DOUBLE CANONICAL BASES

ARKADY BERENSTEIN AND JACOB GREENSTEIN

Abstract. We introduce a new class of bases for quantized universal enveloping algebras $U_q(\mathfrak{g})$ and other doubles attached to semisimple and Kac-Moody Lie algebras. These bases contain dual canonical bases of upper and lower halves of $U_q(\mathfrak{g})$ and are invariant under many symmetries including all Lusztig's symmetries if $\mathfrak{g}$ is semisimple. It also turns out that a part of a double canonical basis of $U_q(\mathfrak{g})$ spans its center.

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1. Introduction and main results

The goal of this paper is to construct a canonical basis $\mathbf{B}_{\mathfrak{g}}$ of a quantized enveloping algebra $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple or a Kac-Moody Lie algebra. For instance, if $\mathfrak{g}=\mathfrak{sl}_2$, then $\mathbf{B}_{\mathfrak{g}}$ is given by

$$\mathbf{B}_{\mathfrak{sl}_2}=\{q^{n(m_--m_+)}K^nC^{(m_0)}F^{m_-}E^{m_+}\,|\,n\in\mathbb{Z},\,m_0,m_\pm\in\mathbb{Z}_{\ge 0},\min(m_-,m_+)=0\}\ , \tag{1.1}$$

where we used a slightly non-standard presentation of $U_q(\mathfrak{sl}_2)$ (obtained from the more familiar one by rescaling generators $E\mapsto(q^{-1}-q)E$, $F\mapsto(q-q^{-1})F$)

$$U_q(\mathfrak{sl}_2):=\langle E,F,K^{\pm1}\,:\,KEK^{-1}=q^2E,\,KFK^{-1}=q^2F,\,EF-FE=(q^{-1}-q)(K-K^{-1})\rangle.$$

Here the $C^{(m)}$ are central elements of $U_q(\mathfrak{sl}_2)$ defined by $C^{(0)}=1$, $C=C^{(1)}=EF-q^{-1}K-qK^{-1}=FE-qK-q^{-1}K^{-1}$ and $C\cdot C^{(m)}=C^{(m+1)}+C^{(m-1)}$ for $m\ge 1$.

We call $\mathbf{B}_{\mathfrak{sl}_2}$ double canonical because of the following remarkable properties (we will explain later, in §4.1 the reason why we must use Chebyshev polynomials $C^{(m)}$ instead of C^m).

1. Each element of $\mathbf{B}_{\mathfrak{sl}_2}$ is homogeneous and is fixed by the bar-involution $u\mapsto\overline{u}$, which is the $\mathbb{Q}$-anti-automorphism of $U_q(\mathfrak{sl}_2)$ given by $\bar q=q^{-1}$, $\bar E=E$, $\bar F=F$, $\bar K=K$.
2. $\mathbf{B}_{\mathfrak{sl}_2}$ is invariant, as a set, under the $\mathbb{Q}(q)$-linear anti-automorphisms $u\to u^*$ and $u\to u^t$ given respectively by $E^*=E$, $F^*=F$, $K^*=K^{-1}$ and $E^t=F$, $F^t=E$, $K^t=K$; and under the rescaled Lusztig's symmetry T given by $T(E)=qFK^{-1}$, $T(F)=q^{-1}KE$, $T(K)=K^{-1}$.
3. Each monomial in E,F, $K^{\pm1}$ is in the $\mathbb{Z}_{\ge0}[q,q^{-1}]$-span of $\mathbf{B}_{\mathfrak{sl}_2}$.
4. $\mathbf{B}_{\mathfrak{sl}_2}$ is compatible with the filtered mock Peter-Weyl components $\mathcal{J}_s=\sum_{r=0}^{s}(\operatorname{ad}U_q(\mathfrak{sl}_2))(K^r)$ (see e.g. [13]), where ad denotes an adjoint action of the Hopf algebra $U_q(\mathfrak{sl}_2)$ on itself.

Remark 1.1. It should be noted that this basis is rather different from Lusztig's canonical basis since the latter is in the *modified* quantized enveloping algebra $\dot{U}_q(\mathfrak{sl}_2)$, as defined in [19, §23.1.1] and we are not aware of any relationship between these bases. It would also be interesting to compare our bases with the ones announced by Fan Qin in [21, 22]. Finally, it should be noted that John Foster constructed in [11] a basis of $U_q(\mathfrak{sl}_2)$ which differs from (1.1) in that Chebyshev polynomials $C^{(m)}$ are replaced by C^m.

We establish properties of $\mathbf{B}_{\mathfrak{sl}_2}$ in §§4.1,4.2 and §5.4.

