $s = ind_{\mathbf{i}}(P)$ and $\mathbf{i}' := \mathbf{j}^s_{i'k'}(\mathbf{i})$. Note that $s + 1 \in Ind_{\mathbf{i}'}(P)$ because $\{i'_{s+1}, i'_{s+2}\} \in \{\{i, k\}, \{j, \ell\}\}$. This and the fact that $\{i'_s, i'_{s+1}, i'_{s+2}, i'_{s+3}\} = \underline{P}$ imply that $s \notin Ind_{\mathbf{i}'}(P)$ and $s + 2 \notin Ind_{\mathbf{i}'}(P)$. Finally, if $s'' \leq s - 1$ (res. $s'' \geq s + 3$), then $s'' \notin Ind_{\mathbf{i}'}(P)$ because $s'' \notin Ind_{\mathbf{i}}(P)$ (resp. because $s'' - 2 \notin Ind_{\mathbf{i}}(P)$).

This proves (2.49). □

Furthermore, let $(\mathbf{i}, \varepsilon) \in \hat{I}_P$, $(\mathbf{i}', \varepsilon') := J_P(\mathbf{i}, \varepsilon)$, $s := Ind_{\mathbf{i}}(P)$, $s' = Ind_{\mathbf{i}'}(P)$. By Lemma 2.77, $s' = s + 1$. This, in particular, implies that $\varepsilon' = (-1)^{(s-1)\chi_{\{i,j\}}(i_s)} \in \{1, (-1)^{s'}\}$. If s is odd, this proves the desired inclusion $J_P(\mathbf{i}, \varepsilon) \in \hat{I}_P$.

It remains to consider the case when s is even. Indeed, $\mathbf{i}' = \mathbf{j}_{i',k'}(\mathbf{i})$, where $i' = i'_{s'}$, $k' = i'_{s'+1}$ are given by the even case of (2.48). Note that

$$P_{\varepsilon'} = \begin{cases} \{i,j\} & \text{if } i_s \in \{i,j\} \\ \{k,\ell\} & \text{if } i_s \in \{k,\ell\} \end{cases}, P_{-\varepsilon'} = \begin{cases} \{i,j\} & \text{if } i_{s+1} \in \{i,j\} \\ \{k,\ell\} & \text{if } i_{s+1} \in \{k,\ell\} \end{cases} \tag{2.50}$$

hence $\{i_s\} = \{i'_{s'-1}\} = P_{\varepsilon'} \setminus \{i'_{s'}, i'_{s'+1}\}$, $\{i_{s+1}\} = \{i'_{s'+2}\} = P_{-\varepsilon'} \setminus \{i'_{s'}, i'_{s'+1}\}$.

Finally, $i'_{s'-2} \neq i'_{s'}$ and $i'_{s'+1} \neq i'_{s'+3}$ if and only if $\{i_{s-1}, i_{s+2}\} \cap \underline{P} = \emptyset$ hence $\{i'\} = \{i, j\} \setminus \{i_s, i_{s+1}\}$.

This proves that $J_P(\mathbf{i}, \varepsilon) \in \hat{I}_P$ for even s as well.

The proposition is proved. □

Denote by $[I_P] \subset I_P$ the set of all $\mathbf{i} \in \hat{I}_P$ such that $\mathbf{i} = [\mathbf{i}]$ is minimal in the partial order $\prec$ and abbreviate $[\hat{I}_P] := \hat{I}_P \cap ([I_P] \times \{-1, 1\})$.

Proposition 2.74 guarantees that the assignments $\mathbf{i} \mapsto [\mathbf{i}]$ define a projection $I_P \to [I_P]$ (resp. $\hat{I}_P \to [\hat{I}_P]$).

Proposition 2.78. *The assignments* $(\mathbf{i}, \varepsilon) \mapsto [J_P(\mathbf{i}, \varepsilon)]$ *define an involution* $[J_P] : \hat{I}_P \to \hat{I}_P$.

Proof. Let $(\mathbf{i}, \varepsilon) \in [\hat{I}_P]$, let $s := ind_{\mathbf{i}}(P)$, $(\mathbf{i}', \varepsilon') := [J_P(\mathbf{i}, \varepsilon)]$, $s' := ind_{\mathbf{i}'}(P)$. By definition,

$$\mathbf{i}' = [(\ldots, i_s, i', k', i_{s+1}, \ldots)] = \begin{cases} (\ldots, i_s, i', k', i_{s+1}, \ldots) & \text{if } i' \neq i_{s-1},\ k' \neq i_{s+2} \\ (\ldots, i_{s-1}, i_{s+2}, \ldots) & \text{if } i' = i_{s-1},\ k' = i_{s+2} \\ (\ldots, i_{s-1}, i_s, i', i_{s+2}, \ldots) & \text{if } i' \neq i_{s-1},\ k' = i_{s+2} \\ (\ldots, i_{s-1}, k', i_{s+1}, i_{s+2}, \ldots) & \text{if } i' = i_{s-1},\ k' \neq i_{s+2} \end{cases} \tag{2.51}$$

in the notation (2.48). In particular, $i'_{s'} = i'$, $i'_{s'+1} = k'$.

Note that, by Lemma 2.77 and Proposition 2.74, $s' \equiv s + 1 \mod 2$.

First, show that $(\mathbf{i}', \varepsilon') \in [\hat{I}_P]$ (i.e., $[J_P]$ is well-defined). If s is odd, this is obvious. Suppose that s is even. Then we have in each of the cases of (2.51):

• $i' \neq i_{s-1}$, $k' \neq i_{s+2}$. Since $s' = s+1$ and $\{i'_{s'-1}, i'_{s'}, i'_{s'+1}, i'_{s'+2}\} = \underline{P}$ and $i' \in \{i, j\}$, clearly, $(\mathbf{i}', \varepsilon') \in [\hat{I}_P]$.

• $i' = i_{s-1}$, $k' = i_{s+2}$. Since $s' = s-1$ and $\{i'_{s'-1}, i'_{s'+2}\} \cap \underline{P} = \emptyset$, clearly, $(\mathbf{i}', \varepsilon') \in [\hat{I}_P]$.

• $i' \neq i_{s-1}$, $k' = i_{s+2}$. Since $s' = s + 1$ and $\{i_s\} = \{i'_{s'-1}\} = \underline{P}_{\varepsilon'} \setminus \{i'_{s'}, i'_{s'+1}\}$, $\{i'_{s'+2}\} = \{i_{s+3}\} \neq \{i_{s+1}\} = \underline{P}_{-\varepsilon'} \setminus \{i'_{s'}, i'_{s'+1}\}$ by (2.50) and $i'_{s'+1} = i_{s+2} \neq i_{s-1} = i'_{s'-2}$, clearly, $(\mathbf{i}', \varepsilon') \in [\hat{I}_P]$.

• $i' = i_{s-1}$, $k' \neq i_{s+2}$. Since $s' = s - 1$ and $\{i'_{s'+2}\} = \underline{P}_{-\varepsilon'} \setminus \{i'_{s'}, i'_{s'+1}\}$ by (2.50), clearly, $(\mathbf{i}', \varepsilon') \in [\hat{I}_P]$.

Furthermore, let $(\mathbf{i}'', \varepsilon'') = J_P(\mathbf{i}', \varepsilon')$. That is,

$$\mathbf{i}'' = \mathbf{j}^{s'}_{i'',k''}(\mathbf{i}') \ ,$$

where $\varepsilon'' = (-1)^{(s'-1)\chi_{\{i,j\}}(i'_{s'})}$, $\{i'', k''\} = \{i_s, i_{s+1}\}$ and one has (note that $\{i'_{s'}, i'_{s'+1}\} = \{i', k'\}$):

• If s is even, then $\{i'\} = \begin{cases} \{i_{s-1}\} & \text{if } i_{s-1} \in \underline{P} \setminus \{i_s, i_{s+1}\} \\ \underline{P} \setminus \{i_s, i_{s+1}, i_{s+2}\} & \text{if } i_{s+2} \in \underline{P} \setminus \{i_{s-1}, i_s, i_{s+1}\} \\ \{i, j\} \setminus \{i_s, i_{s+1}\} & \text{otherwise} \end{cases}$, $\varepsilon' = (-1)^{\chi_{ij}(i_s)}$, and:

$$\{i''\} = \underline{P}_{\varepsilon'} \setminus \{i'_{s'}, i'_{s'+1}\} = \underline{P}_{\varepsilon'} \setminus \{i', k'\} = \begin{cases} \{i, j\} \setminus \{i'\} & \text{if } i_s \in \{i, j\} \\ \{k, \ell\} \setminus \{k'\} & \text{if } i_s \in \{k, \ell\} \end{cases} = \{i_s\} \ . \tag{2.52}$$

• If s is odd, then: $\{i'\} = \underline{P}_\varepsilon \setminus \{i_s, i_{s+1}\}$,

$$\{i''\} = \begin{cases} \{i'_{s'-1}\} & \text{if } i'_{s'-1} \in \underline{P} \setminus \{i', k'\} \\ \underline{P} \setminus \{i', k', i'_{s'+2}\} & \text{if } i'_{s'+2} \in \underline{P} \setminus \{i'_{s'-1}, i', k'\} \\ \{i, j\} \setminus \{i', k'\} & \text{otherwise} \end{cases} . \tag{2.53}$$

First, show that $\varepsilon'' = \varepsilon$. Indeed, by the above, $\varepsilon'' = (-1)^{s\chi_{\{i,j\}}(i')}$. Since $\varepsilon \in \{1, (-1)^s\}$, then the above implies that for even s one has $\varepsilon'' = \varepsilon = 1$. If s is odd, then, by definition, $i' \in \{i, j\}$ iff $\varepsilon = -1$. This proves that $\varepsilon'' = \varepsilon$ in this case as well.

Thus, it remains to prove that

$$\mathbf{i} \preceq \mathbf{i}'' . \tag{2.54}$$

To do so, show that $i'' = i_s$ in each case of (2.51):

• $i' \neq i_{s-1}$, $k' \neq i_{s+2}$, $s' = s + 1$, $\mathbf{i}'' = (\ldots, i_s, i', i'', k'', k', i_{s+1}, \ldots)$, where for even s we have $i'' = i_s$ by (2.52) and for odd s we also have $i'' = i_s$ by (2.53) because $i'_{s'-1} = i_s$ and $i'_{s'+2} = i_{s+1}$.

• $i' \neq i_{s-1}$, $k' = i_{s+2}$, $s' = s + 1$, $\mathbf{i}'' = (\ldots, i_s, i', i'', k'', i_{s+2}, \ldots)$, where for even s, $i'' = i_s$ by (2.52) and for odd s we have $\{i''\} = \underline{P} \setminus \{i', k'\} = \{i_s\}$ by (2.53) because $i_s = i'_{s'-1} \in \underline{P} \setminus \{i', k'\}$.

• $i' = i_{s-1}$, $k' = i_{s+2}$, e.g., $\{i_{s-1}, i_{s+2}\} = \underline{P} \setminus \{i_s, i_{s+1}\}$, $s' = s - 1$, $\mathbf{i}'' = (\ldots, i_{s-1}, i'', k'', i_{s+2}, \ldots)$, where for even s, $i'' = i_s$ by (2.52) and for odd s we have: $i_{s-1} \in P_\varepsilon$, $i_{s+2} \in P_{-\varepsilon}$, $i_{s-2} \neq i_s$, $i_{s+3} \neq i_{s+1}$ hence $i_s \in \{i, j\}$ and: $\{i''\} = \{i, j\} \setminus \{i_{s-1}, i_{s+2}\} = i_s$ by (2.53).

• $i' = i_{s-1}$, $k' \neq i_{s+2}$, $s' = s - 1$, $\mathbf{i}'' = (\ldots, i_{s-1}, i'', k'', k', i_{s+1}, i_{s+2}, \ldots)$, where for even s, $i'' = i_s$ by (2.52) and for odd s we have $\{i''\} = \underline{P} \setminus \{i', k', i'_{s'+2}\} = \{i_s, i_{s+1}\} \setminus \{i_{s+1}\} = \{i_s\}$ by (2.53) because:

• $i_{s-1} \in P_\varepsilon \setminus \{i_s, i_{s+1}\}$, $i_{s+2} \notin P_{-\varepsilon} \setminus \{i_s, i_{s+1}\}$ hence $i_{s-2} \neq i_{s+1}$.

• $i_{s+1} = i'_{s'+2} \in \underline{P} \setminus \{i'_{s'-1}, i', k'\} = \{i_s, i_{s+1}\} \setminus \{i_{s-2}\}$.

Thus, $i'' = i_s$, $k'' = i_{s+1}$ in all cases, which immediately implies (2.54) in all these cases.

This proves that $[J_P]$ is an involution on $[\hat{I}_P]$.

The proposition is proved. $\square$

Now suppose that $P = (i, j, k, \ell)$ where $\Delta \setminus \Delta' = \{(i, k), (k, i)\}$, $\Delta' \setminus \Delta = \{(j, \ell), (\ell, j)\}$, as in Lemma 2.72. In what follows, we assume that $(p, i) \cap (j, \ell) = \emptyset$ and $(p, q) \cap (i, j) \neq \emptyset$ (i.e., informally speaking, (i, j) is closer to p than (k, ℓ)).

By Definition 2.9 of admissible sequences, if $\mathbf{i} \in Adm_\Delta(p, q) \subset [I_P] \sqcup Adm_{\Delta'}(p, q)$ then $[\mathbf{i}] = \mathbf{i}$ is minimal, its index $s := ind_{\mathbf{i}}(P)$ is positive and unique, and $\{i_s, i_{s+1}\} = \begin{cases} \{i, k\} & \text{if } \mathbf{i} \in Adm_\Delta(p, q) \\ \{j, \ell\} & \text{if } \mathbf{i} \in Adm_{\Delta'}(p, q) \end{cases}$.

Proposition 2.79. *Let Δ, Δ' be triangulations of $[n]$ and $P = (i, j, k, \ell)$ as above. Then the restriction of $[J_P]$ to $(Adm_{\Delta'}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P]$ is a bijection:*

$$J_{\Delta,\Delta'} : (Adm_{\Delta'}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P] \widetilde{\to} (Adm_{\Delta}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P] \tag{2.55}$$

Proof. We need the following obvious fact.

Lemma 2.80. *Let $\mathbf{i} \in Adm_\Delta(p, q) \sqcup Adm_{\Delta'}(p, q)$ such that $s := ind_{\mathbf{i}}(P) > 0$. Then*
(i) If s is even, then $(\mathbf{i}, 1)$ belongs to $[\hat{I}_P]$.
(ii) If s is odd, then both $(\mathbf{i}, 1)$ and $(\mathbf{i}, -1)$ belong to $[\hat{I}_P]$.

Furthermore, for any triangulation Δ of $[n]$ and any $p, q \in [n]$ denote by $PreAdm_\Delta(p, q)$ the set of all $\mathbf{i} \in [n]^\bullet$ such that $[\mathbf{i}] \in Adm_\Delta(p, q)$. We need the following fact.

Lemma 2.81. *In the assumptions of Proposition 2.79, let $\mathbf{i} \in Adm_{\Delta'}(p, q)$ and suppose that $s = ind_{\mathbf{i}}(P) > 0$. Then:*
(a) if s is odd, then $\mathbf{j}^s_{i',k'}(\mathbf{i}) \in PreAdm_\Delta(p, q)$ whenever $\{i_s, i_{s+1}, i', k'\} = \{i, j, k, \ell\}$.
(b) if s is even then $[J_P(\mathbf{i}, 1)] \in Adm_\Delta(p, q) \times \{-1, 1\}$.

Proof. In what follows, we will write $\mathbf{p} \le \mathbf{p}'$ for any points $\mathbf{p}, \mathbf{p}'$ in the chord (p, q) such that either $\mathbf{p} = \mathbf{p}'$ or $\mathbf{p}$ is closer to p than $\mathbf{p}'$.

Prove (a). Indeed, it suffices to show that for $\mathbf{i} = (i_1, \ldots, i_{2m}) \in Adm_{\Delta'}(p, q)$, one has

$$\mathbf{i}' := (\ldots, i_s, i', k', i_{s+1}, \ldots) \in PreAdm_\Delta(p, q) \ , \tag{2.56}$$

where $s = ind_{\mathbf{i}}(P)$ is odd (note that $\{i_s, i_{s+1}\} = \{j, \ell\}$ and $\{i', k'\} = \{i, k\}$).

Let $\mathbf{p}_-$ and $\mathbf{p}_+$ be the intersection points of (p, q) respectively with (i_{s-1}, i_s) and (i_{s+1}, i_{s+2}) (with the convention that $\mathbf{p}_- = p$ if $s = 1$ and $\mathbf{p}' = q$ if $s = 2m - 1$). Clearly, $\mathbf{p}_- < \mathbf{p}_+$.

We now consider a number of cases.

Case 1. Suppose that $(p, q) \cap (i_s, i_{s+1}) \neq \emptyset$, $3 \le s \le 2m - 3$ (i.e., $\{p, q\} \cap \{i, j, k, \ell\} = \emptyset$). Since $(i_r, i_{r+1}) \in \Delta$ for $r = s - 1, s, s + 1$, the above and convexity of the n-gon $[n]$ imply that there exist $i'', k'' \in [n]$ such that $\{i'', k''\} = \{i', k'\}$ and $(p, q) \cap (i_s, i'') \neq \emptyset$, $(p, q) \cap (i_s, k'') \neq \emptyset$, $(p, q) \cap (i', k') \neq \emptyset$ and

$$\mathbf{p}_- \le (p, q) \cap (i_s, i'') < (p, q) \cap (i', k') < (p, q) \cap (i_s, k'') \le \mathbf{p}_+ \ .$$

In turn, this immediately implies (2.56) in this case.

Case 2. Suppose that $(p, q) \cap (i_s, i_{s+1}) = \emptyset$, $3 \le s \le 2m - 3$. By definition, $\mathbf{p}_- < \mathbf{p}_0 < \mathbf{p}_+$. Then the convexity of the n-gon $[n]$ implies that there exist $i'', k'' \in [n]$ such that $\{i'', k''\} = \{i', k'\}$ and $(p, q) \cap (i_s, k'') = \emptyset$, $(p, q) \cap (i_{s+1}, k'') = \emptyset$. This and the facts that $(i'', k'') \cap (i_s, i_{s+1}) \neq \emptyset$ and that i'' does not belong to the convex hull of $\mathbf{p}_-$, $\mathbf{p}_+$, i_s, i_{s+1} imply that $(p, q) \cap (i_s, i'') \neq \emptyset$, $(p, q) \cap (i_{s+1}, i'') \neq \emptyset$, $(p, q) \cap (i', k') \neq \emptyset$ and

$$\mathbf{p}_- \leq (p, q) \cap (i_s, i'') < (p, q) \cap (i', k') < (p, q) \cap (i_{s+1}, k'') \leq \mathbf{p}_+ \ .$$

In turn, this immediately implies (2.56) in this case.

Case 3. Suppose that $s = 1$ or $s = 2m - 1$. If $s = 1 = 2m - 1$, we have nothing to prove because $\mathbf{i} = (i_1, i_2) = (p, q)$, $\mathbf{i}' = (p, i', k', q) \in Adm_\Delta(p, q)$. Therefore it remains to consider the sub-case when $s = 1$, $m \geq 2$ (the sub-case $s = 2m - 1 \geq 3$ is identical to it). Indeed, the facts that $(i', k') \cap (i_1, i_2) \neq \emptyset$ implies that there exist $i'', k'' \in [n]$ such that $\{i'', k''\} = \{i', k'\}$ and $(p, q) \cap (i_1, i'') = \emptyset$. This and the facts that $(i'', k'') \cap (i_1, i_2) \neq \emptyset$ and that k'' does not belong to the convex hull of $\mathbf{p}_- = p = i_1$, i_2 $\mathbf{p}_+$ imply that $(p, q) \cap (i_2, k'') \neq \emptyset$, $(p, q) \cap (i_{s+1}, i'') \neq \emptyset$, $(p, q) \cap (i', k') \neq \emptyset$ and

$$(p, q) \cap (i', k') < (p, q) \leq \mathbf{p}_+ \ .$$

In turn, this immediately implies (2.56) in this case.

This finishes the proof of part (a).

Prove (b) now. That is, we have to show that

$$\mathbf{i}' := [(\ldots, i_s, i', k', i_{s+1}, \ldots)] \in Adm_\Delta(p, q) \ , \tag{2.57}$$

where $s = ind_{\mathbf{i}}(P)$ is even and i', k' are as in (2.48) (note that $\{i_s, i_{s+1}\} = \{j, \ell\}$ and $\{i', k'\} = \{i, k\}$).

Denote $\mathbf{p}_0 := (p, q) \cap (i_s, i_{s+1})$ and consider a number of cases.