To construct $\mathbf{B}_{\mathfrak{g}}$ for any symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ we need some notation. Fix a triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ and let $\tilde{\mathfrak{g}}:=\mathfrak{g}\oplus\mathfrak{h}$, which we view as the Drinfeld double of the Borel subalgebra $\mathfrak{b}_+=\mathfrak{n}_+\oplus\mathfrak{h}$. Let $U_q(\tilde{\mathfrak{g}})$ be the quantized enveloping algebra of $\tilde{\mathfrak{g}}$ of *adjoint type* over $\Bbbk=\mathbb{Q}(q^{\frac12})$. Thus, $U_q(\tilde{\mathfrak{g}})$ is the $\Bbbk$-algebra generated by the E_i, F_i, $K_{\pm i}$, $i\in I$ subject to the relations: $\mathcal{K}=\langle K_{+i},K_{-i}\,:\,i\in I\rangle$ is commutative and

$$[E_i,F_j]=\delta_{ij}(q_i^{-1}-q_i)(K_{+i}-K_{-i}),\quad K_{\pm i}E_j=q_i^{\pm a_{ij}}E_jK_{\pm i},\quad K_{\pm i}F_j=q_i^{\mp a_{ij}}F_jK_{\pm i}\ , \tag{1.2}$$

$$\sum_{r,s\ge0,\,r+s=1-a_{ij}}(-1)^sE_i^{\langle r\rangle}E_jE_i^{\langle s\rangle}=\sum_{r,s\ge0,\,r+s=1-a_{ij}}(-1)^sF_i^{\langle r\rangle}F_jF_i^{\langle s\rangle}=0 \tag{1.3}$$

for all $i,j\in I$, where $A=(a_{ij})_{i,j\in I}$ is the Cartan matrix of $\mathfrak{g}$, the d_i are positive integers such that $DA=(d_ia_{ij})_{i,j\in I}$ is symmetric, $q_i=q^{d_i}$, $X_i^{\langle k\rangle}:=\big(\prod\limits_{s=1}^{k}\langle s\rangle_{q_i}\big)^{-1}X_i^k$ and $\langle s\rangle_v=v^s-v^{-s}$.

Remark 1.2. The reason for choosing such a non-standard presentation (1.2)-(1.3) of $U_q(\tilde{\mathfrak{g}})$ is that one can now view $U_q(\tilde{\mathfrak{g}})$ as a quantized coordinate algebra of $\mathcal{O}_q(\tilde{G}^*)$, were $\tilde{G}^*$ is the Poisson dual group of the Lie group $\tilde{G}$ of $\tilde{\mathfrak{g}}$. This agrees with Drinfeld's observation that the dual Hopf algebra of the complete Hopf algebra $U_h(\tilde{\mathfrak{g}}^*)$ (where $\tilde{\mathfrak{g}}^*$ is the Lie dual bialgebra of the Lie bialgebra $\tilde{\mathfrak{g}}$) is, on the one hand, $\mathcal{O}_h(\tilde{G}^*)$ and, on the other hand, is isomorphic to $U_h(\tilde{\mathfrak{g}})$. In particular, our basis $\mathbf{B}_\mathfrak{g}$ will have a "dual canonical" flavor.

Our strategy for constructing $\mathbf{B}_\mathfrak{g}$ is as follows. First, we define quantum Heisenberg algebras $\mathcal{H}_q^\pm(\mathfrak{g})$ by $\mathcal{H}_q^\pm(\mathfrak{g}):=U_q(\tilde{\mathfrak{g}})/\langle K_{\mp i}, i\in I\rangle$. Then we use a variant of Lusztig's Lemma (Proposition 2.3) to construct the double canonical basis $\mathbf{B}_\mathfrak{g}^+$ of $\mathcal{H}_q^+(\mathfrak{g})$ (see Theorem 1.3 below). Furthermore, using a natural embedding of $\Bbbk$-vector spaces $\iota_+:\mathcal{H}_q^+(\mathfrak{g})\hookrightarrow U_q(\tilde{\mathfrak{g}})$, which splits the canonical projection $\pi_+:U_q(\tilde{\mathfrak{g}})\twoheadrightarrow\mathcal{H}_q^+(\mathfrak{g})$ and the Lusztig's lemma variant again, we build the *double canonical basis* $\mathbf{B}_{\tilde{\mathfrak{g}}}$ of $U_q(\tilde{\mathfrak{g}})$ out of $\iota_+(\mathbf{B}_\mathfrak{g}^+)$. Finally, the desired basis $\mathbf{B}_\mathfrak{g}$ is just the image of $\mathbf{B}_{\tilde{\mathfrak{g}}}$ under the canonical projection $U_q(\tilde{\mathfrak{g}})\twoheadrightarrow U_q(\mathfrak{g})=U_q(\tilde{\mathfrak{g}})/\langle K_{+i}K_{-i}-1, i\in I\rangle$.