Case 1. Suppose that $\{i_{s-1}, i_{s+2}\} = \{i, k\}$. Then $i' = i_{s-1}$, $k' = i_{s+2}$ by (2.48) and

$$\mathbf{i}' = (\ldots, i_{s-1}, i_{s+2}, \ldots) \ ,$$

i.e., $\mathbf{i}'$ is obtained from $\mathbf{i}$ by simultaneously replacing i_s with i_{s-1} and i_{s+1} with i_{s+2}. This immediately implies (2.57) in this case.

Case 2. Suppose that $i_{s+2} \in \{i, k\}$, $i_{s-1} \notin \{i, k\}$ (the case $i_{s-1} \in \{i, k\}$, $i_{s+2} \notin \{i, k\}$ is identical to it). Then $k' = i_{s+2}$ by (2.48) and

$$\mathbf{i}' = (\ldots, i_s, i', i_{s+2}, \ldots) \ ,$$

i.e., $\mathbf{i}'$ is obtained from $\mathbf{i}$ by replacing i_{s+1} with i'. Thus, to prove (2.57), it suffices to show that $(p, q) \cap (i_s, i') \neq \emptyset$. Indeed, suppose that $(p, q) \cap (i_s, i') \neq \emptyset$. If $s = 2$, $i_{s-1} = p \notin \{i, k\}$, then taking into account that $(i_s, i') \in \Delta$, we see that i' belongs to the interior of the convex hull of $p, \mathbf{p}_0, i_s$. If $s \geq 4$, $(i_{s-2}, i_{s-1}) \in \Delta$, $(p, q) \cap (i_{s-2}, i_{s-1}) \neq \emptyset$, then i' belongs to the interior of the convex hull of $p, \mathbf{p}_0, i_{s-1}, i_s$. This contradicts to that i' is a vertex of the convex n-gon $[n]$, which immediately implies (2.57) in this case.

Case 3. Suppose that $\{i_{s-1}, i_{s+2}\} \cap \{i, k\} = \emptyset$ Then $i' = i$, $k' = k$ by (2.48) and

$$\mathbf{i}' = (\ldots, i_s, i, k, i_{s+2}, \ldots) \ .$$

Thus, to prove (2.57), it suffices to show that $(p, q) \cap (i_s, i) \neq \emptyset$, $(p, q) \cap (k, i_{s+1}) \neq \emptyset$. Since $(p, i) \cap (j, \ell) = \emptyset$, using the same argument as in **Case 2**, we see that if $(p, q) \cap (i_s, i) = \emptyset$, then i' belongs to the interior of the convex hull of $p, \mathbf{p}_0, i_{s-1}, i_s$; and if $(p, q) \cap (k, i_{s+1}) = \emptyset$, then k belongs to the interior of the convex hull of $q, \mathbf{p}_0, i_s, i_{s+1}$. This finishes the proof of (2.57) in this case.

This finishes the proof of (b).

Lemma 2.81 is proved. $\square$

Using Lemma 2.81(b) with $P = (i, j, k, \ell)$ such that $(p, i) \cap (j, \ell) = \emptyset$ and $(p, q) \cap (i, j) \neq \emptyset$ and Lemma 2.81(a) with any i', k' such that $\{i', k'\} = \{i, k\}$, we see that

$$[J_P]((Adm_{\Delta'}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P]) \subset (Adm_{\Delta}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P]$$

hence $J_{\Delta,\Delta'}$ given by (2.55) is a well-defined map

$$(Adm_{\Delta'}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P] \hookrightarrow (Adm_{\Delta}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P] \ .$$

Interchanging Δ and Δ', taking into account that $(p, j) \cap (i, k) = \emptyset$, and applying Lemma 2.81 again, we see that

$$[J_P]((Adm_{\Delta}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P]) \subset (Adm_{\Delta'}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P] \ .$$

This gives a well-defined map

$$J_{\Delta',\Delta} : (Adm_{\Delta}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P] \hookrightarrow (Adm_{\Delta'}(p, q) \times \{-1, 1\}) \cap [\hat{I}_P] \ .$$

Since $[J_P]$ is an involution by Proposition 2.78, the maps $J_{\Delta,\Delta'}$ and $J_{\Delta',\Delta}$ are inverse of each other, hence each of them is a bijection.

Proposition 2.79 is proved. $\square$

Furthermore, we need the following obvious fact.

Lemma 2.82. *In the assumptions of Lemma 2.72 let $s \in [2m-1]$ be odd and let $\mathbf{i} = (i_1, \ldots, i_{2m}) \in [n]^{2m}$, $m \geq 1$ be such that $\{i_{s'}, i_{s'+1}\} \neq \{j, \ell\}$ for $r \in [2m-1] \setminus \{s\}$.*

(a) If $\{i_s, i_{s+1}\} = \{j, \ell\}$ then $\varphi_{\Delta,\Delta'}(t_{\mathbf{i}}) = t_{\mathbf{j}^s_{ik}(\mathbf{i})} + t_{\mathbf{j}^s_{ki}(\mathbf{i})}$.

(b) If $\{i_s, i_{s+1}\} = \{i, k\}$, then $\varphi_{\Delta,\Delta'}(t_{\mathbf{j}^s_{j\ell}(\mathbf{i})} + t_{\mathbf{j}^s_{\ell j}(\mathbf{i})}) = t_{\mathbf{i}}$.

Now we are ready to prove (2.45). Indeed, $t_{pq}^{\Delta'} = t_0 + t_- + t_+$, where

$$t_0 = \sum_{\mathbf{i}' \in Adm_{\Delta'}(p,q): ind_{\mathbf{i}'}(P)=0} t_{\mathbf{i}'},\ t_- = \sum_{\mathbf{i}' \in Adm_{\Delta'}(p,q): ind_{\mathbf{i}'}(P)\in 2\mathbb{Z}+1} t_{\mathbf{i}'},\ t_+$$
$$= \sum_{\mathbf{i}' \in Adm_{\Delta'}(p,q): ind_{\mathbf{i}'}(P)\in 2\mathbb{Z}_{\geq 1}} t_{\mathbf{i}'} \ .$$

Clearly,

$$\varphi_{\Delta,\Delta'}(t_0) = t_0 = \sum_{\mathbf{i} \in Adm_{\Delta}(p,q): ind_{\mathbf{i}}(P)=0} t_{\mathbf{i}} \ .$$

Furthermore, combining Proposition 2.79 and Lemma 2.82, we obtain:

$$\varphi_{\Delta,\Delta'}(t_-) = \sum_{\mathbf{i}' \in Adm_{\Delta'}(p,q): ind_{\mathbf{i}}(P)\in 2\mathbb{Z}+1} t_{J_{\Delta,\Delta'}(\mathbf{i},1)} + t_{J_{\Delta,\Delta'}(\mathbf{i},-1)}$$
$$= \sum_{\mathbf{i} \in Adm_{\Delta}(p,q): ind_{\mathbf{i}}(P)\in 2\mathbb{Z}_{\geq 1}} t_{\mathbf{i}} \ ,$$
$$\varphi_{\Delta,\Delta'}(t_+) = \sum_{\mathbf{i} \in Adm_{\Delta}(p,q): ind_{\mathbf{i}}(P)\in 2\mathbb{Z}+1} \varphi_{\Delta,\Delta'}(t_{J_{\Delta',\Delta}(\mathbf{i},1)} + t_{J_{\Delta',\Delta}(\mathbf{i},-1)})$$
$$= \sum_{\mathbf{i} \in Adm_{\Delta}(pq): ind_{\mathbf{i}}(P)\in 2\mathbb{Z}+1} t_{\mathbf{i}} \ .$$

This finishes the proof of (2.45).

Furthermore, we define a homomorphism $\psi_{\Delta,\Delta'}$ as follows. First, composing $\varphi_{\Delta,\Delta'}$ with the universal localization by S_Δ and taking into the account that $t_{j\ell}^\Delta \in S_\Delta$, we obtain a homomorphism of algebras:

$$\varphi'_{\Delta,\Delta'} : \mathbb{QT}_{\Delta'} \to \mathbb{QT}_\Delta[(t_{j\ell}^\Delta)^{-1}]$$

such that $\varphi'_{\Delta,\Delta'}(t_{ij}^{\Delta'}) = t_{ij}^\Delta$ for all i,j. Since $t_{ij}^\Delta \in S_\Delta$ is invertible in the image, $\varphi'_{\Delta,\Delta'}$ canonically extends to a homomorphisms of algebras

$$\psi_{\Delta,\Delta'} : \mathbb{QT}_{\Delta'}[S_{\Delta'}^{-1}] \to \mathbb{QT}_\Delta[S_\Delta^{-1}] \ .$$

Switching Δ and Δ' we obtain a homomorphism $\psi_{\Delta',\Delta} : \mathbb{QT}_\Delta[S_\Delta^{-1}] \to \mathbb{QT}_\Delta[S_{\Delta'}^{-1}]$, which is, clearly, inverse of $\psi_{\Delta,\Delta'}$.

This proves Theorem 2.71 for neighboring triangulations Δ, Δ'.

Now we prove Theorem 2.71 for any (non-neighboring) triangulations Δ, Δ' of $[n]$ as follows. We say that the distance $dist(\Delta, \Delta')$ is the minimal number $d \geq 0$ such that there is a sequence of triangulations $\Delta = \Delta^{(0)}, \Delta^{(1)}, \ldots, \Delta^{(d)} = \Delta'$ of $[n]$ such that $\Delta^{(s)}, \Delta^{(s+1)}$, $s \in [r-1]$ are neighboring.

We construct appropriate $\varphi_{\Delta,\Delta'}$ by induction in $dist(\Delta,\Delta')$. If $dist(\Delta,\Delta')=1$, then Δ and Δ' are neighboring and we have nothing to prove. Suppose that $d=dist(\Delta,\Delta')>1$. Then there is a triangulation Δ'' of $[n]$ with $dist(\Delta,\Delta'')<d$ and $dist(\Delta'',\Delta')<d$.

By the inductive hypothesis, there are isomorphisms

$$\psi_{\Delta,\Delta''}:\mathbb{QT}_{\Delta''}[S_{\Delta''}^{-1}]\to\mathbb{QT}_{\Delta}[S_{\Delta}^{-1}],\ \psi_{\Delta'',\Delta'}:\mathbb{QT}_{\Delta'}[S_{\Delta'}^{-1}]\to\mathbb{QT}_{\Delta''}[S_{\Delta''}^{-1}]$$

such that $\psi_{\Delta,\Delta''}(t_{ij}^{\Delta''})=t_{ij}^{\Delta}$ and $\psi_{\Delta'',\Delta'}(t_{ij}^{\Delta'})=t_{ij}^{\Delta''}$ for all i,j.

Define $\psi_{\Delta,\Delta'}:=\psi_{\Delta,\Delta''}\circ\psi_{\Delta'',\Delta'}$. By definition, $\psi_{\Delta,\Delta'}$ is an isomorphism $\mathbb{QT}_{\Delta'}[S_{\Delta'}^{-1}]\to\mathbb{QT}_{\Delta}[S_{\Delta}^{-1}]$ such that $\psi_{\Delta,\Delta''}(t_{ij}^{\Delta'})=t_{ij}^{\Delta}$ for all i,j. In particular, $\psi_{\Delta,\Delta'}$ does not depend on the choice of Δ''. This finishes the induction.

The transitivity (2.42) also follows.

Theorem 2.71 is proved. $\square$

Furthermore, we need the following result.

Proposition 2.83. *In the notation of Theorem 2.71, for each triangulation Δ of $[n]$ the homomorphism $\mathbf{i}_\Delta:\mathbb{QT}_\Delta\to\mathcal{A}_\Delta\subset\mathcal{A}_n$ given by (2.4) extends to an isomorphism of algebras $\mathbb{QT}_\Delta[S_\Delta^{-1}]\widetilde{\to}\mathcal{A}_n$.*

Proof. We need the following result.

Lemma 2.84. *Let Δ be any triangulation of $[n]$. Then*
(i) For any distinct $i,j,k\in[n]$, the elements $x'_{ab}:=t_{ab}^{\Delta}$, $\{a,b\}\subset\{i,j,k\}$ satisfy the triangle relations (2.1).
(ii) For any cyclic quadruple (i,j,k,ℓ) the elements $x'_{ab}:=t_{ab}^{\Delta}$, $\{a,b\}\subset\{i,j,k,\ell\}$ satisfy the exchange relations (2.2).

Proof. Indeed, to prove (i) note that for any distinct $i,j,k\in[n]$ there exists a triangulation Δ_0 such that (i,j,k) is a triangle in Δ_0 therefore, the elements $t_{ab}\in\mathbb{T}_{\Delta'}$, $\{a,b\}\subset\{i,j,k\}$ satisfy (2.1). Applying the isomorphism ψ_{Δ,Δ_0} given by (2.71), we finish the proof of (i).

To prove (ii) note that for any cyclic (i,j,k,ℓ) there exists a triangulation Δ_0 such that both triangles (i,j,k) and (j,k,ℓ) belong to Δ_0 (hence $(j,\ell)\notin\Delta_0$). By (2.43) for Δ_0, we see that $t_{ab}^{\Delta_0}$, $\{a,b\}\subset\{i,j,k,\ell\}$ satisfy (2.2).

Thus applying the isomorphism ψ_{Δ,Δ_0}, we finish the proof of (ii).

The lemma is proved. $\square$

By Lemma 2.84, the assignments $x_{pq}\mapsto t_{pq}^{\Delta}$ for all distinct $p,q\in[n]$ define an epimorphism of algebras

$$\mathcal{A}_n\twoheadrightarrow\mathbb{QT}_\Delta[S_\Delta^{-1}]\ .$$

On the other hand, by (already proved) Theorem 2.10, for each triangulation Δ of $[n]$ and any distinct $i, j \in [n]$ the element $x_{ij} \in \mathbf{i}_\Delta(\mathbb{Q}\mathbb{T}_\Delta)$. Therefore, by the universality of localizations, $\mathbf{i}_\Delta$ extends to an epimorphism of algebras $\hat{\mathbf{i}}_\Delta : \mathbb{Q}\mathbb{T}[S_\Delta^{-1}] \twoheadrightarrow \mathcal{A}_n$. Clearly, these two homomorphisms are mutually inverse.

This finishes the proof of Proposition 2.83. $\square$

Furthermore, denote by $\mathbf{S}$ the submonoid of $\mathcal{A}_\Delta \setminus \{0\}$ generated by all x_{ij}. Clearly, $\mathbf{S} = \mathbf{i}_\Delta(S_\Delta)$ and $\mathcal{A}_\Delta = \mathbf{i}_\Delta(\mathbb{Q}\mathbb{T}_\Delta)$. Therefore, $\mathcal{A}_n = \mathcal{A}_\Delta[\mathbf{S}^{-1}]$.

This proves Theorem 2.8. $\square$

Finally, Theorem 2.3 follows from Theorem 2.8 and that $\mathcal{A}'_n := \mathcal{A}_\Delta = \mathbf{i}_\Delta(\mathbb{Q}\mathbb{T}_\Delta)$ is the group algebra of $\mathbb{T}_\Delta$, which is a free group in $3n-4$ generators by (already proved) Theorem 2.7. $\square$

2.15. Self-similarity implies injectivity

In this section we prove the following result.

Proposition 2.85. *If Conjecture A.18 holds for $m = 3n-4$, $n \geq 4$ and $k = 2, \ldots, n-2$, then for each triangulation Δ of $[n]$ the homomorphism of algebras*

$$\mathcal{A}_n \to \mathcal{F}_{3n-4} ,$$

which is the canonical (by Proposition 2.83 and Lemma A.1) extension to $\mathcal{A}_n \cong \mathbb{Q}\mathbb{T}_\Delta[S_\Delta^{-1}]$ of the natural inclusion $\mathbb{Q}\mathbb{T}_\Delta \hookrightarrow Frac(\mathbb{Q}\mathbb{T}_\Delta) \cong \mathcal{F}_{3n-4}$ is a also a monomorphism (hence $\mathcal{A}_n$ has no zero divisors).

Proof. it suffices to show that for at lest one triangulation Δ of $[n]$ the submonoid $\hat{S}_\Delta \subset \mathbb{Q}\mathbb{T}_\Delta \setminus \{0\}$ generated by all t_{ij}^Δ and by $(\mathbb{Q}\mathbb{T}_\Delta)^\times = \mathbb{Q}^\times \cdot \mathbb{T}_\Delta$ is factor-closed in the sense of Definition A.4. Since $\mathbb{T}_\Delta$ is a free group by Theorem 2.7, in view of Proposition A.15, it suffices to verify that each t_{ij}^Δ, $(i,j) \notin \Delta$ is prime in $\mathbb{Q}\mathbb{T}_\Delta$ and all primes similar to t_{ij} belong to $\hat{S}_\Delta$. Now let $\Delta = \Delta_1$ be the starlike triangulation as in (2.6) with $i = 1$.

We need the following obvious fact.

Lemma 2.86. *For all $n \geq 2$ the group $\mathbb{T}_{\Delta_1}$ is freely generated by $\tau_j := T_1^{j,j+1}$, $j = 2, \ldots, n-1$, $t_{1,k}$, $t_{k,1}$, $k = 2, \ldots, n$.*

Proof. Clearly, $\mathbb{T}_{\Delta_1}$ has a presentation $t_{j,j+1} = t_{j,1}\tau_j t_{1,j+1}$, $t_{j+1,j} = t_{j+1,1}\tau_j t_{1j}$ for $j = 2, \ldots, n-1$.

This proves the lemma. $\square$

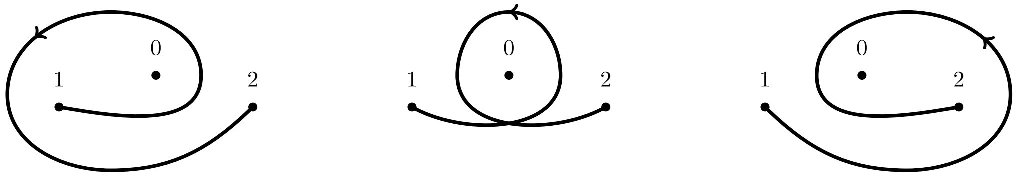


Fig. 1. Pairwise non-equivalent curves from puncture 1 to puncture 2.

Furthermore, Corollary 2.23 implies that the monoid $\hat{S}_{\Delta_1}$ is generated by $\mathbb{T}_{\Delta_1}$ and noncommutative angles

$$T_1^{ij} = \tau_i + \ldots + \tau_j$$

for $2 \leq i < j \leq n$. Clearly, each T_1^{ij}, $i < j-1$ is prime in $\mathbb{Q}\mathbb{T}_{\Delta_1}$. Let $P_{ij} := \mathbb{Q}^\times \cdot \mathbb{T}_{\Delta_1} \cdot T_1^{ij} \cdot \mathbb{T}_{\Delta_1}$ for $2 \leq i < j \leq n$. By Conjecture A.18 with $m = 3n-4$, $k = j$, that the only primes similar to T_1^{ij} are elements of P_{ij}. This together with Proposition A.15 and Remark A.14 proves that the submonoid $\mathbb{Q}^\times \cdot \hat{S}_{\Delta_1}$ of $\mathbb{Q}\mathbb{T}_{\Delta_1} \setminus \{0\}$ is factor-closed because it is generated by $\mathbb{Q}^\times \cdot \mathbb{T}_{\Delta_1}$ and $P = \bigcup\limits_{2\leq i<j\leq n} P_{ij}$. Therefore, Corollary A.13 guarantees that $\mathbb{Q}\mathbb{T}_{\Delta_1}[S^{-1}] = \mathbb{Q}\mathbb{T}_{\Delta_1}[\mathbb{Q}^\times \cdot \hat{S}_{\Delta_1}^{-1}]$ is a subalgebra of $\mathcal{F}_{3n-4} = Frac(\mathbb{Q}\mathbb{T}_{\Delta_1})$.

Using this and Proposition 2.83 with $\Delta = \Delta_1$, we finish the proof of Proposition 2.85. $\square$

3. Noncommutative surfaces

In this section we extend all the constructions and results of Section 2 to *marked surfaces* i.e., (connected compact smooth) surfaces Σ possibly with boundary equipped with a non-empty finite set $I = I(\Sigma) = I_b \sqcup I_p$ of marked points with a subset $I_b = I_b(\Sigma) \subset I$ of marked boundary points, the set $I_p = I_p(\Sigma) = I \setminus I_b$ of *ordinary punctures* and a set $I_s = I_s(\Sigma)$ of *special punctures* (which were called *orbifold point of order* 2 in [10], however, we will not use this terminology). We also require that each boundary component contains at least one point from I_b. We denote by $\underline{\Sigma}$ the underlying topological space.

3.1. Multi-groupoid of curves on Σ

Given points $p_1, p_2 \in I(\Sigma)$, consider connected smooth directed curves C in $\underline{\Sigma} \setminus I_p(\Sigma)$ starting at p_1 and terminating at p_2. For a curve C denote by $\overline{C}$ the same curve traversed from p_2 to p_1. We say that curves C and C' in Σ from p_1 to p_2 are *equivalent* if C and C' are homotopy equivalent as (connected smooth directed) curves in $\underline{\Sigma} \setminus I_p(\Sigma)$ (Fig. 1).

Denote by $\Gamma_{ij} = \Gamma_{ij}(\Sigma)$ the set of equivalence classes of curves C in Σ which originate at i and terminate at j then let $\Gamma = \Gamma(\Sigma) := \bigsqcup\limits_{i,j \in I(\Sigma)} \Gamma_{ij}$. For $\gamma \in \Gamma_{ij}$ we denote by $s(\gamma) \in I(\Sigma)$ (resp. by $t(\gamma) \in I(\Sigma)$) the source i (resp. the target j).

Thus we have a natural involution $\bar{\cdot} : \Gamma \widetilde{\to} \Gamma$ $(\gamma \mapsto \overline{\gamma})$. By definition, $\overline{\Gamma}_{ij} = \Gamma_{j,i}$ for all $i, j \in I(\Sigma)$.

Involution $\gamma \mapsto \overline{\gamma}$

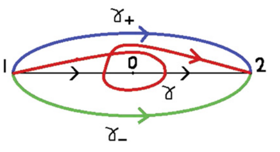
For $j \in I(\Sigma)$ denote by id_j the *trivial loop* at j. Clearly, $\gamma = \overline{\gamma}$ iff γ is trivial.

It is easy to see that $\Gamma(\Sigma)$ is finite iff Σ is homeomorphic to an n-gon, i.e., a disk with $n \geq 1$ marked points and no punctures. In that case, the assignments $\gamma \mapsto (s(\gamma), t(\gamma))$ define a bijection $\Gamma \widetilde{\to} \{(i, j) \in [n], i \neq j\}$.

We say that $\gamma \in \Gamma(\Sigma)$ is *simple* if it has a non-self-intersecting representative. Denote by $\Gamma^0(\Sigma)$ the set of all simple $\gamma \in \Gamma(\Sigma)$.

Definition 3.1. We say that a pair (γ, γ') in $\Gamma(\Sigma)$ is *composable* if $t(\gamma) = s(\gamma')$ and define the *composition* $\gamma'' = \gamma \circ \gamma'$ to be the pullback, under the natural projection $\Gamma(\Sigma) \twoheadrightarrow \Gamma(\Sigma \setminus (I_p(\Sigma) \setminus \{t(\gamma)\}))$ of the concatenation of γ and γ'.

Clearly, the multi-composition $\gamma \circ \gamma'$ is a 1-element set iff $t(\gamma) = s(\gamma') \in I_b(\Sigma)$. Otherwise $\gamma \circ \gamma'$ is countable.



Multi-composition: $\{\gamma_-, \gamma, \gamma_+\} \in (1, 0) \circ (0, 2)$.

The following is immediate.

Lemma 3.2. *For each marked surface Σ the set $\Gamma(\Sigma)$ is a multi-groupoid with the object set $I(\Sigma)$ and the inverse given by $\gamma^{-1} := \overline{\gamma}$.*

Remark 3.3. A multi-category (e.g., a multi-groupoid) is a natural generalization of a category (e.g., of a groupoid) where we allow the composition of two morphisms to be a set of arrow and require the associativity $(\gamma \circ \gamma') \circ \gamma'' = \gamma \circ (\gamma' \circ \gamma'')$, which is an equality of sets, see e.g. [7] (where the term *polygroupoid* was introduced).

Remark 3.4. If $I_p(\Sigma) = \emptyset$, then $\Gamma(\Sigma)$ is an ordinary groupoid (cf. [4, Section 2.2]).

3.2. Category of surfaces and reduced curves

Definition 3.5. Given a continuous map $f : \underline{\Sigma} \to \underline{\Sigma}'$ with discrete fibers, we say that f is a *morphism of marked surfaces* $\Sigma \to \Sigma'$ if:

Fig. 2. Crossing resolution.

- $f^{-1}(I(\Sigma')) = I(\Sigma)$, $f(I_s(\Sigma)) \subset I_s(\Sigma')$ (we abbreviate $I^f := f^{-1}(I_s(\Sigma')) \setminus I_s(\Sigma)$).
- For each point $p \in \underline{\Sigma} \setminus I^f$ there is a neighborhood $\mathcal{O}_p$ of p in $\underline{\Sigma}$ such that the restriction of f to $\mathcal{O}_p$ is injective (if $p \in \partial\underline{\Sigma}$ is a boundary point, then $\mathcal{O}_p$ is a "half-neighborhood").
- For each $p \in I^f$ there is a neighborhood $\mathcal{O}_p$ of p in $\underline{\Sigma}$ such that the restriction of f to $\mathcal{O}_p$ is a two-fold cover of $f(\mathcal{O}_p)$ ramified at $f(p)$.

Theorem 3.6. *For any morphisms of marked surfaces $f : \Sigma \to \Sigma'$ and $f' : \Sigma' \to \Sigma''$ the composition $f' \circ f : \underline{\Sigma} \to \underline{\Sigma}''$ is also a morphism of marked surfaces $\Sigma' \to \Sigma''$.*

We prove Theorem 3.6 in Section 3.11.

In what follows, denote by **Surf** the category whose objects are marked surfaces and arrows are morphisms of marked surfaces.

Note that if $f : \Sigma \to \Sigma'$ is a morphism in **Surf** with $I^f = \emptyset$, then f respects (homotopy) equivalence of curves and, in particular, defines a map $\Gamma(\Sigma) \to \Gamma(\Sigma')$. In general, this is no longer true. To fix it, we define below a stronger equivalence relation than the equivalence for curves in Σ'.

Indeed, given $i \in I_s(\Sigma)$, we say that a curve C in Σ is *i-reducible* if there is a self-intersection point $p \in C$ such that the loop $C_0 \subset C$ defined by p encloses exactly one special puncture i; otherwise, C is *i-reduced*. Respectively, $\gamma \in \Gamma(\Sigma)$ is i-reducible (resp. i-reduced) if γ has an i-reducible (resp. i-reduced) representative. Denote by $[\Gamma(\Sigma)]_i$ the set of all i-reduced $\gamma \in \Gamma(\Sigma)$, abbreviate $[\Gamma(\Sigma)] := \bigcap\limits_{i \in I_s(\Sigma)} [\Gamma(\Sigma)]_i$ and refer to elements of $[\Gamma(\Sigma)]$ as *reduced*. Clearly, $[\Gamma(\Sigma)] = \Gamma(\Sigma)$ iff $I_s(\Sigma) = \emptyset$. It is also clear that and each $\gamma \in \Gamma^0(\Sigma)$ is reduced.

For each i-reducible $\gamma \in \Gamma(\Sigma)$ denote by $[\gamma]_i$ the class in $\Gamma(\Sigma)$ obtained by resolving the self-intersecting simple loop around i in (a generic representative C of) γ so that the resulting curve is connected (the "wrong" crossing resolution would result in creating two connected components, one of which is a circle around i) (Fig. 2).

The following is obvious.

Lemma 3.7.

(a) The assignments $\gamma \to \begin{cases} [\gamma]_i & \textit{if } \gamma \textit{ is } i\textit{-reducible} \\ \gamma & \textit{if } \gamma \textit{ is } i\textit{-reduced} \end{cases}$ *define a map $\pi_i : \Gamma(\Sigma) \to \Gamma(\Sigma)$.*

(b) $\pi_i \circ \pi_j = \pi_j \circ \pi_i$ for all $i, j \in I_s(\Sigma)$.

(c) The assignments $\gamma \mapsto \pi_i^N(\gamma)$ *for sufficiently big* N *define a projection* $\pi_i^\infty : \Gamma(\Sigma) \to [\Gamma(\Sigma)]_i$.

(d) The composition $\pi^\infty := \prod\limits_{i \in I_s(\Sigma)} \pi_i^\infty$ *is a projection* $\Gamma(\Sigma) \to [\Gamma(\Sigma)]$.

This, in particular, defines an equivalence relation on $[\Gamma(\Sigma)]$, namely for $\gamma, \gamma' \in \Gamma(\Sigma)$ we say that any representatives $C \in \gamma$ and $C' \in \gamma'$ are $I_s(\Sigma)$-*equivalent* iff $\pi^\infty(\gamma) = \pi^\infty(\gamma')$. We naturally identify $I_s(\Sigma)$-equivalence classes with elements of $[\Gamma(\Sigma)]$.

For each $i \in I_s(\Sigma)$ and $j \in I(\Sigma)$ let λ_{ij} denote a (unique up to $\bar{\cdot}$) simple loop at j around i in $[\Gamma(\Sigma)]$. We refer to such loops as *special*.

For $n \geq 1$, $h \geq 0$ denote by $P_n(h)$ the n-gon (i.e., a disk with n marked boundary points) with h special punctures and abbreviate $P_n := P_n(0)$. Clearly, each special loop λ determines a (homeomorphic) copy of $P_1(1)$ with the marked point set $\{\underline{j}\}$ and the special puncture set $\{i\}$.

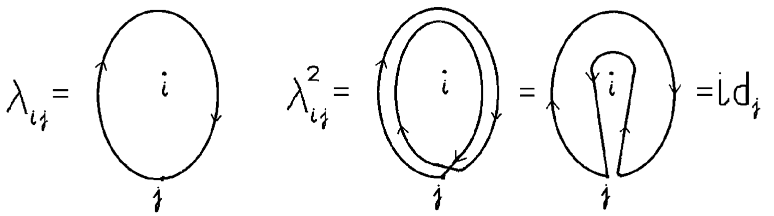
Lemma 3.8. *For any marked surface* Σ *the set* $[\Gamma(\Sigma)]$ *has a natural multi-groupoid structure:*

$$[\gamma] \circ [\gamma'] := [\gamma \circ \gamma']$$

for any composable (γ, γ') *in the multi-groupoid* $\Gamma(\Sigma)$ *with the object set* $I(\Sigma)$*. Moreover,*

(i) the assignments $\gamma \mapsto [\gamma]$ *define a surjective homomorphism of multi-groupoids* $\Gamma(\Sigma) \twoheadrightarrow [\Gamma(\Sigma)]$.

(ii) For each $i \in I_s(\Sigma)$ *and* $j \in I(\Sigma)$ *each special loop satisfies* $\overline{\lambda}_{ij} = \lambda_{ij}$.



Special loops are involutions in $[\Gamma(\Sigma)]$

The following result asserts functoriality of the multi-groupoid under morphisms of surfaces.

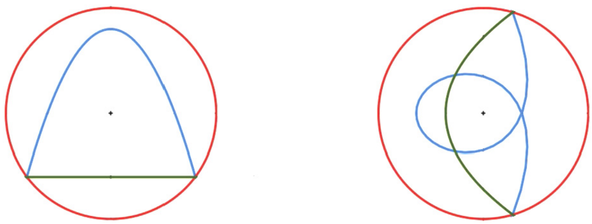
Theorem 3.9. *Let* f *be any morphism of marked surfaces* $\Sigma \to \Sigma'$ *and let* $\gamma \in [\Gamma(\Sigma)]$*. Then*

(a) for any generic representatives $C, C' \in \gamma$*, their images* $f(C)$ *and* $f(C')$ *are* $I_s(\Sigma')$*-equivalent.*

(b) For each $\gamma \in [\Gamma(\Sigma)]$ *there exists a unique* $I_s(\Sigma')$*-equivalence class* $f(\gamma) \in [\Gamma(\Sigma')]$ *such that* $f(C) \in f(\gamma)$ *for any generic curve* $C \in \gamma$.

(c) $f : [\Gamma(\Sigma)] \to [\Gamma(\Sigma')]$ $(\gamma \mapsto f(\gamma))$ *is a homomorphism of multi-groupoids.*

(d) The assignments $\Sigma \mapsto [\Gamma(\Sigma)]$ define a functor from **Surf** *to the category of multi-groupoids.*



$I_s(\Sigma)$-equivalence of images of curves under the ramified double cover $z \mapsto z^2$ of $\mathbb{C}$

We prove Theorem 3.9 in Section 3.11.

It is well-known that marked surfaces can be glued out of polygons, i.e., for any Σ there exists a surjective gluing morphism $f : P_n(h) \twoheadrightarrow \Sigma$ in **Surf** with $h = |I_s(\Sigma)|$, $n \geq 1$ such that all $f(i, i^+) \in \Gamma^0(\Sigma)$ and the restriction of f to the interior of $P_n(h)$ is injective. For readers' convenience we construct such a gluing morphism f in Lemma 3.47 for any triangulation of Σ.

The following fact is obvious.

Lemma 3.10. *Let Σ be a marked surface. Then $[\Gamma(\Sigma)]$ is finite if an only if Σ is homeomorphic either a once punctured sphere or to $[n] = P_n$ or to $P_n(1)$ for some $n \geq 1$. More precisely, the assignments*

$$\gamma \mapsto \begin{cases} (s(\gamma), t(\gamma), +) & \textit{if the special puncture is to the right of } \gamma \\ (s(\gamma), t(\gamma), -) & \textit{if the special puncture is to the left of } \gamma \end{cases}$$

define a bijection $[\Gamma(P_n(1))] \widetilde{\to} \{(i, j) \in [n]\} \times \{-, +\}$.

3.3. Polygons in surfaces, noncommutative surfaces and functoriality

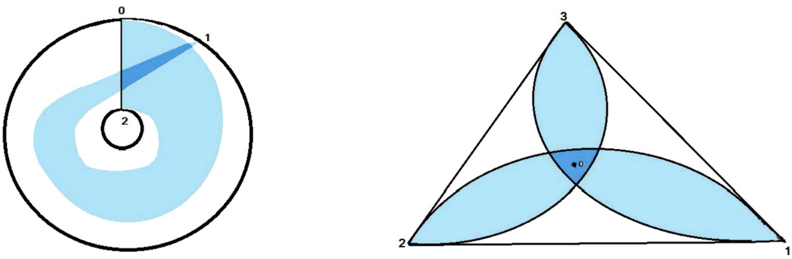
We say that a sequence $P = (\gamma_1, \ldots, \gamma_r)$ of not necessarily distinct $\gamma_i \in [\Gamma(\Sigma)]$, $i \in [r]$, is *cyclic* if each pair (γ_i, γ_{i^+}), $i \in [r]$ is composable.

Definition 3.11. We say that a sequence $P = (\gamma_1, \ldots, \gamma_n)$ is an n-gon in Σ if there exists a morphism $f : P_n \to \Sigma$ such that $f(i, i^+) = \gamma_i$ for $i \in [n]$. We also denote $\gamma_{ij} := f(i, j)$ for all distinct $i, j \in [n]$ (clearly, γ_{ij} is nontrivial for all distinct $i, j \in [n]$). We will refer to such an f as an *accompanying to P* morphism.

Clearly, each n-gon $P = (\gamma_1, \ldots, \gamma_n)$ in Σ is cyclic and for any $\gamma \in [\Gamma(\Sigma)]$ the pair $(\gamma, \overline{\gamma})$ is a 2-gon in Σ. It is convenient to define the *interior* P^0 of an n-gon $P = (\gamma_1, \ldots, \gamma_n)$ to be the image of the interior of P_n under an accompanying morphism (to do so we choose generic representatives $C_i \in \gamma_i$ so that $f(i, i^+) = C_i$ for $i \in [i]$). It is also clear that P^0

does not depend on the choice of f, and different choices of $C_i \in \gamma_i$ result in homotopic to each other morphisms $f : P_n \to \Sigma$. We say that P is *simple* if P^0 is homeomorphic to a disk.

We will sometimes refer to an 3-gon in Σ respectively as a triangle and to a 4-gon – as a quadrilateral.



Non-simple triangles in an annulus and in $P_3(1)$

Definition 3.12. For a marked surface Σ let $\mathcal{A}_\Sigma$ be the $\mathbb{Q}$-algebra generated by all x_γ, $\gamma \in [\Gamma(\Sigma)]$ subject to

(i) $x_\gamma = 1$ if γ is trivial.

(ii) (triangle relations) For any triangle $(\gamma_1, \gamma_2, \gamma_3)$ in Σ one has

$$x_{\gamma_1} x_{\overline{\gamma}_2}^{-1} x_{\gamma_3} = x_{\overline{\gamma}_3} x_{\gamma_2}^{-1} x_{\overline{\gamma}_1} \ . \tag{3.1}$$

(iii) (exchange relations) For any quadrilateral $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ in Σ:

$$x_{\gamma_{24}} = x_{\gamma_{21}} x_{\gamma_{31}}^{-1} x_{\gamma_{34}} + x_{\gamma_{23}} x_{\gamma_{13}}^{-1} x_{\gamma_{14}} \ . \tag{3.2}$$

Likewise (similarly to Section 2.5), we define the *big triangle group* $\mathbb{T}_\Sigma$ of Σ to be generated by all t_γ, $\gamma \in [\Gamma(\Sigma)]$ subject to:

- $t_\gamma = 1$ if γ is trivial.
- (triangle relations) $t_{\gamma_1} t_{\overline{\gamma}_2}^{-1} t_{\gamma_3} = t_{\overline{\gamma}_3} t_{\gamma_2}^{-1} t_{\overline{\gamma}_1}$ for all triangles $(\gamma_1, \gamma_2, \gamma_3)$ in Σ.

The following fact is obvious.

Lemma 3.13. *For each marked surface Σ the assignments $t_\gamma \mapsto x_\gamma$ define a homomorphism of groups:*

$$\mathbb{T}_\Sigma \to \mathcal{A}_\Sigma^\times \ . \tag{3.3}$$

It is natural to conjecture that this homomorphism is an isomorphism.

The following result is also obvious.

Lemma 3.14. *(a) For each marked surface Σ there is a unique involutive anti-automorphism $\bar{\cdot}$ of $\mathcal{A}_\Sigma$ (resp. of $\mathbb{T}_\Sigma$) such that $\overline{x}_\gamma = x_{\overline{\gamma}}$ (resp $\overline{t}_\gamma = t_{\overline{\gamma}}$) for all $\gamma \in [\Gamma(\Sigma)]$.*

(b) If γ is a simple special loop around $i \in I_p(\Sigma) \sqcup I_s(\Sigma)$, then $\overline{x}_\gamma = x_\gamma$ (resp. $\overline{t}_\gamma = t_\gamma$).

Remark 3.15. This bar anti-involution is analogous to the one in quantum algebras. Also, Lemma 3.14(b) asserts that simple loops around an ordinary and special punctures are "close relatives."

The following result, in fact, asserts that the assignments $\Sigma \mapsto \mathbb{T}_\Sigma$ and $\Sigma \mapsto \mathcal{A}_\Sigma$ are respectively functors **Surf** $\to$ **Groups** and **Surf** $\to \mathbb{Q} -$ **Alg**.

Theorem 3.16. *For any morphism $f : \Sigma \to \Sigma'$ in **Surf** the assignments $t_\gamma \mapsto t_{f(\gamma)}$ (resp. $x_\gamma \mapsto x_{f(\gamma)}$) define a homomorphism of groups $f_\star : \mathbb{T}_\Sigma \to \mathbb{T}_{\Sigma'}$ (resp. of algebras $f_* : \mathcal{A}_\Sigma \to \mathcal{A}_{\Sigma'}$) and the following diagram is commutative*

$$\begin{array}{ccc} \mathbb{T}_\Sigma & \longrightarrow & \mathcal{A}_\Sigma \\ {\scriptstyle f_\star}\big\downarrow & & {\scriptstyle f_*}\big\downarrow \\ \mathbb{T}_{\Sigma'} & \longrightarrow & \mathcal{A}_{\Sigma'} \end{array} . \tag{3.4}$$

We prove Theorem 3.16 in Section 3.11.

Definition 3.17. For a marked surface Σ denote by $\hat{\Sigma}$ the marked surface obtained from Σ by turning each special puncture into the ordinary one, i.e., $\underline{\hat{\Sigma}} = \underline{\Sigma}$, $I(\hat{\Sigma}) = I(\Sigma) \sqcup I_s(\Sigma)$, $I_s(\hat{\Sigma}) = \emptyset$.

Clearly, $[\Gamma(\Sigma)] \subseteq [\Gamma(\hat{\Sigma})] = \Gamma(\hat{\Sigma})$ and the complement $[\Gamma(\hat{\Sigma})] \setminus [\Gamma(\Sigma)]$ consists of classes of curves originating or terminating in formerly special punctures.

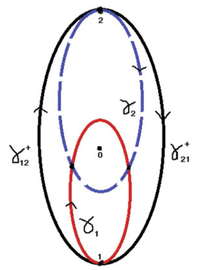
Proposition 3.18. *The assignments $t_\gamma \mapsto t_\gamma$ for $\gamma \in [\Gamma(\Sigma)]$ define a homomorphism of groups*

$$\mathbb{T}_\Sigma \to \mathbb{T}_{\hat{\Sigma}} \ , \tag{3.5}$$

where $\hat{\Sigma}$ is as in Definition 3.17.