More precisely, by a slight abuse of notation we denote by $E_i,F_i,\ K_{+i}$ (respectively K_{-i}) the images of E_i,F_i,K_{+i} (respectively K_{-i}) under the canonical projection $\pi_+:U_q(\tilde{\mathfrak{g}})\twoheadrightarrow\mathcal{H}_q^+(\mathfrak{g})$ (respectively under $\pi_-:U_q(\tilde{\mathfrak{g}})\twoheadrightarrow\mathcal{H}_q^-(\mathfrak{g})$). It is obvious (and well-known) that, applying $\pi_\pm$ to the triangular decomposition $U_q(\tilde{\mathfrak{g}})=\mathcal{K}_-\otimes\mathcal{K}_+\otimes U_q^-\otimes U_q^+$, where $U_q^-=\langle F_i\,:\,i\in I\rangle$, $U_q^+=\langle E_i\,:\,i\in I\rangle$, $\mathcal{K}_\pm=\langle K_{\pm i}\,:\,i\in I\rangle$, one obtains a triangular decomposition

$$\mathcal{H}_q^\pm(\mathfrak{g})=\mathcal{K}_\pm\otimes U_q^-\otimes U_q^+\ .$$

Let $\Gamma=\bigoplus_{i\in I}\mathbb{Z}_{\ge0}\alpha_i$ be a free abelian monoid, $\widehat{\Gamma}=\Gamma\oplus\Gamma$ and set $\alpha_{-i}=(\alpha_i,0),\alpha_{+i}=(0,\alpha_i)\in\widehat{\Gamma}$. Then it is easy to see that $U_q(\tilde{\mathfrak{g}})$ and $\mathcal{H}_q^\pm(\mathfrak{g})$ are graded by $\widehat{\Gamma}$ via $\deg_{\widehat{\Gamma}}E_i=\alpha_{+i}$, $\deg_{\widehat{\Gamma}}F_i=\alpha_{-i}$ and $\deg_{\widehat{\Gamma}}K_{\pm i}=\alpha_{+i}+\alpha_{-i}$. Denote by $\mathbf{K}_+$ (respectively, $\mathbf{K}_-$) the submonoid of $\mathcal{K}$ generated by the K_{+i} (respectively, the K_{-i}), $i\in I$ and let $\mathbf{K}=\mathbf{K}_-\mathbf{K}_+$. Sometimes it is convenient to regard U_q^+ as graded by Γ.

Denote by $\mathbf{B}_{\mathfrak{n}_\pm}$ the *dual canonical basis* of $U_q^\pm$ (see [19, Chapter 14] and Section 3 for the details) i.e. the upper global crystal basis of [18]. By definition, each element of $\mathbf{B}_{\mathfrak{n}_\pm}$ is homogeneous and is fixed under the involutive $\mathbb{Q}$-linear anti-automorphism $\bar{\cdot}$ of $U_q(\tilde{\mathfrak{g}})$ determined by $\overline{q^{\frac12}}=q^{-\frac12}$, $\overline{E}_i=E_i$, $\overline{F}_i=F_i$, $\overline{K_{\pm i}}=K_{\pm i}$. For instance, if $\mathfrak{g}=\mathfrak{sl}_2$ then $\mathbf{B}_{\mathfrak{n}_+}=\{E^r\,:\,r\in\mathbb{Z}_{\ge0}\}$ and $\mathbf{B}_{\mathfrak{n}_-}=\{F^r\,:\,r\in\mathbb{Z}_{\ge0}\}$.

We have an action $\diamond$ of the algebra $\mathcal{K}$ on $U_q(\tilde{\mathfrak{g}})$ defined by

$$K_{\pm i}\diamond x:=q_i^{\mp\frac12\alpha_i^\vee(\deg_{\widehat{\Gamma}}x)}K_{\pm i}x,\tag{1.4}$$

where $\alpha_i^\vee\in\operatorname{Hom}_\mathbb{Z}(\widehat{\Gamma},\mathbb{Z})$ is defined by $\alpha_i^\vee(\alpha_{\pm j})=\pm a_{ij}$ and $x\in U_q(\tilde{\mathfrak{g}})$ is homogeneous. This action is more suitable for our purposes than the left multiplication due to the following easy property

$$\overline{K\diamond x}=\overline{K}\diamond\overline{x},\qquad K\in\mathcal{K},\ x\in U_q(\tilde{\mathfrak{g}}).\tag{1.5}$$

Note that this action, as well as the involution $\bar{\cdot}$, factors through to a $\mathcal{K}_\pm$-action and an anti-involution $\bar{\cdot}$ on $\mathcal{H}_q^\pm(\mathfrak{g})$ via the canonical projection $\pi_\pm:U_q(\tilde{\mathfrak{g}})\twoheadrightarrow\mathcal{H}_q^\pm(\mathfrak{g})$ and (1.5) holds.