Remark 3.19. It is natural to conjecture that (3.5) is injective. Note, however, that the natural identification $Id : \Sigma \hookrightarrow \hat{\Sigma}$ is not a morphism in **Surf** since it takes $I_s(\Sigma)$ to $I_p(\hat{\Sigma})$, so we expect that there is **no** homomorphisms $\mathcal{A}_\Sigma \to \mathcal{A}_{\hat{\Sigma}}$, which together with (3.5) would make the diagram (3.4) commutative, and illustrate this with the following example.

Example 3.20. Let $\Sigma = P_2(1)$ with the vertex set $I = \{1, 2\}$ and a single special puncture 0. For $i \in I$ denote by γ_i the clockwise loop at i around 0 inside Σ. For $i, j \in I$, $i \neq j$ denote by γ_{ij}^+ (resp. γ_{ij}^-) the boundary curve from i to j so that 0 is to the right (resp. to the left).



A quadrilateral in $P_2(1)$

We abbreviate $x_i := x_{\gamma_i}$, $\overline{x}_i := x_{\overline{\gamma}_i}$, $x_{ij}^+ := x_{\gamma_{ij}^+}$, $x_{ij}^- := x_{\gamma_{ij}^-}$ for the corresponding generators of $\mathcal{A}_\Sigma$.

Then, according to Definition 3.12, $\mathcal{A}_\Sigma$ has a presentation:

$$\overline{x}_1 = x_1,\ \overline{x}_2 = x_2,\ x_{21}^+ x_1^{-1} x_{12}^+ = x_{21}^- x_1^{-1} x_{12}^-,\ x_{12}^+ x_2^{-1} x_{21}^+ = x_{12}^- x_2^{-1} x_{21}^- \ ,$$
$$x_2 = x_{21}^+ x_1^{-1} x_{12}^- + x_{21}^- x_1^{-1} x_{12}^+,\ x_1 = x_{12}^+ x_2^{-1} x_{21}^- + x_{12}^- x_2^{-1} x_{21}^+ \ .$$

Let $\hat{\Sigma}$ be obtained from Σ by converting all special punctures into ordinary ones (as in Definition 3.17). Therefore, curves on $\hat{\Sigma}$ are those on Σ plus four additional ones: directed intervals $\gamma_{0,i}$ from 0 to each i and $\gamma_{i,0} := \gamma_{0,i}^{-1}$. We abbreviate the generators of $\mathcal{A}_{\hat{\Sigma}}$ same way as in $\mathcal{A}_\Sigma$ and $x_{0,i} := x_{\gamma_{0,i}}$, $x_{i,0} := x_{\gamma_{i,0}}$.

Then, according to Definition 3.12, $\mathcal{A}_{\hat{\Sigma}}$ has a presentation:

$$\overline{x}_1 = x_1,\ \overline{x}_2 = x_2,\ x_{21}^+ x_1^{-1} x_{12}^+ = x_{21}^- x_1^{-1} x_{12}^-,$$
$$x_{12}^+ x_2^{-1} x_{21}^+ = x_{12}^- x_2^{-1} x_{21}^-,\ x_{01}(x_{21}^\pm)^{-1} x_{20} = x_{02}(x_{12}^\mp)^{-1} x_{10} \ ,$$
$$x_1 = x_{10} x_{20}^{-1}(x_{21}^+ + x_{21}^-),\ x_2 = x_{20} x_{10}^{-1}(x_{12}^+ + x_{12}^-) \ .$$

In particular,

$$x_2 = x_{21}^- x_1^{-1} x_{12}^- + x_{21}^+ x_1^{-1} x_{12}^+ + x_{21}^- x_1^{-1} x_{12}^+ + x_{21}^+ x_1^{-1} x_{12}^-,$$
$$x_1 = x_{12}^- x_2^{-1} x_{21}^- + x_{12}^+ x_2^{-1} x_{21}^+ + x_{12}^- x_2^{-1} x_{21}^+ + x_{12}^+ x_2^{-1} x_{21}^- \ .$$

Therefore, there is no homomorphism $\mathcal{A}_\Sigma \to \mathcal{A}_{\hat{\Sigma}}$ or $\mathcal{A}_{\hat{\Sigma}} \to \mathcal{A}_\Sigma$ which would send $x_i \mapsto x_i$, $x_{ij}^\pm \mapsto x_{ij}^\pm$ (which justifies Remark 3.19).

3.4. Triangulations of marked surfaces

Let Σ be a marked surface, given distinct $\gamma, \gamma' \in [\Gamma(\Sigma)]$, define their *intersection number* $n_{\gamma,\gamma'} \in \mathbb{Z}_{\geq 0}$ to be the number of intersection points in the interiors of their generic representatives minus the endpoints of γ and γ'. Clearly, $n_{\gamma,\gamma'}$ is well-defined, i.e., does not depend on the choice of representatives. By definition, $n_{\gamma,\gamma'} = n_{\gamma',\gamma} = n_{\overline{\gamma},\gamma'}$

for all γ, γ'. Note that $n_{\gamma,\gamma'} = 0$ iff γ and γ' do not intersect (and may have only endpoints in common).

Given a marked surface Σ, we say that a subset $\Gamma' \subset \Gamma^0(\Sigma)$ is *non-crossing* if $n_{\gamma,\gamma'} = 0$ for all distinct $\gamma, \gamma' \in \Gamma'$, i.e., one can simultaneously choose generic representatives of classes in Γ' such that they pairwise do not intersect in Σ and do not self-intersect (i.e., may have only endpoints in common). Furthermore, we say that Δ is a *triangulation* of Σ if Δ is a maximal non-crossing subset of $\Gamma^0(\Sigma)$ such that $\overline{\Delta} = \Delta$.

Clearly, if $I_s(\Sigma) \neq \emptyset$, then any triangulation Δ of Σ has a special loop λ_{ij} at some $j \in I_s(\Sigma)$ around each $i \in I_s(\Sigma)$, i.e., λ_{ij} defines a 2-gon $(\lambda_{ij}, \lambda_{ij})$ in Δ homeomorphic to $P_1(1)$. It is customary to fix a generic representative of each $\gamma^0 \in \Delta$ so that Σ is literally cut into triangles and $P_1(1)$'s.

It is well-known that all triangulations of Σ are finite of same cardinality. Moreover, any triangulation Δ' can be obtained from a given triangulation Δ by a sequence of flips of diagonals in quadrilaterals in Δ (see e.g., [12, Proposition 7.10] and [10, Theorem 4.2]).

Given an r-gon $Q = (\gamma_1, \ldots, \gamma_r)$ in Σ and a triangulation Δ of Σ, we say that $\gamma^0 \in \Delta$ is *attracted* to Q if either γ^0 intersects Q or there is a triangle $\tau = (\gamma^-, \gamma^0, \gamma^+)$ in Δ such that γ^- intersects Q; denote by $\Delta_0 = \Delta_0(Q, \Delta)$ the set of all $\gamma^0 \in \Delta$ attracted to Q.

The following is immediate.

Theorem 3.21. *Let Δ be a triangulation of Σ. Then for each r-gon $Q = (\gamma'_1, \ldots, \gamma'_r)$ in Σ there exists an n-gon $P = (\gamma_1, \ldots, \gamma_n) \in (\Delta_0(Q, \Delta))^n$ for some $n \geq r$, a triangulation Δ^0 of $[n]$, and an order-preserving embedding $\iota : [r] \hookrightarrow [n]$ such that:*

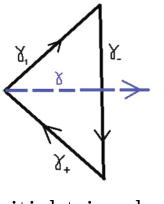
(a) $\gamma_{ij} \in \Delta_0(Q, \Delta)$ iff $(i, j) \in \Delta^0$.

(b) $\gamma'_k = \gamma_{\iota(k),\iota(k^+)}$ for all $k \in [r]$ (i.e., Q is a "sub-polygon" of P).

In fact, if $Q = (\gamma, \overline{\gamma})$, $\gamma \in [\Gamma(\Sigma)]$, we will construct a *canonical* polygon $P_\Delta(\gamma)$ as follows.

We need the following obvious fact.

Lemma 3.22. *Let Δ be a triangulation of Σ and let $\gamma \in [\Gamma(\Sigma)] \setminus \Delta$. Then there exists a unique (up to relabeling) triangle $\tau_1 = (\gamma_1, \gamma_-, \gamma_+) \in \Delta^3$ such that $n_{\gamma,\gamma_-} > 0$ and the closest to $s(\gamma)$ intersection point of γ with Δ is the intersection point of γ and γ_-.*

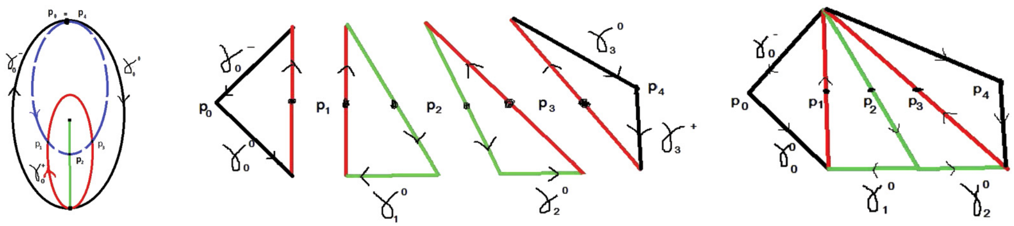


The initial triangle for γ

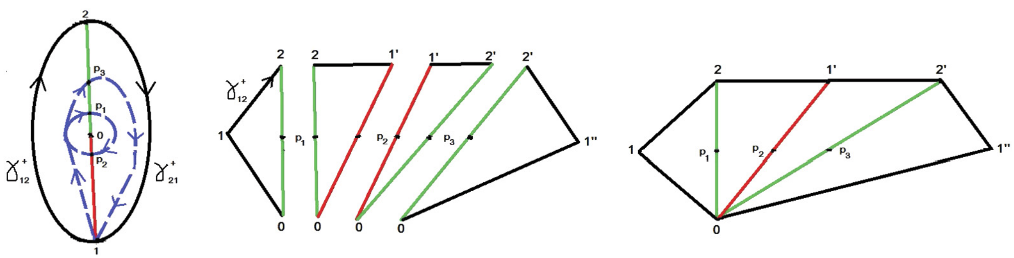
We refer to such a triangle as *initial* for γ. Fix the initial triangle τ as in Lemma 3.22 and denote by $\gamma^{(1)}$ the unique (class of) curve which starts as γ_-, follows this "route" until the first intersection point of γ_- and γ and then "becomes" γ. Repeating this process, we obtain a new initial triangle $\tau_s = (\gamma_s, \gamma_-^{(s)}, \gamma_+^{(s)})$ for $\gamma^{(s)}$, $s = 1, \ldots, j-1$, where $j \geq 2$ is unique with $\gamma^{(j)} = \gamma_j \in \Delta$. This process converges by induction in $n_{\gamma,\Delta} := \sum\limits_{\gamma_0 \in \Delta} n_{\gamma,\gamma_0}$ because $n_{\gamma,\Delta} > n_{\gamma^{(1)},\Delta} > \cdots > n_{\gamma^{(j)},\Delta} = 0$. Denote $F_\Delta(\gamma) := (\gamma_1, \ldots, \gamma_j) \in \Delta^j$ and refer to this sequence as a Δ-*factorization* of γ. By definition, $\gamma \in \gamma_1 \circ \cdots \circ \gamma_j$ in the multi-groupoid $[\Gamma(\Sigma)]$, which justifies the terminology.

Finally, we set $P_\Delta(\gamma) := (F_\Delta(\gamma), F_\Delta(\overline{\gamma}))$ and refer to it as the *canonical polygon* of γ in Δ due to the following obvious result.

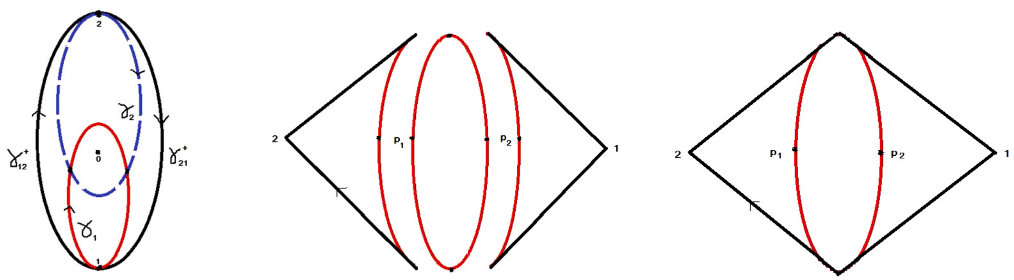
Lemma 3.23. *Each $P_\Delta(\gamma) = (\gamma_1, \ldots, \gamma_n)$ is an n-gon in Δ.*



Canonical polygon, no special punctures



Canonical polygon, no special punctures, γ self-intersects



Canonical polygon, one special puncture

3.5. Triangle groups and their topological invariance

For each triangulation Δ of Σ we define the *triangle group* $\mathbb{T}_\Delta = \mathbb{T}_\Delta(\Sigma)$ to be generated by all $t_\gamma^{\pm 1}$, $\gamma \in \Delta$ subject to (same relations as in $\mathbb{T}_\Sigma$):

- $t_\gamma = 1$ if γ is trivial.
- $t_{\gamma_1} t_{\overline{\gamma}_2}^{-1} t_{\gamma_3} = t_{\overline{\gamma}_3} t_{\gamma_2}^{-1} t_{\overline{\gamma}_1}$ for any triangle $T = (\gamma_1, \gamma_2, \gamma_3)$ in Δ.

Also, for each triangulation Δ of Σ denote by $\mathbb{Y}_\Delta$ the subgroup of $\mathbb{T}_\Delta$ generated by:

$$y_{\gamma,\gamma'} := t_{\overline{\gamma}}^{-1} t_{\gamma'}$$

for all $\gamma, \gamma' \in \Delta$ such that $(\gamma, \gamma', \gamma'')$ is a triangle in Δ for some $\gamma'' \in \Delta$.

Theorem 3.24. *For any two triangulations Δ and Δ' of a marked surface Σ there exists a group isomorphism:*

$$f_{\Delta,\Delta'} : \mathbb{T}_\Delta \cong \mathbb{T}_{\Delta'}$$

such that $f_{\Delta,\Delta'}(\mathbb{Y}_\Delta) = \mathbb{Y}_{\Delta'}$.

We prove Theorem 3.24 in Section 3.11.

Remark 3.25. Theorem 3.24 implies that isomorphism classes of groups $\mathbb{T}_\Delta$ and $\mathbb{Y}_\Sigma$ are topological invariants of surfaces. However, by contrast with Theorem 3.16, we do not expect the assignments $\Sigma \mapsto \mathbb{T}_\Delta$ to be functorial.

Our next result is classification of triangle groups of marked surfaces.

Theorem 3.26. *Let Σ be a marked surface with the Euler characteristic $\chi(\Sigma)$, the set $I = I(\Sigma) \neq \emptyset$ of marked points, the set $I_b \subseteq I$ of marked boundary points, and $h = |I_s|$ special punctures. Then for any triangulation Δ of Σ one has:*

(a) If Σ has a boundary or special punctures, then $\mathbb{T}_\Delta$ is a free group in:

- $|I| + 1$ *generators if Σ is a disk with $|I| + |I_b| = 2$, $h = 0$.*
- $2h + 3|I| - 4$ *generators if Σ is a disk with $|I| + |I_b| = 2$, $h > 0$.*
- $2h + 4(|I| - \chi(\Sigma)) - |I_b|$ *generators otherwise.*

(b) If Σ is a closed surface without special punctures, then $\mathbb{T}_\Delta$ is:

- *Trivial if Σ is the sphere with $|I| = 1$.*
- *A free group in $3|I| - 4$ generators if Σ is the sphere with $|I| \in \{2, 3\}$.*
- *A free group in 2 generators if Σ is the real projective plane with $|I| = 1$.*
- *A 1-relator torsion free group (in the sense of Definition A.6) in $4(|I| - \chi(\Sigma)) + 1$ generators otherwise.*

We prove Theorem 3.26 in Section 3.12 by choosing an appropriate triangulation of Σ.

Remark 3.27. If Σ has r boundary components, then it is homotopy equivalent to a bunch of $g \geq r$ circles and $\chi(\Sigma) = 1 - g$. If Σ is a closed orientable (resp. non-orientable) surface, then it is homeomorphic the connected sum of g copies of the torus (resp. of the real projective plane) and $\chi(\Sigma) = 2 - 2g$ (resp. $\chi(\Sigma) = 2 - g$).

Example 3.28 *(See Example 3.51 below for details).* If Δ is a triangulation of the torus, the Klein bottle, the real projective plane respectively with one, one, two (ordinary) punctures, then $\mathbb{T}_\Delta$ is generated by variables a, b, c, d, e subject to, respectively the following relations:

(i) for the torus with one puncture: $abcde = cbeda$;

(ii) for the Klein bottle with one puncture: $abcdc = ebeda$;

(iii) for the real projective plane with two punctures: $abcbc = ededa$.

Example 3.29. If Δ is a triangulation of the sphere with $m+1$ punctures, we can view it as glued from a regular $2m$-gon with $S = [m] \subset [2m]$, $\sigma(k) = m+1-k$, $\varepsilon(k) = +$ for $k \in [m]$ (with some notation from the proof of Theorem 3.26 in Section 3.12). Then $\mathbb{T}_\Delta$ is generated by $c_1, \ldots, c_{2m}$, $t_3, \ldots, t_{2m-1}$ subject to the relation $t_3 c_3 t_4 \cdots c_{2m-2} t_{2m-1} = t_{2m-1} c_3 t_{2m-2} \cdots c_{2m-2} t_3$.

3.6. Noncommutative Laurent Phenomenon for surfaces

The following result extends Noncommutative Laurent Phenomenon for n-gons (Theorem 2.10) to all marked surfaces.

Theorem 3.30 *(Noncommutative Laurent Phenomenon for surfaces). Let Σ be a marked surface and let Δ be a triangulation of Σ. Then for each $\gamma \in [\Gamma(\Sigma)]$ the element x_γ of $\mathcal{A}_\Sigma$ belongs to the subalgebra of $\mathcal{A}_\Sigma$ generated by $x_{\gamma_0}^{\pm 1}$, $\gamma_0 \in \Delta$. More precisely, in the notation of Theorem 2.10, one has*

$$x_\gamma = \sum_{\mathbf{i} \in Adm_{\Delta^0}(1,j)} x_{\mathbf{i}} \ , \tag{3.6}$$

where Δ^0 is the triangulation of $[n]$ assigned (as in Theorem 3.21(a)) to the canonical polygon $P_\Delta(\gamma) = (\gamma_1, \ldots, \gamma_n)$ in Δ with $\gamma = \gamma_{1,j}$, and we abbreviated

$$x_{\mathbf{i}} := x_{\gamma_{i_1,i_2}} x^{-1}_{\gamma_{i_3,i_2}} x_{\gamma_{i_3,i_4}} \cdots x^{-1}_{\gamma_{i_{2m-1},i_{2m-2}}} x_{\gamma_{i_{2m-1},i_{2m}}}$$

for any sequence $\mathbf{i} = (i_1, \ldots, i_{2m}) \in [n]^{2m}$, $m \geq 1$.

We prove Theorem 3.30 in Section 3.11.

Remark 3.31. Theorem 3.30 is a noncommutative generalization of [23, Theorem 6.1].

Example 3.32. Let Σ be a regular triangle with the clockwise vertex set $I = \{1, 2, 3\}$ and a special puncture 0 in the center. For $i \in I$ denote by λ_i the special loop at i around 0. As in Example 3.7, for $i, j \in I$, $i \neq j$ denote by γ_{ij}^+ (resp. γ_{ij}^-) the curve from i to j so that 0 is to the right (resp. to the left) of the curve and abbreviate $x_i := x_{\lambda_i}$, $x_{ij}^{\pm} := x_{\gamma_{ij}^{\pm}}$ for the corresponding generators of $\mathcal{A}_\Sigma$.

Clearly, every triangulation of Σ contains $\gamma_{12}^+, \gamma_{21}^-, \gamma_{23}^+, \gamma_{32}^-, \gamma_{31}^+, \gamma_{13}^-$. Let Δ be the triangulation of Σ containing also γ_1 and γ_{12}^-. Then (3.6) reads:

$$x_2 = x_{21}^+ x_1^{-1} x_{12}^- + x_{21}^- x_1^{-1} x_{12}^+, \;\; x_{23}^- = x_{21}^-(x_{21}^+)^{-1} x_{23}^+ + x_{21}^+ x_1^{-1} x_{13}^- + x_{21}^- x_1^{-1} x_{12}^+ (x_{12}^-)^{-1} x_{13}^- ,$$

$$x_3 = x_{31}^+ x_1^{-1} x_{13}^- + x_{31}^+ (x_{21}^+)^{-1} x_{21}^- (x_{21}^+)^{-1} x_{23}^+ + x_{32}^- (x_{12}^-)^{-1} x_1 (x_{21}^+)^{-1} x_{23}^+$$
$$+ x_{32}^- (x_{12}^-)^{-1} x_{12}^+ (x_{12}^-)^{-1} x_{13}^- + x_{31}^+ (x_{21}^+)^{-1} x_{21}^- x_1^{-1} x_{12}^+ (x_{12}^-)^{-1} x_{13}^- .$$

Let $\hat{\Sigma}$ be as in Definition 3.17. Therefore, simple curves on $\hat{\Sigma}$ are those on Σ plus six additional ones: directed intervals $\gamma_{0,i}$ from 0 to each i and $\gamma_{i,0} := \overline{\gamma}_{0,i}$. We abbreviate the generators of $\mathcal{A}_{\hat{\Sigma}}$ same way as in $\mathcal{A}_\Sigma$ and $x_{0,i} := x_{\gamma_{0,i}}$, $x_{i,0} := x_{\gamma_{i,0}}$.

Let $\hat{\Delta}$ be the triangulation of $\hat{\Sigma}$ obtained from Δ by adding the intervals $\gamma_{0,1}$ and $\gamma_{1,0}$. Then (3.6) reads:

$$x_3 = x_{31}^+ x_1^{-1} x_{13}^- + x_{31}^+ (x_{21}^+)^{-1} x_{21}^- (x_{21}^+)^{-1} x_{23}^+ + x_{32}^- (x_{12}^-)^{-1} x_1 (x_{21}^+)^{-1} x_{23}^+$$
$$+ x_{32}^- (x_{12}^-)^{-1} x_{12}^+ (x_{12}^-)^{-1} x_{13}^- + x_{31}^+ (x_{21}^+)^{-1} x_{21}^- x_1^{-1} x_{12}^+ (x_{12}^-)^{-1} x_{13}^- +$$
$$+ x_{31}^+ (x_{21}^+)^{-1} x_{23}^+ + x_{32}^- (x_{12}^-)^{-1} x_{13}^- + x_{31}^+ x_1^{-1} x_{12}^+ (x_{12}^-)^{-1} x_{13}^- + x_{31}^+ (x_{21}^+)^{-1} x_{21}^- x_1^{-1} x_{13}^- .$$

3.7. Noncommutative $(n,1)$-gon

In this section we consider the $(n,1)$-gon $\Sigma = P_n(1)$ (with the clockwise ordering of the set $[n] = I_b(P(n,1)))$. We abbreviate $\mathcal{A}_{n,1} := \mathcal{A}_\Sigma$ and refer to it as the *noncommutative $(n,1)$-gon*. Clearly, $\mathcal{A}_{n,1}$ is generated by $x_{ij}^\pm := x_{\gamma_{ij}^\pm}$ and $(x_{ij}^\pm)^{-1}$, $i,j \in [n]$, where $\gamma_{ij}^\pm$ is the curve corresponding to $(i,j,\pm)$ under the bijection in Lemma 3.10 where $x_{ii}^+ = x_{ii}^-$ for $i \in [n]$ (we abbreviate $x_i := x_{ii}^+ = x_{ii}^-$). The following is immediate.

Lemma 3.33. *The algebra $\mathcal{A}_{n,1}$ is generated by $(x_{ij}^\pm)^{\pm 1}$, $i,j \in [n]$ subject to:*

(i) (triangle relations) For any distinct $i,j,k \in [n]$:

$$x_{ij}^+ (x_{kj}^+)^{-1} x_{ki}^+ = x_{ik}^- (x_{jk}^-)^{-1} x_{ji}^-, x_{ij}^+ (x_{kj}^-)^{-1} x_{ki}^+ = x_{ik}^- (x_{jk}^+)^{-1} x_{ji}^- .$$

(ii) (2-gon exchange relations) For any distinct $i,j \in [n]$:

$$x_j = x_{ji}^+ x_i^{-1} x_{ij}^- + x_{ji}^- x_i^{-1} x_{ij}^+ .$$

(iii) (4-gon exchange relations) For any cyclic (i,j,k,ℓ) in $[n]$ and $\varepsilon \in \{-,+\}$:

$$x_{j\ell}^+ = x_{jk}^+ (x_{ik}^\varepsilon)^{-1} x_{i\ell}^\varepsilon + x_{ji}^{-\varepsilon} (x_{ki}^{-\varepsilon})^{-1} x_{k\ell}^+ \;, x_{i\ell}^+ = x_{ik}^\varepsilon (x_{jk}^{-\varepsilon})^{-1} x_{j\ell}^{-\varepsilon} + x_{ij}^{-\varepsilon} (x_{ki}^\varepsilon)^{-1} x_{k\ell}^\varepsilon$$
$$x_{j\ell}^- = x_{jk}^\varepsilon (x_{ik}^\varepsilon)^{-1} x_{i\ell}^- + x_{ji}^- (x_{kj}^{-\varepsilon})^{-1} x_{k\ell}^{-\varepsilon} \;, x_{\ell i}^- = x_{\ell j}^\varepsilon (x_{kj}^\varepsilon)^{-1} x_{ki}^{-\varepsilon} + x_{\ell k}^{-\varepsilon} (x_{jk}^{-\varepsilon})^{-1} x_{ji}^\varepsilon$$

Clearly, the assignments $x_{ij}^\pm \mapsto x_{ji}^\mp$ define an involutive anti-automorphism of $\mathcal{A}_{n,1}$. One can easily show

For each $n \geq 1$ define a map $\pi : [2n] \to [n]$ by $\pi(i) = \begin{cases} i & \text{if } i \in [n] \\ i-n & \text{if } i \notin [n] \end{cases}$.

Also for distinct $i, j \in [2n]$ define the sign $\varepsilon_{ij} \in \{-,+\}$ by setting $\varepsilon_{ij} := +$ if the clockwise arc from i to j is shorter than the clockwise arc from i to $i+n$ and $\varepsilon_{ij} := -$ otherwise.

Note that the restriction of the function $\hat{f} : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$ to the unit disk $D \subset \mathbb{C}$ centered at 0 is a map $f : D \to D$ hence for each $n \geq 1$ it is a morphism $f_n : P_{2n} \to P_n(1)$ in **Surf** for all $n \geq 1$ (where the marked boundary points are appropriate roots of unity and the special puncture in $P_n(1)$ is the center 0 of D). The following is immediate corollary of Theorems 3.9 and 3.16.

Corollary 3.34. *For each $n \geq 1$ one has:*

- *The morphism f_n in **Surf** defines a surjective map $\Gamma(P_{2n}) = [2n] \times [2n] \twoheadrightarrow [n] \times [n] \times \{-,+\} = [\Gamma(P(n,1))]$ given by $(ij) \mapsto \gamma_{\pi(i),\pi(j)}^{\varepsilon_{ij}}$ for all distinct $i, j \in [2n]$.*
- *The assignments $x_{ij} \mapsto x_{\pi(i),\pi(j)}^{\varepsilon_{ij}}$ for all distinct $i, j \in [2n]$, define an epimorphism of algebras $(f_n)_* : \mathcal{A}_{2n} \twoheadrightarrow \mathcal{A}_{n,1}$.*

Remark 3.35. For any $1 \leq i < j < k \leq n$, the triple $(\gamma_{ij}^-, \gamma_{jk}^-, \gamma_{ki}^-)$ is a triangle in $\Sigma = P(n,1)$ because it is the image of the triangle $(i, j+n, k)$ in $[2n]$ under the above morphism $f_n : P_{2n} \to P_n(1)$. Note, however, that all intersections $\gamma_{ij}^- \cap \gamma_{jk}^-$, $\gamma_{ij}^- \cap \gamma_{ki}^-$, $\gamma_{jk}^- \cap \gamma_{ki}^-$ are non-empty.

3.8. Universal localizations of noncommutative surfaces

Generalizing (2.4), for any triangulation Δ of any marked surface Σ let $\mathcal{A}_\Delta$ be the subalgebra of $\mathcal{A}_\Sigma$ generated by all x_γ, $\gamma \in [\Gamma(\Sigma)]$ and all $x_{\gamma_0}^{-1}$, $\gamma_0 \in \Delta$.

Clearly, the assignments $t_\gamma \mapsto x_\gamma$, $\gamma \in \Delta$ define a homomorphisms of algebras:

$$\mathbf{i}_\Delta : \mathbb{QT}_\Delta \to \mathcal{A}_\Delta \ . \tag{3.7}$$

The following result is a generalization of Theorem 2.8 to all marked surfaces.

Theorem 3.36. *For each triangulation Δ of Σ one has:*

(a) The homomorphism $\mathbf{i}_\Delta$ given by (3.7) is an isomorphism of algebras.

(b) $\mathcal{A}_\Sigma = \mathcal{A}_\Delta[\mathbf{S}^{-1}]$, where $\mathbf{S}$ is the submonoid of $\mathcal{A}_\Delta \setminus \{0\}$ generated by all x_γ, $\gamma \in [\Gamma(\Sigma)]$.

We prove Theorem 3.36 in Section 3.13.

Theorems 3.26, 3.36, and A.7 imply the following.

Corollary 3.37. *For each triangulation Δ of Σ the homomorphism (3.7) is injective.*

Theorem 3.36 implies that for each Σ the natural homomorphism $\mathbb{Q}\mathbb{T}_\Delta \hookrightarrow Frac(\mathbb{Q}\mathbb{T}_\Delta)$ defines a homomorphism of algebras:

$$\mathcal{A}_\Sigma \to Frac(\mathbb{Q}\mathbb{T}_\Delta) \ . \tag{3.8}$$

In view of Theorem A.7, we propose the following conjecture.

Conjecture 3.38. *For each Σ the homomorphism (3.8) is injective, e.g., the submonoid S_Δ of $\mathbb{Q}\mathbb{T}_\Delta \setminus \{0\}$ is divisible in the sense of Definition A.4.*

Remark 3.39. Conjecture 3.38 generalizes the expected injectivity of (2.3). To prove Conjecture 3.38 for non-closed surfaces (i.e., with free $\mathbb{T}_\Delta$ according to Theorem 3.26) it would suffice to show that the monoid S_Δ is generated by $\mathbb{Q}^\times \cdot \mathbb{T}_\Delta$ and a subset of prime elements in $\mathbb{Q}\mathbb{T}_\Delta$.

3.9. Noncommutative angles and regular elements in noncommutative surfaces

Similarly to Section 2.3, for each triangle $(\gamma_1, \gamma_2, \gamma_3)$ denote by $T_{\gamma_1,\gamma_2,\gamma_3}$ the element of $\mathcal{A}_\Sigma$ given by:

$$T_{\gamma_1,\gamma_2,\gamma_3} = x_{\overline{\gamma}_1}^{-1} x_{\gamma_2} x_{\overline{\gamma}_3}^{-1} \tag{3.9}$$

and refer to it as a *noncommutative angle* of $(\gamma_1, \gamma_2, \gamma_3)$ at $s(\gamma_1) = t(\gamma_3)$.

Given a triangulation Δ of Σ, for any $i \in I$ define the *total angle* T_i^Δ at $i \in I$ by:

$$T_i^\Delta := \sum T_{\gamma_1,\gamma_2,\gamma_3} \ , \tag{3.10}$$

where the summation is over all clockwise triangles $(\gamma_1, \gamma_2, \gamma_3)$ in Δ such that $s(\gamma_1) = i$.

Theorem 3.40. *For any triangulations Δ, Δ' of Σ and $i \in I$ one has:*

$$T_i^\Delta = T_i^{\Delta'} \ .$$

Therefore, in what follows, we simply denote $T_i := T_i^\Delta$ for any triangulation Δ of Σ.

Furthermore, denote by $\mathcal{U}_\Sigma$ the subalgebra of $\mathcal{A}_\Sigma$ generated by all x_γ, $\gamma \in [\Gamma(\Sigma)]$, $x_{\gamma_0}^{-1}$, $\gamma_0 \in \partial\Gamma(\Sigma)$ and all total angles T_i.

In particular, the algebra $\mathcal{U}_n$ from 2.3 is naturally isomorphic to $\mathcal{U}_{P_n}$. The following is an analogue of Lemma 2.18.

Lemma 3.41. *The algebra $\mathcal{U}_\Sigma$ satisfies the following relations:*

(a) (reduced triangle relations) for all triangles $(\gamma_1, \gamma_2, \gamma_3)$ in $[\Gamma(\Sigma)]$ such that γ_2 is a boundary curve:

$$x_{\gamma_1} x_{\overline{\gamma}_2}^{-1} x_{\gamma_3} = x_{\overline{\gamma}_3} x_{\gamma_2}^{-1} x_{\overline{\gamma}_1} \ . \tag{3.11}$$

(b) (reduced exchange relations) for all quadrilaterals $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ *in* Σ *such that* γ_2, γ_3 *are boundary curves:*

$$x_{\gamma_{13}} x_{23}^{-1} x_{\gamma_{24}} = x_{\gamma_{14}} + x_{\gamma_{12}} x_{\gamma_{32}}^{-1} x_{\gamma_{34}} \ . \tag{3.12}$$

Remark 3.42. It is natural to conjecture that the relations (3.11) and (3.12) are defining for $\mathcal{U}_\Sigma$.

Noncommutative Laurent phenomenon (3.6) guarantees that $\mathcal{U}_\Sigma$ belongs to each subalgebra $\mathcal{A}_\Delta \subset \mathcal{A}_\Sigma$.

The following is an analogue of Conjecture 2.20.

Conjecture 3.43. *For each* $n \geq 2$ *one has:*

$$\mathcal{U}_\Sigma = \bigcap_\Delta \mathcal{A}_\Delta \ , \tag{3.13}$$

where the intersection is over all triangulations Δ *of* Σ.

We say that an element of $\mathcal{A}_\Sigma$ is *regular* if it belongs to each subalgebra $\mathcal{A}_\Delta$ as Δ runs over all triangulations of Σ. Thus, similarly to Section 2.3, Conjecture 3.43 asserts that regular elements of $\mathcal{A}_\Sigma$ belong to $\mathcal{U}_\Sigma$.

3.10. Noncommutative cohomology of surfaces

Given a surface Σ, for each triangle $(\gamma_1, \gamma_2, \gamma_3)$ in Σ we define the element $\tau_{\gamma_1,\gamma_2,\gamma_3} \in \mathcal{A}_\Sigma$ (in notation (3.9)) by:

$$\tau_{\gamma_1,\gamma_2,\gamma_3} = T_{\gamma_1,\gamma_2,\gamma_3} + T_{\gamma_2,\gamma_3,\gamma_1} + T_{\gamma_3,\gamma_1,\gamma_2} \ .$$

That is, $\tau_{\gamma_1,\gamma_2,\gamma_3}$ is the sum of all noncommutative angles of the triangle $(\gamma_1, \gamma_2, \gamma_3)$.

Then define the algebra $\mathcal{H}(\Sigma)$ to be the quotient of $\mathcal{A}_\Sigma$ by the ideal generated by all $\tau_{(\gamma_1,\gamma_2,\gamma_3)} - \tau_{(\gamma_1',\gamma_2',\gamma_3')}$ as $(\gamma_1, \gamma_2, \gamma_3)$ and $(\gamma_1', \gamma_2', \gamma_3')$ run independently over all triangles of Σ. We refer to $\mathcal{H}(\Sigma)$ as the *noncommutative cohomology* of Σ.

This notation is justified by the following construction.

Fix a triangulation Δ of Σ. For each loop θ in Σ which does not pass through marked points, define the element $[\theta]'_\Delta \in \mathcal{A}_\Delta$ by:

$$[\theta]'_\Delta = \sum \varepsilon_{\gamma_1,\gamma_2,\gamma_3}(\theta) \cdot T_{\gamma_1,\gamma_2,\gamma_3} \ ,$$

the summation is over all clockwise triangles $(\gamma_1, \gamma_2, \gamma_3)$ in Δ such that θ intersects γ_1 and γ_2 (but not γ_3) and $\varepsilon_{\gamma_1,\gamma_2,\gamma_3}(\theta) := \begin{cases} 1 & \text{if } \gamma_3 \text{ is to the right of } \theta \\ -1 & \text{if } \gamma_3 \text{ is to the left of } \theta \end{cases}$.

Note that if $\theta = \theta_i$ is a (small) clockwise loop around a puncture $i \in I$, then $[\theta]'_\Delta = T_i^\Delta$, the total angle at i (defined in (3.10)).

Furthermore, define $[\theta]_\Delta \in \mathcal{H}(\Sigma)$ by

$$[\theta]_\Delta := \pi(\mathbf{i}_\Delta([\theta]'_\Delta)) ,$$

where $\mathbf{i}_\Delta$ is the homomorphism $\mathbb{Q}\mathbb{T}_\Delta \to \mathcal{A}_\Sigma$ given by (3.7) and $\pi : \mathcal{A}_\Sigma \to \mathcal{H}(\Sigma)$ is the canonical epimorphism.

The following immediate result is an analogue of Theorem 3.40.

Theorem 3.44. *Given a loop on Σ not passing through marked points, then for any triangulations Δ and Δ' of Σ one has:*

$$[\theta]_{\Delta'} = [\theta]_\Delta .$$

This allows us to define a noncommutative loop $[\theta] \in \mathcal{H}(\Sigma)$ by $[\theta] := [\theta]_\Delta$ for any triangulation Δ of Σ.

3.11. Proof of Theorems 3.6, 3.9, 3.16, 3.24, and 3.30

Proof of Theorem 3.6. Clearly, the composition $f' \circ f : \underline{\Sigma} \to \underline{\Sigma}''$ is a continuous map with finite fibers. Also,

$$(f' \circ f)^{-1}(I(\Sigma'')) = f^{-1}(f'^{-1}(I(\Sigma''))) = f^{-1}(I(\Sigma'')) = I(\Sigma) ,$$
$$(f' \circ f)(I_s(\Sigma)) = f'(f(I_s(\Sigma)) \subset f'(I_s(\Sigma')) \subset I_s(\Sigma'') .$$

This verifies the first requirement of Definition 3.5 for $f' \circ f$.

Furthermore, prove that $I^{f'\circ f} = I^f \sqcup f^{-1}(I^{f'})$. Indeed,

$$\begin{aligned} I^{f'\circ f} &= (f' \circ f)^{-1}(I_s(\Sigma'')) \setminus I_s(\Sigma) = f^{-1}(f'^{-1}(I_s(\Sigma'')) \setminus I_s(\Sigma) \\ &= f^{-1}(I_s(\Sigma') \sqcup I^{f'}) \setminus I_s(\Sigma) = (f^{-1}(I_s(\Sigma')) \sqcup f^{-1}(I^{f'})) \setminus I_s(\Sigma) = I^f \sqcup f^{-1}(I^{f'}) \end{aligned}$$

since $f'^{-1}(I_s(\Sigma'')) = I_s(\Sigma') \sqcup I^{f'}$, $f^{-1}(I_s(\Sigma')) = f^{-1}(I_s(\Sigma')) \setminus I^f$, $f^{-1}(I^{f'}) \cap I_s(\Sigma) = \emptyset$, and $f^{-1}(A \sqcup B) = f^{-1}(A) \sqcup f^{-1}(B)$ for any disjoint subsets A and B of $\underline{\Sigma}'$.

Let now $p \in \underline{\Sigma} \setminus I^{f'\circ f}$. By above, this is equivalent to that $p \in \underline{\Sigma} \setminus I^f$ and $f(p) \in \underline{\Sigma}' \setminus I^{f'}$. Hence there is a neighborhood $\mathcal{O}_p$ of p in $\underline{\Sigma}$ ($\mathcal{O}_p$ is a half-neighborhood if $p \in \partial\underline{\Sigma}$) such that the restriction of f to $\mathcal{O}_p$ is injective and a (half-)neighborhood $\mathcal{O}_{f(p)}$ of $f(p)$ in $\underline{\Sigma}'$ such that the restriction of f' to $\mathcal{O}_{f(p)}$ is injective. In particular, $\mathcal{O}'_p := f^{-1}(\mathcal{O}_{f(p)})$ is a neighborhood of p in $\underline{\Sigma}$ and the restriction of $f' \circ f$ to $\mathcal{O}'_p$ is injective. This verifies the second requirement of Definition 3.5 for $f' \circ f$.

Let now $p \in I^{f'\circ f}$. By above, this is equivalent to that either $p \in I^f$ or $f(p) \in I^{f'}$.

In the first case, clearly, $f(p) \in \underline{\Sigma}' \setminus I^{f'}$, therefore there is a neighborhood $\mathcal{O}_{f(p)}$ of $f(p)$ in $\underline{\Sigma}'$ such that the restriction of f' to $\mathcal{O}_{f(p)}$ is injective and a neighborhood $\mathcal{U}_p$ of p in $\underline{\Sigma}$ such that the restriction of f to $\mathcal{U}_p$ is a two-fold cover of the neighborhood $\mathcal{O}'_p = f(\mathcal{U}_p)$ ramified at $f(p)$. Therefore, the restriction of f to the neighborhood $\mathcal{U}'_p = f^{-1}(\mathcal{O}_p \cap \mathcal{O}'_p)$ is a two-fold cover of $\mathcal{O}_p \cap \mathcal{O}'_p$ ramified at $f(p)$ and the restriction of f' to $\mathcal{O}_p \cap \mathcal{O}'_p$ is injective. Thus, the restriction of $f' \circ f$ to $\mathcal{U}'_p$ is a two-fold cover of $f(\mathcal{O}_p \cap \mathcal{O}'_p)$ ramified at $(f' \circ f)(p)$.

In the second case, clearly, $p \in \underline{\Sigma} \setminus I^f$, therefore there is a neighborhood $\mathcal{O}_p$ of p in $\underline{\Sigma}$ such that the restriction of f to $\mathcal{O}_p$ is injective and a neighborhood $\mathcal{U}_{f(p)}$ of $f(p)$ in $\underline{\Sigma}'$ such that the restriction of f' to $\mathcal{U}_{f(p)}$ is a two-fold cover of the neighborhood $\mathcal{O}_{f'(f(p))} = f(\mathcal{U}_p)$ ramified at $f(f'(p))$. Therefore, the restriction of f' to the neighborhood $\mathcal{U}'_{f(p)} = f(\mathcal{O}_p) \cap \mathcal{U}_{f(p)}$ is a two-fold cover of $f'(\mathcal{U}'_{f(p)})$ ramified at $f'(f(p))$ and the restriction of f to $\mathcal{O}'_p = f^{-1}(\mathcal{U}'_{f(p)})$ is injective. Thus, the restriction of $f' \circ f$ to $\mathcal{O}'_p$ is a two-fold cover of $f'(\mathcal{U}'_{f(p)})$ ramified at $(f' \circ f)(p)$.

This verifies the last requirement of Definition 3.5 for $f' \circ f$.

The theorem is proved. □

Proof of Theorem 3.9. Without loss of generality, it suffices to prove the first assertion in the case when $C \subset C'$ and $C' \setminus C$ is a single loop around $i \in I^f$ not enclosing any points $I(\Sigma) \cup I_s(\Sigma) \cup I^f \setminus \{i\}$ (where we regard C and C' as subsets of $\underline{\Sigma}$). Moreover, it suffices to take $C = \{p\}$ for some $p \in \underline{\Sigma}$, $p \neq i$, so that C' is a simple loop at p around i (e.g., C' is contractible to p in $\underline{\Sigma} \setminus I(\Sigma)$).

By definition, there is a neighborhood $\mathcal{U}_p$ of p such that the restriction of f to $\mathcal{U}_p$ is a two-fold cover of $f(\mathcal{U}_p)$ ramified at $f(p)$. Once again, without loss of generality, we may assume that C' intersects $\mathcal{U}_p$ and there exist exactly two distinct points $p', p'' \in C$ such that $f(p') = f(p'')$. This implies that $f(C') \subset \underline{\Sigma}'$ is a (self-intersecting) loop at $f(p)$ with a single self-intersection point $f(p') = f(p'')$. If we denote by γ' the equivalence class of $f(C')$ in $\underline{\Sigma}' \setminus (I(\Sigma) \cup I_s(\Sigma)) \cup \{f(p)\}$, then, clearly, $[\gamma']_i$ is trivial.

This proves (a).

Parts (b), (c) and (d) follow.

The theorem is proved. □

Proof of Theorem 3.16. We need the following fact.

Lemma 3.45. *In the notation of Theorem 3.9, for any polygon $P = (\gamma_1, \ldots, \gamma_n)$ in Σ the tuple $f(P) = (f(\gamma_1), \ldots, f(\gamma_n))$ is a polygon in Σ'.*

Proof. Indeed, let $P = (\gamma_1, \ldots, \gamma_n)$ be a polygon in Σ and let $g : P_n \to \Sigma$ be an accompanying morphism. Then $g' = f \circ g$ is a morphism $P_n \to \Sigma'$ in **Surf** such that $g'(i, i^+) = f(\gamma_i)$ for $i \in [n]$, i.e., $f(P)$ is an n-gon in Σ').

The lemma is proved. □

Thus the triangle relations in $\mathbb{T}_\Sigma$ are carried by $f_\star$ to those in $\mathbb{T}_{\Sigma'}$. This proves the assertion for groups.

Likewise, the triangle and exchange relations in $\mathcal{A}_\Sigma$ are carried by f_* to those in $\mathcal{A}_{\Sigma'}$. This proves the assertion for algebras. The commutativity of the diagram (3.4) follows.

The theorem is proved. □

Proof of Theorem 3.24. It suffices to prove the assertion only for neighboring triangulations Δ and Δ', i.e., for a quadrilateral $(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ in Δ such that $\Delta\setminus\Delta'=\{\gamma_{13},\gamma_{31}\}$ and $\Delta'\setminus\Delta=\{\gamma_{24},\gamma_{42}\}$.

The following result is obvious.

Lemma 3.46. *In the notation as above, the assignment* $t_\gamma\mapsto\begin{cases} t_{\gamma_{12}}t_{\gamma_{42}}^{-1}t_{\gamma_{43}} & \text{if } \gamma=\gamma_{13}\\ t_{\gamma_{34}}t_{\gamma_{24}}^{-1}t_{\gamma_{21}} & \text{if } \gamma=\gamma_{31}\\ t_\gamma & \text{otherwise}\end{cases}$ *for* $i,j\in[4]$, $i\neq j$, *defines an isomorphism* $\varphi_{\Delta,\Delta'}:\mathbb{T}_\Delta\widetilde{\to}\mathbb{T}_{\Delta'}$.

The second assertion follows immediately because one has for $\gamma,\gamma'\in\Delta$, $\gamma'\notin\{\gamma_{13},\gamma_{31}\}$:

$$f_{\Delta,\Delta'}(y_{\gamma,\gamma'})=\begin{cases} t_{\gamma_{21}}^{-1}t_{\gamma_{24}}t_{\gamma_{34}}^{-1}t_{\gamma'} & \text{if } \gamma=\gamma_{13}\\ t_{\gamma_{43}}^{-1}t_{\gamma_{42}}t_{\gamma_{12}}^{-1}t_{\gamma'} & \text{if } \gamma=\gamma_{31}\\ t_\gamma^{-1}t_{\gamma'} & \text{otherwise}\end{cases}=\begin{cases} y_{\gamma_{12},\gamma_{24}}y_{\gamma_{43},\gamma'} & \text{if } \gamma=\gamma_{13}\\ y_{\gamma_{34},\gamma_{42}}y_{\gamma_{21},\gamma'} & \text{if } \gamma=\gamma_{31}\\ y_{\gamma,\gamma'} & \text{otherwise}\end{cases}\in\mathbb{Y}_{\Delta'}\,.$$

This proves the theorem. □

Proof of Theorem 3.30. Indeed, let $f:P_n\to\Sigma$ be an accompanying map for the canonical polygon $P_\Delta(\gamma)=(\gamma_1,\ldots,\gamma_n)$. Then, by Theorem 3.16, the assignments $x_{ij}\mapsto x_{\gamma_{ij}}$ define an algebra homomorphism $f_*:\mathcal{A}_n\to\mathcal{A}_\Sigma$, where $\mathcal{A}_n=\mathcal{A}_{P_n}$ is the noncommutative n-gon as in Section 2.2. Applying f_* to (2.5) with $i=1$ yields (3.6).

The theorem is proved. □

3.12. Noncommutative triangle groups and proof of Theorem 3.26

We need the following immediate result.

Lemma 3.47. *For any marked surface* Σ *there is* $n\geq 1$, *a subset* $S\subset[n]$, *an injective map* $\sigma:S\to[n]\setminus S$, *and a function* $\varepsilon:S\to\{-,+\}$ *such that* Σ *is obtained from* $P_n(h)$, $h=|I_s(\Sigma)|$ *by gluing the chord* (i,i^+) *to the chord* $\begin{cases}(\sigma(i)^+,\sigma(i)) & \text{if } \varepsilon(i)=+\\ (\sigma(i),\sigma(i)^+) & \text{if } \varepsilon(i)=-\end{cases}$ *for all* $i\in S$.

Remark 3.48. Clearly, for any $n \geq 2$ and any pair (σ, ε) as in Lemma 3.47, there is a marked surface $\Sigma_{\sigma,\varepsilon}$ obtained from $P_n(h)$ by such a gluing procedure.

The following is an obvious version of Theorem 3.16.

Lemma 3.49. *Let $f : \Sigma \to \Sigma'$ be as in Theorem 3.16 and let Δ and Δ' be triangulations of Σ and Σ' respectively such that $f(\Delta) \subset \Delta'$. Then the assignments $t_\gamma \mapsto t_{f(\gamma)}$ for $\gamma \in \Delta$ define homomorphism of groups $f_\star : \mathbb{T}_\Delta \to \mathbb{T}_{\Delta'}$.*

Combining Lemmas 3.47 and 3.49 and taking into account that under the gluing map $f : P_n(h) \to \Sigma$, the image $f(\Delta)$ of any triangulation Δ of $P_n(h)$ is a triangulation of $\Sigma = \Sigma_{\sigma,\varepsilon}$, we see that the quotient group of $\mathbb{T}_\Delta$ by the relations

$$t_{i,i^+} = \begin{cases} t_{\sigma(i)^+,\sigma(i)} & \text{if } \varepsilon(i) = + \\ t_{\sigma(i),\sigma(i)^+} & \text{if } \varepsilon(i) = - \end{cases}, t_{i^+,i} = \begin{cases} t_{\sigma(i),\sigma(i)^+} & \text{if } \varepsilon(i) = + \\ t_{\sigma(i)^+,\sigma(i)} & \text{if } \varepsilon(i) = - \end{cases}, \tag{3.14}$$

$i \in S$, is naturally isomorphic to $\mathbb{T}_{f(\Delta)}$ (of course, $\mathbb{T}_{f(\Delta)} \cong \mathbb{T}_{\Delta''}$ for any triangulation Δ'' of Σ by Theorem 3.24).

We will use this observation with the appropriately modified starlike triangulation $\Delta = \tilde{\Delta}_1$ of $P_n(h)$, where Δ_1 is the starlike triangulation of $[n]$ as in (2.6) with $i = 1$.

Namely, for all $n \geq 2$, $\tilde{\Delta}_1$ is obtained from Δ_1 by adding h curves $\gamma_{12}^{(s)}$, $s \in [h]$ from the vertex 1 to the vertex 2 *outside* of Δ_1 so that each 2-gon $((\gamma_{12}^{(s)})^{-1}, \gamma_{12}^{(s-1)})$, $s \in [h]$ contains exactly one special puncture (here, with a slight abuse of notation, $\gamma_{12}^{(0)}$ is the chord $(1,2)$ in $[n]$) and a clockwise loop $\gamma_1^{(s)}$ around each special puncture inside $((\gamma_{12}^{(s)})^{-1}, \gamma_{12}^{(s-1)})$, $s \in [h]$.

Lemma 3.50. *Suppose that $n \geq 2$. Then the group $\mathbb{T}_{\tilde{\Delta}_1}$ is generated by $t_j = T_j^{1,j^+}$, $j = 3, \dots, n-1$, $c_k = t_{k,k^+}$, $\overline{c}_k = t_{k^+,k}$, $k \in [n]$, $y_s = t_{\gamma_{12}^{(s)}}$, $z_s = t_{\gamma_1^{(s)}}$, $s \in [h]$, and $\overline{y}_h = t_{(\gamma_{12}^{(h)})^{-1}}$, subject to (if $n \geq 4$):*

$$c_2 t_3 c_3 \cdots t_{n-1} c_{n-1} \overline{c}_n^{-1} c_1 = \overline{c}_1 c_n^{-1} \overline{c}_{n-1} t_{n-1} \overline{c}_{n-2} \cdots t_3 \overline{c}_2 \tag{3.15}$$

and (if $h > 0$):

$$\overline{y}_h = \overline{c}_1 (z_1^{-1} y_1 c_1^{-1} z_1)(z_2^{-1} y_2 y_1^{-1} z_2) \cdots (z_h^{-1} y_h y_{h-1}^{-1} z_h) \,. \tag{3.16}$$

Proof. Clearly, $t_{1,j} = c_1 t_2 c_2 \cdots t_{j-1} c_{j-1}$, $t_{j,1} = \overline{c}_{j-1} t_{j-1} \cdots \overline{c}_2 t_2 \overline{c}_1$ in $\mathbb{T}_{\Delta_1}$ for $j = 1, \dots, n$. Thus, $\mathbb{T}_{\Delta_1}$ is generated by $t_2, \dots, t_{n-1}$, c_k, $\overline{c}_k$, $k = 1, \dots, n$ subject to the relations:

$$\overline{c}_n = c_1 t_2 c_2 \cdots c_{n-2} t_{n-1} c_{n-1}, c_n = \overline{c}_{n-1} t_{n-1} \cdots \overline{c}_2 t_2 \overline{c}_1 \,.$$

By eliminating t_2, we see that $\mathbb{T}_{\Delta_1}$ is subject to the relation (3.15). Furthermore, the 1-gon relations in the 1-gons $(\gamma_1^{(s)})$ and triangle relations in the triangles $((\gamma_{12}^{(s)})^{-1}, \gamma_1^{(s)}, \gamma_{12}^{(s-1)})$ for the remaining generators $y_s = t_{\gamma_{12}^{(s)}}$, $\overline{y}_s = t_{(\gamma_{12}^{(s)})^{-1}}$ $z_s = t_{\gamma_1^{(s)}}$, $\overline{z}_s = t_{(\gamma_1^{(s)})^{-1}}$, $s \in \{0\} \sqcup [h]$ of $\mathbb{T}_{\tilde{\Delta}_1}$ read:

$$\overline{z}_s = z_s,\ \overline{y}_s \overline{z}_s^{-1} y_{s-1} = \overline{y}_{s-1} z_s^{-1} y_s$$

for $s \in [h]$ (here $y_0 = c_1$, $\overline{y}_0 = \overline{c}_1$). That is, one can eliminate all $\overline{z}_s$, $s \in [h]$ and one can solve recursively for all $\overline{y}_s$, $s \in [h]$:

$$\overline{y}_s = \overline{c}_1(z_1^{-1} y_1 y_0^{-1} z_1)(z_2^{-1} y_2 y_1^{-1} z_2) \cdots (z_s^{-1} y_s y_{s-1}^{-1} z_s) ,$$

so that the remaining generators z_s and y_s, $s \in [h]$ are free.

The lemma is proved. $\square$

Combining Lemmas 3.47 and 3.50, we see that for $n \geq 3$ the group $\mathbb{T}_{f(\Delta)}$ is generated by t_j, $j = 3, \ldots, n-1$, c_k, $\overline{c}_k$, $k = 1, \ldots, n$, y_s, z_s, $s \in [h]$, and $\overline{y}_h$ subject to (3.15) and the following relations for all $i \in S$:

$$c_{\sigma(i)} = \begin{cases} \overline{c}'_i & \text{if } \varepsilon(i) = + \\ c'_i & \text{if } \varepsilon(i) = - \end{cases}, \overline{c}_{\sigma(i)} = \begin{cases} c'_i & \text{if } \varepsilon(i) = + \\ \overline{c}'_i & \text{if } \varepsilon(i) = - \end{cases},$$

$$\text{where } c'_i := \begin{cases} c_i & \text{if } i \neq 1 \\ y_h & \text{if } i = 1 \end{cases}, \overline{c}'_i := \begin{cases} \overline{c}_i & \text{if } i \neq 1 \\ \overline{y}_h & \text{if } i = 1 \end{cases}.$$

Thus, if $n \geq 3$, then the group $\mathbb{T}_{\tilde{\Delta}_1}$ has $(n-3) + 2(n - |S|) + 2h = 3n - 3 - 2|S| + 2h$ generators t_j, $j = 3, \ldots, n-1$, c_k, $\overline{c}_k$, $k \in [n] \setminus \sigma(S)$, y_s, z_s, $s \in [h]$ and exactly one relation (3.15). Now compute the Euler characteristic of Σ using the triangulation Δ'' of Σ obtained by removing all h loops around special punctures from $f(\Delta)$. By definition,

$$\chi(\Sigma) = |I| - E + T ,$$

where E is the number of edges and T is the number of triangles in Δ''. Clearly, $T = n-2$ and $E = (n-3) + (n - |S|)$, therefore,

$$\chi(\Sigma) = |I| - ((n-3) + (n - |S|)) + n - 2 = |I| + 1 - n + |S| = |I| + 1 - \frac{n}{2} - \frac{|I_b|}{2}$$

because $n - 2|S| = |I_b|$. Therefore, the number of generators of $\mathbb{T}_{f(\Delta)}$ is:

$$\begin{aligned} 3n - 3 - 2|S| + 2h = 2n - 3 + |I_b| + 2h &= 4(|I| + 1 - \chi(\Sigma) - \frac{|I_b|}{2}) - 3 + |I_b| + 2h \\ &= 4(|I| - \chi(\Sigma)) + 1 - |I_b| + 2h . \end{aligned}$$

We now consider several cases.

Case 1. $n \geq 3$ and either Σ has boundary, i.e., $S \cup \sigma(S) \neq [n]$ or $h > 0$. The above implies that $\mathbb{T}_{f(\Delta)}$ is free in $4(|I| - \chi(\Sigma)) - |I_b| + 2h$ generators.

Case 2. $n = 2$ (hence $h > 0$). Then, clearly, $\mathbb{T}_{\tilde{\Delta}_1}$ is a free group in $2h + 2$ generators. Therefore:

- If $n = 2$, $h \geq 2$, then $\mathbb{T}_{f(\Delta)}$ is free in $2h + 2 - 2|S|$ generators, where $|S| \in \{0, 1\}$.
- If $n = 2$, $h = 1$, then $\mathbb{T}_{f(\Delta)}$ is free in $4 - |S|$ generators, where $|S| \in \{0, 1\}$.

Case 3. $n = 1$, then f is the identity map and Σ is a disk with $|I| = |I_b| = 1$ and h special punctures. If $h = 0$, then, clearly, $\mathbb{T}_\Delta$ is free in two generators t_γ and $t_{\overline{\gamma}}$, where γ is the clockwise loop. Suppose that $h > 0$. Then one can choose a triangulation Δ of Σ in such a way that, in addition to γ it consists of a special loop λ_s, $s \in [h]$ around each special puncture and a clockwise loop γ_s enclosing first s special punctures (from the left to the right), $s = 2, \ldots, h$ (so that $\lambda_1 = \gamma_1$ and $\gamma_h = \gamma$). Then $\mathbb{T}_\Delta$ is generated by $z_s = t_{\lambda_s}$, $y_s = t_{\gamma_s}$, $\overline{y}_s = t_{\overline{\gamma}_s}$, $s \in [h]$ subject to the following triangle relations in the $h - 1$ triangles $(\gamma_{s-1}, \lambda_s, \overline{\gamma}_s)$, $s = 2, \ldots, h$:

$$y_{s-1} z_s^{-1} \overline{y}_s = y_s z_s^{-1} \overline{y}_{s-1}$$

for $s = 2, \ldots, h$ if $h \geq 2$. That is, similarly to the equations (3.16), one can solve recursively for all $\overline{y}_s$, $s = 2, \ldots, h$:

$$\overline{y}_s = (z_s y_{s-1}^{-1} y_s) \cdots (z_3 y_2^{-1} z_3 y_2) \cdot (z_2 z_1^{-1} z_2 z_1)$$

(since $y_1 = \overline{y}_1 = z_1$) so that $\mathbb{T}_\Delta$ is freely generated by z_s, $s \in [h]$ and y_s, $s = 2, \ldots, h$.

This finishes the proof of Theorem 3.26(a).

Case 4. $n \geq 3$ and Σ has no boundary, i.e., $|I_b| = 0$ hence $S \cup \sigma(S) = [n]$ and $h = 0$. Then $n = 2|S|$ is even and $\mathbb{T}_{f(\Delta)}$ is a 1-relator torsion-free group in $4(|I| - \chi(\Sigma)) + 1$ generators t_k, $k = 3, \ldots, n - 1$, $c_k, \overline{c}_k$, $k \in S$. Suppose that Σ is a sphere with $|I| \leq 3$ punctures. Then $\mathbb{T}_{f(\Delta)}$ trivial for $|I| = 1$ because all loops are contractible, is free in 2 generators t_γ and $t_{\gamma^{-1}}$ if $|I| = 2$, where γ is an arc between these two punctures, and if $|I| = 3$, it is free in 5 generators, because we can take $S = \{1, 3\} \subset [4]$, $\sigma(1) = 2$, $\sigma(3) = 4$, $\varepsilon(1) = \varepsilon(3) = +$ so that $\mathbb{T}_{f(\Delta)}$ is freely generated by $c_1, \overline{c}_1, c_3, \overline{c}_3, t_1$. Otherwise, it is, clearly, non-free. This finishes the proof of Theorem 3.26(b).

The theorem is proved. □

Example 3.51. If Δ is a triangulation of the torus, the Klein bottle, the real projective plane respectively with one, one, two (ordinary) punctures, then $\mathbb{T}_\Delta$ is generated by $c_1, c_2, \overline{c}_1, \overline{c}_2, t_3$ subject to, respectively (with some notation from the proof of Theorem 3.26 in Section 3.12):

(i) for the torus with one puncture: $c_2 t_3 \overline{c}_1 c_2^{-1} c_1 = \overline{c}_1 \overline{c}_2^{-1} c_1 t_3 \overline{c}_2$, because Δ is glued from a square with diagonal $(1, 3)$, where: $S = \{1, 2\} \subset [4]$, $\sigma(1) = 3$, $\sigma(2) = 4$, $\varepsilon(1) = \varepsilon(2) = +$ (equivalently, $abcde = cbeda$ after substitution $a = t_3$, $b = \overline{c}_1$, $c = c_2^{-1}$, $d = c_1$, $e = \overline{c}_2^{-1}$).

(ii) for the Klein bottle with one puncture: $c_2t_3\overline{c}_1\overline{c}_2^{-1}c_1 = \overline{c}_1c_2^{-1}c_1t_3\overline{c}_2$, because Δ is glued from a square with diagonal $(1,3)$, where: $S = \{1,2\} \subset [4]$, $\sigma(1) = 3$, $\sigma(2) = 4$, $\varepsilon(1) = +$, $\varepsilon(2) = -$ (equivalently, $abcdc = ebeda$ after substitution $a = t_3$, $b = \overline{c}_1$, $c = \overline{c}_2^{-1}$, $d = c_1$, $e = c_2^{-1}$).

(iii) for the real projective plane with two punctures: $c_2t_3c_1\overline{c}_2^{-1}c_1 = \overline{c}_1c_2^{-1}\overline{c}_1t_3\overline{c}_2$, because Δ is glued from a square with diagonal $(1,3)$, where: $S = \{1,2\} \subset [4]$, $\sigma(1) = 3$, $\sigma(2) = 4$, $\varepsilon(1) = \varepsilon(2) = -$ (equivalently, $abcbc = ededa$ after substitution $a = t_3$, $b = c_1$, $c = \overline{c}_2^{-1}$, $d = \overline{c}_1$, $e = c_2^{-1}$).

3.13. Noncommutative curves and proof of Theorem 3.36

For each $\gamma \in [\Gamma(\Sigma)]$, a triangulation Δ of Σ define the elements $t_{\gamma,\Delta} \in \mathbb{QT}_\Delta$ same way as in Theorem 2.10:

$$t_\gamma^\Delta := \sum_{\mathbf{i} \in Adm_{\Delta^0}(1,j)} t_\mathbf{i} \ , \tag{3.17}$$

where Δ^0 is the triangulation of $[n]$ assigned (as in Theorem 3.21(a)) to Δ and the canonical polygon $P_\Delta(\gamma) = (\gamma_1, \ldots, \gamma_n)$ with $\gamma = \gamma_{1,j}$ and we abbreviated

$$t_\mathbf{i} := t_{\gamma_{i_1,i_2}} t_{\gamma_{i_3,i_2}}^{-1} t_{\gamma_{i_3,i_4}} \cdots t_{\gamma_{i_{2m-1},i_{2m-2}}}^{-1} t_{\gamma_{i_{2m-1},i_{2m}}}$$

for any sequence $\mathbf{i} = (i_1, \ldots, i_{2m}) \in [n]^{2m}$, $m \geq 1$.

We refer each t_γ^Δ as it as a *noncommutative triangulated curve.*

Clearly, if $\Sigma = P_n(0)$ is an n-gon (i.e., a disk with $I(\Sigma) = I_b(\Sigma) = [n]$) so that $\gamma = (p,q) \in [n] \times [n]$, then $t_\gamma^\Delta = t_{pq}^\Delta$ is as in (2.40).

To finish the proof of Theorem 3.36, we need the following result.

Proposition 3.52. *The assignments $x_\gamma \mapsto t_\gamma^\Delta$ for $\gamma \in [\Gamma(\Sigma)]$ define an epimorphism of algebras*

$$\mathcal{A}_\Sigma \to \mathbb{QT}_\Delta[\mathbf{S}_\Delta^{-1}] \ , \tag{3.18}$$

where $\mathbf{S}_\Delta$ is the sub-monoid of $\mathbb{QT}_\Delta$ generated by all t_γ^Δ.

Proof. It suffices to show that the elements t_γ^Δ satisfy the defining relations of $\mathcal{A}_\Sigma$ from Definition 3.12.

We need the following result.

Lemma 3.53. *Let $Q = (\gamma'_1, \ldots, \gamma'_r)$ be an n-gon in Σ and let Δ be any triangulation of Σ. Then the assignments $x_{ij} \mapsto x_{\gamma_{ij}}$, $i,j \in [r]$ define a homomorphism of algebras*

$$\mathcal{A}_r \to \mathbb{QT}_\Delta[\mathbf{S}_\Delta^{-1}] \ . \tag{3.19}$$

Proof. Let $P = (\gamma_1, \ldots, \gamma_n)$, Δ^0, and $\iota : [r] \hookrightarrow [n]$ be as in Theorem 3.21. Then, in view of Theorem, for any accompanying morphism $f : P_n \to \Sigma$ Therefore, the assignments $(i,j) \mapsto f(i,j) = \gamma_{ij}$ restricted to Δ^0 define a homomorphism of algebras $f_\Delta : \mathbb{Q}\mathbb{T}_{\Delta^0} \to \mathbb{Q}\mathbb{T}_\Delta$ such that $f_\Delta(t_{ij}^{\Delta^0}) = t_{\gamma_{ij}}^{\Delta}$ for $i, j \in [n]$. Since $f_\Delta(\mathbf{S}_{\Delta^0}) \subset \mathbf{S}_{\boldsymbol{\Delta}}$, then passing to the universal localizations, this gives an algebra homomorphism

$$\mathbb{Q}\mathbb{T}_{\Delta^0}[\mathbf{S}_{\Delta^0}^{-1}] \to \mathbb{Q}\mathbb{T}_\Delta[\mathbf{S}_\Delta^{-1}] \ .$$

Composing it with the isomorphism $\mathcal{A}_n \cong \mathbb{Q}\mathbb{T}_{\Delta^0}[\mathbf{S}_{\Delta^0}^{-1}]$ given by Theorem 2.8(b) and the homomorphism $\mathcal{A}_r \to \mathcal{A}_n$ given by $x_{k,\ell} \mapsto x_{\iota(k),\iota(\ell)}$ give the desired homomorphism (3.19). The lemma is proved. $\square$

Using the Lemma with $r = 3, 4$, we finish the proof of the proposition. $\square$

Since each x_γ, $\gamma \in [\Gamma(\Sigma)]$ is invertible in $\mathcal{A}_\Sigma$, the universality of localization $\mathbb{Q}\mathbb{T}_\Delta[\mathbf{S}_\Delta^{-1}]$ implies that (3.7) extends to a homomorphism of algebras

$$\mathbb{Q}\mathbb{T}_\Delta[\mathbf{S}_\Delta^{-1}] \to \mathcal{A}_\Sigma \ . \tag{3.20}$$

By the construction and Theorem 3.30, (3.7) takes each t_γ^Δ to x_Δ and therefore is an epimorphism $\mathbb{Q}[\mathbb{T}_\Delta] \twoheadrightarrow \mathcal{A}_\Delta$. In turn, (3.20) is an epimorphism as well.

Thus, we obtained two mutually inverse epimorphisms (3.19) and (3.20), which implies that they are isomorphisms of algebras.

Therefore, (3.18) is an isomorphism, which proves Theorem 3.36(b). Theorem 3.36(a) also follows because (3.7) is a restriction to $\mathbb{Q}\mathbb{T}_\Delta$ of the isomorphism (3.20) and $\mathbf{i}_\Delta(\mathbb{Q}\mathbb{T}_\Delta) = \mathcal{A}_\Delta$.

Theorem 3.36 is proved. $\square$

4. Noncommutative discrete integrable systems

4.1. An integrable system on a cylinder

Denote by $\Sigma_{1,r}$, an annulus (i.e., a cylinder) with no punctures, one marked point p on the outer circle and r marked points $p_1, \ldots, p_r$ on the inner circle (listed clockwise).

It is easy to see that equivalence classes of curves from p to $\{p_1, \ldots, p_r\}$ in $\Sigma_{1,r}$ are in a natural bijection with $\mathbb{Z}$: the n-th curve γ_n goes (without self-intersections) from p to p_s where $s \equiv n \mod r$ and γ_n has the winding number q such that $n = rq + s$ (so that the arc is winding clockwise if $q \geq 0$ and counterclockwise if $q < 0$).

We also denote γ_i^- (resp. $\overline{\gamma}_i^-$) the short counterclockwise boundary arc in the inner circle from p_i to the previous point p_{i^-} (resp. from p_{i^-} to p_i), $i \in [r]$; and by γ^+ (resp. $\overline{\gamma}^+$) the clockwise (resp. counterclockwise) loop in the outer circle.

We abbreviate in the algebra $\mathcal{A}_{\Sigma_{1,r}}$: $x_n := x_{\gamma_n}$, $\overline{x}_n := x_{\overline{\gamma}_n}$, $c_n := x_{\gamma_n^-}$, $\overline{c}_n := x_{\overline{\gamma}_n^-}$, $d := x_{\gamma^+}$, $\overline{d} := x_{\overline{\gamma}^+}$ for $n \in \mathbb{Z}$ (where we extend γ_n^- periodically so that $\gamma_{n+r}^- = \gamma_r^-$ for all $n \in \mathbb{Z}$).

Since $(\overline{\gamma}_{n-1}, \gamma^+, \gamma_{n-r}, \gamma_n^-)$ is a 4-gon in $\Gamma(\Sigma_{1,r}) = [\Gamma(\Sigma_{1,r})]$ containing triangles $(\gamma_{n-1}, \overline{\gamma}_{n-1}^-, \overline{\gamma}_n)$ and $(\overline{\gamma}_n, \gamma^+, \gamma_{n-r})$ for all $n \in \mathbb{Z}$, the following fact is immediate from Definition 3.12.

Lemma 4.1. *For each $r \geq 1$ one has in $\mathcal{A}_{\Sigma_{1,r}}$:*

(i) (triangle relations)

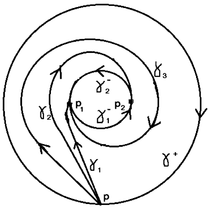
$$x_{n-1}\overline{c}_n^{-1}\overline{x}_n = x_n c_n^{-1}\overline{x}_{n-1},\ \overline{x}_n\overline{d}^{\,-1}x_{n-r} = \overline{x}_{n-r}d^{-1}x_n\ . \tag{4.1}$$

(ii) (exchange relations) For each $n \in \mathbb{Z}$:

$$\overline{x}_{n-r-1}d^{-1}x_n = c_n + \overline{x}_{n-1}\overline{d}^{\,-1}x_{n-r},\ \overline{x}_n\overline{d}^{\,-1}x_{n-r-1} = \overline{c}_n + \overline{x}_{n-r}d^{-1}x_{n-1}\ . \tag{4.2}$$

Note that for each $m \in \mathbb{Z}$ the annulus $\Sigma_{1,r}$ has a triangulation

$$\Delta_m := \{\gamma^+, \overline{\gamma}^+; \gamma_1^-, \overline{\gamma}_1^- \ldots, \gamma_r^-, \overline{\gamma}_r^-; \gamma_m, \overline{\gamma}_m, \ldots, \gamma_{m+r}, \overline{\gamma}_{m+r}\}\ .$$



Triangulation Δ_1 of the cylinder $\Sigma_{1,2}$

Hence the group $\mathbb{T}_r$ generated by $x_n, \overline{x}_n$, $n = 1, \ldots, r+1$, $c_i, \overline{c}_i$, $i = 1, \ldots, r$, $d, \overline{d}$ subject to the triangle relations

$$\overline{x}_{r+1}\overline{d}^{\,-1}x_1 = \overline{x}_1 d^{-1}x_{r+1},\ x_{s-1}\overline{c}_s^{-1}\overline{x}_s = x_s c_s^{-1}\overline{x}_{s-1}, \tag{4.3}$$

$s = 2, \ldots, r+1$ (with the convention $p_{r+n} = p_r$ hence $c_{r+n} = c_r$, $\overline{c}_{r+n} = \overline{c}_n$ for $n \in \mathbb{Z}$) is naturally isomorphic to the triangle group $\mathbb{T}_{\Delta_1}$. Moreover, in the notation of Section 3.8, the subalgebra $\mathcal{A}_{\Delta_1}$ of $\mathcal{A}_\Sigma$ (generated by all x_γ, $\gamma \in \Gamma(\Sigma_{1,r})$ and all $x_{\gamma_0}^{-1}$, $\gamma_0 \in \Delta_1$) is the group algebra $\mathbb{Z}\mathbb{T}_r$ by Theorem 3.36(a).

Proposition 4.2. *For each $r \geq 1$ we have:*

(a) Each $x_n, \overline{x}_n$, $n \in \mathbb{Z}$ is sum of elements of $\mathbb{T}_r$ in $\mathbb{Z}\mathbb{T}_r$.

(b) The total angle $T_p \in \mathbb{Z}\mathbb{T}_r$ at p is given by $T_p = \overline{d}^{\,-1}x_{n-r}x_n^{-1} + d^{-1}x_{n+r}x_n^{-1} = \overline{x}_n^{-1}\overline{x}_{n-r}d^{-1} + \overline{x}_n^{-1}\overline{x}_{n+r}\overline{d}^{\,-1}$ for each $n \in \mathbb{Z}$.

Proof. Part (a) follows directly from Theorem 3.30 and Corollary 3.37.

Prove (b). Consider a triangle $(\gamma^+, \gamma_n, \overline{\gamma}_{n-r})$ in Δ_{n-r} and $(\gamma^+, \gamma_n, \overline{\gamma}_{n+r})$ in Δ_n.

The following is an immediate corollary of Theorem 3.40.

Lemma 4.3. $T_p = T_{\gamma^+,\gamma_n,\overline{\gamma}_{n-r}} + T_{\gamma^+,\gamma_n,\overline{\gamma}_{n+r}}$.

Using this and taking into account that

$$T_{\gamma^+,\gamma_n,\overline{\gamma}_{n-r}} = \overline{d}^{-1} x_{n-r} x_n^{-1} = \overline{x}_n^{-1} \overline{x}_{n-r} d^{-1},\ T_{\gamma^+,\gamma_n,\overline{\gamma}_{n+r}} = \overline{d}^{-1} x_{n+r} x_n^{-1} = \overline{x}_n^{-1} \overline{x}_{n+r} d^{-1}$$

in the notation (3.9), we obtain (b).

The proposition is proved. □

Remark 4.4. Using the triangulation Δ_n, it is easy see that

$$T_p = d^{-1} x_n x_{n-r}^{-1} + \overline{d}^{-1} x_{n-r} x_n^{-1} + \sum_{m=n+1-r}^{n} \overline{x}_{m-1}^{-1} c_m x_m^{-1}$$

for all $n \in \mathbb{Z}$.

Clearly, by Theorem 3.40, T_p does not depend on n.

If r is even, we can refine these observations and thus recover the recursion (1.4).

Indeed, set $U_n := \begin{cases} x_n & \text{if } n \text{ is even} \\ \overline{x}_n & \text{if } n \text{ is odd} \end{cases}, C_n := \begin{cases} c_n & \text{if } n \text{ is even} \\ \overline{c}_n & \text{if } n \text{ is odd} \end{cases}$, and $D := d^{-1}$, $\overline{D} := \overline{d}^{-1}$.

By definition, $\mathbb{T}_r$ is freely generated by D, $\overline{D}$ and C_i, $i \in [r]$, U_j, $j \in [r+1]$ and, by Proposition 4.2, $U_n \in \mathbb{Q}\mathbb{T}_r$ is a sum of elements of $\mathbb{T}_r$. This and Proposition 4.2 imply the following result.

Theorem 4.5. *Let $r \geq 1$ be even. Then each element $U_n \in \mathbb{Z}\mathbb{T}_r$, $n \in \mathbb{Z}$ satisfies the recursion:*

$$\begin{cases} U_{n-r-1} D U_n = C_n + U_{n-1} \overline{D} U_{n-r} & \text{if } n \text{ is even} \\ U_n \overline{D} U_{n-r-1} = C_n + U_{n-r} D U_{n-1} & \text{if } n \text{ is odd} \end{cases}. \tag{4.4}$$

(with the convention $C_{n+r} = C_r$). Furthermore, the element $H_n \in Frac(\mathbb{Z}\mathbb{T}_r)$, $n \in \mathbb{Z}$, given by

$$H_n := \begin{cases} \overline{D} U_{n-r} U_n^{-1} + D U_{n+r} U_n^{-1} & \text{if } n \text{ is even} \\ U_n^{-1} U_{n-r} D + U_n^{-1} U_{n+r} \overline{D} & \text{if } n \text{ is odd} \end{cases} \tag{4.5}$$

does not depend on n and belongs to $\mathbb{Z}\mathbb{T}_r$.

The recursion (4.4) clearly coincides with the recursion (1.4) with $k = r + 1$ and the element H_n given by (4.5) coincides with the element given by (1.5).

Remark 4.6. In fact, Remark 4.4 implies that the "conserved quantity" $H = H_n$ is equal (for any $n \in \mathbb{Z}$) to

$$\begin{cases} DU_nU_{n-r}^{-1} + \overline{D}U_{n-r}U_n^{-1} + \sum\limits_{m=(n+2-r)/2}^{n/2} U_{2m-1}^{-1}C_{2m}U_{2m}^{-1} + U_{2m-1}^{-1}C_{2m-1}U_{2m-2}^{-1} \\ \quad \text{if } n \text{ is even} \\ U_n^{-1}U_{n-r}D + U_{n-r}^{-1}U_n\overline{D} + \sum\limits_{m=(n+1-r)/2}^{(n-1)/2} U_{2m-1}^{-1}C_{2m}U_{2m}^{-1} + U_{2m+1}^{-1}C_{2m+1}U_{2m}^{-1} \\ \quad \text{if } n \text{ is odd} \end{cases}.$$

4.2. An integrable system on an infinite strip

In this section we establish Laurentness of another noncommutative recursion (which specializes to the discrete integrable system recently studied by P. Di Francesco in [9]). Indeed, let Σ_∞ be a horizontal strip with marked boundary points $I = I_- \sqcup I_+$, where $I_+ = \{i_+, i \in \mathbb{Z}\}$ (resp. $I_- = \{i_-, i \in \mathbb{Z}\}$) is the marked point set on the left (resp on the right) boundary line. Then, clearly, $\Gamma(\Sigma_\infty) = [\Gamma(\Sigma_\infty)] = \{(i_\varepsilon, j_{\varepsilon'}) : i, j \in \mathbb{Z},\ \varepsilon, \varepsilon' \in \{-,+\}, i \neq j \text{ if } \varepsilon = \varepsilon'\}$. Clearly, $\Sigma_\infty = \bigcup\limits_{m^-,m^+\in\mathbb{Z}, n\in\mathbb{Z}_{>0}} \Sigma^n_{m^-,m^+}$, where $\Sigma^n_{m^-,m^+} \subset \Sigma$ is the convex hull of the real intervals $[m^- + 1, m^- + n]_- \subset I_-$, $[m^+ + 1, m^+ + n]_+ \subset I_+$. Clearly, it is an $2n$-gon embedded (as a parallelogram) into Σ, where we identify its vertex set $[2n]$ with $\{(m^- + 1)_- \dots, (m^- + n)_-\} \sqcup \{(m^+ + 1)_+ \dots, (m^+ + n)_+\}$ via

$$k \mapsto \begin{cases} (m^- + k)_- & \text{if } k \le n \\ (m^+ + 2n + 1 - k)_+ & \text{if } k > n \end{cases}.$$

We denote by $\mathcal{A}_{\Sigma^n_{m^-,m^+}}$ a copy of $\mathcal{A}_{2n}$ under the above identification of the vertex set $[2n]$.

Then the natural inclusions $\Sigma^n_{m^-,m^+} \subset \Sigma^{n'}_{m'^-,m'^+}$ for $m'^- \le m^-$, $m'^+ \le m^+$, $m'^- + n' \ge m^- + n$, $m'^+ + n' \ge m^+ + n$ are morphisms in **Surf** so they define (by Theorem 3.16) the appropriate homomorphisms of algebras $\mathcal{A}_{\Sigma^n_{m^-,m^+}} \to \mathcal{A}_{\Sigma^{n'}_{m'^-,m'^+}}$, so we denote by $\mathcal{A}_\Sigma$ the direct limit $\varinjlim \mathcal{A}_{\Sigma^n_{m^-,m^+}}$ under these homomorphisms.

Clearly, the following noncommutative Ptolemy relations (in the form (2.14)) hold in $\mathcal{A}_{\Sigma_\infty}$:

$$x_{(i+1)_\pm, j_\mp} x^{-1}_{(j+1)_\mp, j_\mp} x_{(j+1)_\mp, i_\pm} = x_{(i+1)_\pm, i_\mp} + x_{(i+1)_\pm, (j+1)_\mp} x^{-1}_{j_\mp, (j+1)_\mp} x_{j_\mp, i_\pm} \tag{4.6}$$

together with the triangle relations:

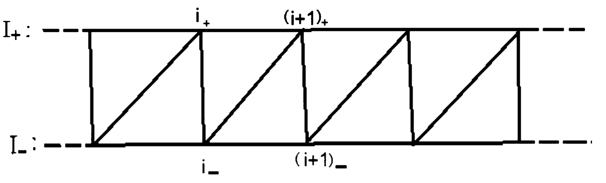
$$x_{i_\pm,j_\mp}x^{-1}_{(j+1)_\mp,j_\mp}x_{(j+1)_\mp,i_\pm}=x_{i_\pm,(j+1)_\mp}x^{-1}_{j_\mp,(j+1)_\mp}x_{j_\mp,i_\pm} \tag{4.7}$$

for all $i,j\in\mathbb{Z}$.

Remark 4.7. It is natural to conjecture that the relations (4.6) and (4.7) are defining for $\mathcal{A}_{\Sigma_\infty}$ and (in view of Remark 2.6 that) all natural homomorphisms $\mathcal{A}_{\Sigma^n_{m^-,m^+}}\hookrightarrow\mathcal{A}_{\Sigma_\infty}$ are injective.

Note that Σ_∞ has a triangulation

$$\Delta_\infty=\{(i_\pm,(i+1)_\pm),((i+1)_\pm,i_\pm);(i_-,i_+),(i_+,i_-),(i_-,(i+1)_+),((i+1)_+,i_-):i\in\mathbb{Z}\}\ .$$



Triangulation Δ_∞ of Σ_∞

Hence the group $\mathbb{T}_\infty$ generated by $d_{i,\pm}:=x_{i_\pm,(i+1)_\pm}$, $\overline{d}_{i,\pm}:=x_{i_\pm,(i+1)_\pm}$, $x_i:=x_{i_-,i_+}$, $\overline{x}_i:=x_{i_+,i_-}$, $y_i=x_{i_-,(i+1)_+}$, $\overline{y}_i=x_{i_-,(i+1)_+}$, $i\in\mathbb{Z}$ subject to the triangle relations

$$x_i\overline{d}^{-1}_{i,+}\overline{y}_i=y_id^{-1}_{i,+}\overline{x}_i,\ \overline{y}_i\overline{d}^{-1}_{i,-}x_{i+1}=\overline{x}_{i+1}d^{-1}_{i,-}y_i \tag{4.8}$$

for $i\in\mathbb{Z}$ is naturally isomorphic to the triangle group $\mathbb{T}_{\Delta_\infty}$. Corollary 3.37 implies that the subalgebra $\mathcal{A}_{\Delta_\infty}$ of $\mathcal{A}_{\Sigma_\infty}$ (generated by all x_γ, $\gamma\in\Gamma(\Sigma_\infty)$ and all $x^{-1}_{\gamma_0}$, $\gamma_0\in\Delta_\infty$) is the group algebra $\mathbb{Z}\mathbb{T}_\infty$.

Proposition 4.8. *In $\mathcal{A}_{\Sigma_\infty}$ we have:*

(a) Each $x_{i_\pm,j_\mp}$, $i,j\in\mathbb{Z}$ is sum of elements of $\mathbb{T}_\infty$ in $\mathbb{Z}\mathbb{T}_\infty$.

(b) The total angle $T_{i_\pm}\in\mathbb{Z}\mathbb{T}_r$ at $i_\pm$ is given by

$$\begin{aligned}T_{i_\pm}&=x^{-1}_{j_\mp,i_\pm}(x_{j_\mp,(i-1)_\pm}x^{-1}_{i_\pm,(i-1)_\pm}+x_{j_\mp,(i+1)_\pm}x^{-1}_{i_\pm,(i+1)_\pm})\\&=(x^{-1}_{(i-1)_\pm,i_\pm}x_{(i-1)_\pm,j_\mp}+x^{-1}_{(i+1)_\pm,i_\pm}x_{(i-1)_\pm,j_\mp})x^{-1}_{i_\pm,j_\mp}\end{aligned}$$

for each $j\in\mathbb{Z}$.

Proof. Part (a) follows directly from Theorem 3.30.

Prove (b). Consider triangles in the vertices $(i_\pm,j_\mp,(i-1)_\pm)$ and $(i_\pm,j_\mp,(i+1)_\pm)$ in Σ_∞.

The following is an immediate corollary of Theorem 3.40.

Lemma 4.9. $T_{i_\pm} = T_{(i_\pm,j_\mp),(j_\mp,(i-1)_\pm),((i-1)_\pm,i_\pm)} + T_{(i_\pm,j_\mp),(j_\mp,(i+1)_\pm),((i+1)_\pm,i_\pm)}$.

Using this and taking into account that

$$T_{(i_\pm,j_\mp),(j_\mp,(i-1)_\pm),((i-1)_\pm,i_\pm)} = x_{j_\mp,i_\pm}^{-1} x_{j_\mp,(i-1)_\pm} x_{i_\pm,(i-1)_\pm}^{-1} = x_{(i-1)_\pm,i_\pm}^{-1} x_{(i-1)_\pm,j_\mp} x_{i_\pm,j_\mp}^{-1} ,$$
$$T_{(i_\pm,j_\mp),(j_\mp,(i+1)_\pm),((i+1)_\pm,i_\pm)} = x_{j_\mp,i_\pm}^{-1} x_{j_\mp,(i+1)_\pm} x_{i_\pm,(i+1)_\pm}^{-1} = x_{(i+1)_\pm,i_\pm}^{-1} x_{(i-1)_\pm,j_\mp} x_{i_\pm,j_\mp}^{-1}$$

in the notation (3.9), we obtain (b).

The proposition is proved. □

Remark 4.10. Using the triangulation Δ_∞, it is easy see that

$$T_{i_-} = d_{i-1,-}^{-1} y_{i-1} x_i^{-1} + \overline{x}_i^{-1} d_{i,+} y_i^{-1} + \overline{d}_{i+1,-}^{-1} x_{i+1} y_i^{-1},$$
$$T_{i_+} = y_i^{-1} x_{i-1} \overline{d}_{i-1,+}^{-1} + y_i^{-1} d_{i-1} x_i^{-1} + x_i^{-1} y_i d_{i,+}^{-1} .$$

We can refine these observations and thus recover the recursions (1.6), (1.7). Indeed, set

$$U_{ij} = x_{i_-,j_+},\ V_{ij} := x_{i_+,j_-},\ A_j := x_{(j+1)_+,j_+}^{-1},$$
$$\overline{A}_j = x_{j_+,(j+1)_+}^{-1},\ B_j := x_{(j+1)_-,j_-}^{-1},\ \overline{B}_j = x_{j_-,(j+1)_-}^{-1} .$$

By definition, $\mathbb{T}_\infty$ is freely generated by $A_i, \overline{A}_i, B_i, \overline{B}_i, U_{i,i}, V_{i,i}, U_{i,i+1}$, $i \in \mathbb{Z}$ and, by Proposition 4.8, each $U_{ij}^\pm \in \mathbb{Q}\mathbb{T}_\infty$ is a sum of elements of $\mathbb{T}_r$. This and Proposition 4.8 imply the following result.

Theorem 4.11. *The elements $U_{ij}, V_{ij} \in \mathbb{Z}\mathbb{T}_\infty$ $i, j \in \mathbb{Z}$ satisfy (1.6), (1.7). Furthermore, the elements $H_{ij}^\pm \in Frac(\mathbb{Z}\mathbb{T}_\infty)$, $i \in \mathbb{Z}$, given by (1.8) do not depend on j and belong to $\mathbb{Z}\mathbb{T}_\infty$.*

Appendix A. Noncommutative localizations

Recall that for a multiplicative monoid S its *linearization* $\mathbb{Z}S$ is the ring $\mathbb{Z}S = \bigoplus_{s\in S} \mathbb{Z}\cdot [s]$ with the natural extension of multiplication on S.

Given a multiplicative submonoid S of a unital ring R, define the *universal localization* $R[S^{-1}]$ of R by S to be quotient of the free product $R * (\mathbb{Z}S^{op})$ by the ideal generated by all elements of the form $s * [s] - 1$, $[s] * s - 1$ for any $s \in S$.

By definition, one has a canonical ring homomorphism

$$R \to R[S^{-1}] . \tag{A.1}$$

In other words, $R[S^{-1}]$ is the unital ring R' with the universal property that one has a ring homomorphism $R \to R'$ under which the image of each element of S in invertible.

Note that (A.1) in not always injective. For each unital ring R denote by $R^\times$ the set of all units (i.e., invertible elements) in R.

The following fact is obvious.

Lemma A.1. *For any ring homomorphism $\varphi : R \to R'$ and any submonoid $S \subset R \setminus \{0\}$ such that $\varphi(S) \subset (R')^\times$ there is a unique ring homomorphism $\varphi_S : R[S^{-1}] \to R'$ such that the composition $R \to R[S^{-1}] \to R'$ is φ.*

For each submonoid $S \subset R \setminus \{0\}$ define its *saturation* $\hat{S}$ to be the set of all $r \in R$ such that the image of r in $R[S^{-1}]$ is invertible. Clearly, $\hat{S}$ is a submonoid of $R \setminus \{0\}$ containing S. We say that S is saturated if $\hat{S} = S$. The following obvious fact justifies this definition.

Lemma A.2. *For any submonoid $S \subset R \setminus \{0\}$ one has $R[S^{-1}] = R[\hat{S}^{\,-1}]$. Moreover, $\hat{S}$ is the largest submonoid of $R \setminus \{0\}$ with this property.*

Following Malcev and Cohn, we say that a unital ring is of class $\mathcal{E}$ if it can be embedded into a skew-field.

Lemma A.3. *Let R be any ring of class $\mathcal{E}$. Then for any multiplicative submonoid S of $R \setminus \{0\}$ the canonical homomorphism (A.1) is injective.*

Proof. Indeed, let $\mathcal{F}$ be a skew field and $\varphi : R \to \mathcal{F}$ be a monomorphism. By definition, for any submonoid S of $R \setminus \{0\}$, φ factors as $\varphi = g \circ f$, where $f : R \to R[S^{-1}]$ and $g : R[S^{-1}] \to \mathcal{F}$ are canonical homomorphisms. Since φ is a monomorphism, then f is also a monomorphism. $\square$

Definition A.4. For a ring R of class $\mathcal{E}$ we say that a submonoid S of $R \setminus \{0\}$ is *divisible* if $R[S^{-1}]$ is also of class $\mathcal{E}$.

Following Cohn, we say that a submonoid S of $R \setminus \{0\}$ is *factor-closed* if for any $a, b \in R \setminus \{0\}$, $ab \in S$ implies that $a, b \in S$.

Proposition A.5. *Let R be of class $\mathcal{E}$ and S be a divisible submonoid of $R \setminus \{0\}$. Then the saturation $\hat{S}$ of S is a factor-closed submonoid of $R \setminus \{0\}$.*

Proof. Since S is divisible, in particular, the canonical homomorphism $R \to R' = R[S^{-1}] = R[\hat{S}^{-1}]$ is injective. It suffices to prove that if $x, y \in R$ such that $xy \in \hat{S}$, then $x \in \hat{S}$, $y \in \hat{S}$. Indeed, let $z := (xy)^{-1}$ and $t := yzx - 1$ in R'. By definition, $xyz = 1 = zxy$. This implies that $xt = xyzx - x = 1 \cdot x - x = 0$. Since R' has no zero divisors and $x \neq 0$, then $t = 0$, i.e., $(yz)x = 1$. Since $x(yz) = 1$, we see that x is invertible in R' hence $x \in \hat{S}$. Similarly, $y \in \hat{S}$ as well.

The proposition is proved. $\square$

Below we provide a sufficient criterion for a group algebra of a group to belong to class $\mathcal{E}$ and for divisibility of some of its submonoids.

Definition A.6. A group G is called 1-*relator torsion-free* if G is isomorphic to $F/\langle x\rangle$ where F is a finitely generated free group, $x \in F \setminus \{1\}$ is not a proper power in F, and $\langle x\rangle$ denotes the normal subgroup of F generated by x.

Results of Malcev, Newman, J. Lewin and T. Lewin (see e.g., [5, Section 8.7], [22]) imply the following.

Theorem A.7. *Let G be any finitely generated free group or any* 1*-relator torsion free group. Then the group algebra $R = \mathbb{Q}G$ is of class $\mathcal{E}$. In particular, for any submonoid $S \subset \mathbb{Q}G \setminus \{0\}$ the canonical homomorphism* (A.1) *is injective.*

We will need the following result, which is a particular case of [26, Theorem 10.11] (here $\mathcal{F}_\ell$ denotes a free skew field freely generated by ℓ elements).

Proposition A.8. *Let $\ell \geq 1$ and assume that ℓ elements $t_1, \ldots, t_\ell$ of $\mathcal{F}_\ell$ generate $\mathcal{F}_\ell$. Then $t_1, \ldots, t_\ell$ are free generators. In particular, the assignments $c_i \mapsto t_i$ for $i = 1, \ldots, \ell$ define an injective homomorphism of algebras $\mathbb{Q}F_\ell \hookrightarrow \mathcal{F}_\ell$.*

Following Cohn, we say that a ring R is a left (resp. right) *semifir* if each finitely generated left (resp. right) ideal J is isomorphic to R^n for a unique $n = n_J$. R is called a semifir if it is both left and right semifir. We use below the standard definition of a universal R-field, see [5, Section 7.2].

Theorem A.9. *Let R be a semifir. Then:*

(a) There exists a universal skew field $Frac(R)$ containing R as a subalgebra and generated by R.

(b) For any factor-closed submonoid S of $R \setminus \{0\}$ the canonical homomorphism $R_S \to Frac(R)$ is injective.

Proof. Recall from [5] that:

- an $n \times n$ matrix A over a unital ring R is *full* if for any factorization $A = BC$ for some $n \times p$ matrix B and a $p \times n$ matrix C one has $p \geq n$;
- A homomorphism $f : R \to R'$ is *honest* if the image of each full matrix is full.
- A set Σ of square matrices over a unital ring R is *multiplicative* if any upper block-triangular matrix with diagonal in Σ also belongs to Σ and Σ is closed under simultaneous permutation of rows and columns.
- A set Σ of matrices over a unital ring R is called *factor-closed* if $AB \in \Sigma$ for some $n \times n$ matrices A and B over R implies that $A, B \in \Sigma$.
- For any set Σ of square matrices over a unital ring R, R_Σ denotes the *universal localization* ([5, Theorem 2.1]) so that the image of each element of Σ under the canonical

homomorphism $R \to R_\Sigma$ is an invertible matrix (e.g., $R_S = R[S^{-1}]$ in the notation as above).

Then Theorem A.9(a) immediately follows from the following result.

Theorem A.10. *[5, Section 7.5, Corollary 5.11]) For each semifir R the universal localization $Frac(R) := R_\Phi$, where Φ is the set of full matrices over R, is a skew field and the canonical homomorphism $R \to Frac(R)$ is honest (hence injective).*

To prove (b) we need following results from [5].

Proposition A.11. *([5, Section 7.5, Proposition 5.7(ii)]) Given unital rings R and R' and a honest homomorphism $f : R \to R'$, then for any factor-closed multiplicative set Σ of square matrices over R, the canonical homomorphism $f_\Sigma : R_\Sigma \to R'$ is injective.*

For any $S \subset R$ denote by Σ_S the set of all matrices over R of the form PMQ where P and Q are invertible matrices over R and M is an upper triangular matrix over R with diagonal entries in S.

Lemma A.12. *([5, Section 7.5, Lemma 10.1]) Let R be a semifir. Then for any factor-closed submonoid S of $R \setminus \{0\}$ the set Σ_S is factor-closed and multiplicative.*

Finally, letting R be a semifir and $R' = Frac(R)$ in Proposition A.11, $\Sigma = \Sigma_S$ as in Lemma A.12 and taking into account that $R[S^{-1}] = R_S = R_{\Sigma_S}$, we finish the proof of part (b).

Theorem A.9 is proved. $\square$

It is well-known (see e.g., [8]) that for any finitely generated free group F its group algebra $R = \mathbb{Q}F$ is a semifir. Therefore, Theorem A.9 implies the following corollary.

Corollary A.13. *Let F be a finitely generated free group and $R = \mathbb{Q}F$. Then any factor-closed submonoid S of $R \setminus \{0\}$ is divisible, more precisely, $R[S^{-1}] \subset Frac(R)$.*

Remark A.14. Based on Theorem A.7, we expect that an analogue of Corollary A.13 also holds for $R = \mathbb{Q}G$, where G is a torsion-free 1-relator group.

Given a unital ring R, following Cohn, we say that:

• Elements $a, b \in R$ are *similar* if the right R-modules R/aR and R/bR are isomorphic (clearly, similarity is an equivalence relation on R).

• An element $p \in R \setminus R^\times$ is *prime* if for any factorization $p = p'p''$ one has: either $p' \in R^\times$ or $p'' \in R^\times$.

• A unital ring R is a (noncommutative) *unique factorization domain (UFD)* if each nonzero non-unit admits a prime factorization and for any two prime factorizations of a non-unit $x \in R$:

$$x = p_1 \cdots p_r = q_1 \cdots q_s$$

one has $s = r$ and q_i is similar to $p_{\sigma(i)}$ for $i = 1, \ldots, r$ where σ is a permutation of $\{1, \ldots, r\}$.

Proposition A.15. *Let R be a UFD and S be a submonoid of $R \setminus \{0\}$. Then S is factor-closed iff it is generated by $R^\times$ and a (empty or not) set P which is the union of similarity classes of prime elements in R.*

Proof. Denote by P the set of all primes in S and by S_P the submonoid of $R \setminus \{0\}$ generated by $R^\times$ and P. Clearly, $S_P \subset S$.

Suppose that S is factor-closed. Let us show that $S = S_P$. We proceed by contradiction, i.e., suppose that there is at least one element $x \in S \setminus S_P$. Then x is not a unit hence x has a prime factorization $x = p_1 \cdots p_r$. If $r = 1$, then $x = p_1 \in S$ hence $x \in S_P$ and we arrive at the contradiction. If $r \geq 2$, then since S is factor-closed, we have $p_i \in S$ for $i = 1, \ldots, r$. Hence $x \in S_P$ and we arrive at the contradiction once again.

Suppose that P is a union of similarity classes and $S = S_P$. Let us prove that S is factor-closed. Suppose that $ab \in S$ for some $a, b \in R$. Let us show that $a, b \in S$. If either a or b is a unit, we have nothing to prove because $R^\times \subset S$. Thus, suppose that $a, b \in R \setminus R^\times$ and let

$$a = p_1 \cdots p_{r'}, \ b = p_{r'+1} \cdots p_r$$

be respective prime factorizations with $1 \leq r' < r$, where $p_1, \ldots, p_r$ are some primes in R. On the other hand, since ab is a non-unit element of S, it admits a prime factorization $ab = q_1 \cdots q_s$ in S, where $q_1, \ldots, q_s \in P$. Comparing the factorizations $p_1 \cdots p_r = q_1 \cdots q_s$ and using the fact that R is UFD, we obtain: $r = s$ and each p_i is similar to one of q_j. Since all primes similar to each q_j belongs to P, we obtain $p_1, \ldots, p_r \in P$ hence $a \in S$, $b \in S$.

The proposition is proved. $\square$

Remark A.16. The class of noncommutative UFD's is rather large: it contains group rings $\mathbb{Q}F$, where F is any finitely generated free group (see e.g., [6, Theorem 3.4, Proposition 3.5 and Corollary]).

Note however, that similarity classes of primes may contain some "unexpected" elements. For instance, if R is the free ring in x, y then $xy + 1$ and $yx + 1$ are similar (see e.g. [6]). This motivates the following definition.

Definition A.17. Given a ring R, we say that an element $a \in R \setminus \{0\}$ is *self-similar* if all elements similar to a are of the form uau', where $u, u' \in R^\times$.

Taking into account that $(\mathbb{Q}F)^\times = \mathbb{Q}^\times \cdot F$ for a free (or, more generally, an ordered) group F (see e.g., [21, Theorem 6.29]), we obtain the following conjectural characterization of certain self-similar primes in $\mathbb{Q}F$.

Conjecture A.18. *Let F be a free group freely generated by $t_1, \ldots, t_m$, $m \geq 2$. Then for $k = 2, \ldots, m$ the element $\tau_k := t_1 + \ldots + t_k$ is a self-similar prime, e.g., all elements of $\mathbb{Q}F$ similar to τ_k belong to $\mathbb{Q}^\times \cdot F \cdot \tau_k \cdot F$.*

Remark A.19. This conjecture was shaped during our discussions with George Bergman, Dolors Herbera, and Alexander Lichtman. We are immensely grateful to these mathematicians.

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