We will show (Propositions 2.7 and 3.13) that for each pair $(b_-,b_+)\in\mathbf{B}_{\mathfrak{n}_-}\times\mathbf{B}_{\mathfrak{n}_+}$ there exists a unique monic $\mathbf{d}_{b_-,b_+}\in\mathbb{Z}[q+q^{-1}]$ of minimal degree such that in $U_q(\tilde{\mathfrak{g}})$ one has

$$\mathbf{d}_{b_-,b_+}(b_+b_--b_-b_+)\in\sum_{K\in\mathbf{K}\setminus\{1\},\,b'_\pm\in\mathbf{B}_{\mathfrak{n}_\pm}}\mathbb{Z}[q,q^{-1}]\mathbf{d}_{b'_-,b'_+}K\diamond(b'_-b'_+).$$

It turns out all $\mathbf{d}_{b_-,b_+}$ are, up to a power of q, products of cyclotomic polynomials in q (Proposition 3.9) and that for $\mathfrak{g}$ semisimple $\mathbf{d}_{b_-,b_+}=1$ for all $b_\pm\in\mathbf{B}_{\mathfrak{n}_\pm}$ (Theorem 3.11). Some examples are shown in §4.3.

Main Theorem 1.3. *For any $(b_-,b_+)\in\mathbf{B}_{\mathfrak{n}_-}\times\mathbf{B}_{\mathfrak{n}_+}$ there is a unique element $b_-\circ b_+\in\mathcal{H}_q^+(\mathfrak{g})$ fixed by $\bar{\cdot}$ and satisfying*

$$b_-\circ b_+-\mathbf{d}_{b_-,b_+}b_-b_+\in\sum q\mathbb{Z}[q]\mathbf{d}_{b'_-,b'_+}K_+\diamond(b'_-b'_+)$$

where the sum is over $K_+\in\mathbf{K}_+\setminus\{1\}$, $b'_\pm\in\mathbf{B}_{\mathfrak{n}_\pm}$ such that $\deg_{\widehat{\Gamma}}b'_-b'_++\deg_{\widehat{\Gamma}}K_+=\deg_{\widehat{\Gamma}}b_-b_+$.

We prove this theorem in Section 3 using a variant of Lusztig's Lemma (Proposition 2.3) which we refer to as the equivariant Lusztig's Lemma.

Corollary 1.4. *The set $\mathbf{B}_{\mathfrak{g}}^+:=\{K_+\diamond(b_-\circ b_+)\,:\,(b_-,b_+)\in\mathbf{B}_{\mathfrak{n}_-}\times\mathbf{B}_{\mathfrak{n}_+},\,K_+\in\mathbf{K}_+\}$ is a $\bar{\cdot}$-invariant $\mathbb{Q}(q^{\frac12})$-linear basis of $\mathcal{H}_q^+(\mathfrak{g})$.*

We call $\mathbf{B}_{\mathfrak{g}}^+$ the *double canonical basis* of $\mathcal{H}_q^+(\mathfrak{g})$ (the double canonical basis $\mathbf{B}_{\mathfrak{g}}^-$ of $\mathcal{H}_q^-(\mathfrak{g})$ is defined verbatim, with q replaced by q^{-1}).

Furthermore, we have a natural, albeit not $\bar{\cdot}$-equivariant, inclusion $\iota_+:\mathcal{H}_q^+(\mathfrak{g})=\mathcal{K}_+\otimes U_q^-\otimes U_q^+\hookrightarrow\mathcal{K}_-\otimes(\mathcal{K}_+\otimes U_q^-\otimes U_q^+)=U_q(\tilde{\mathfrak{g}})$.

Main Theorem 1.5. *For any $(b_-,b_+)\in\mathbf{B}_{\mathfrak{n}_-}\times\mathbf{B}_{\mathfrak{n}_+}$ there is a unique element $b_-\bullet b_+\in U_q(\tilde{\mathfrak{g}})$ fixed by $\bar{\cdot}$ and satisfying*

$$b_-\bullet b_+-\iota_+(b_-\circ b_+)\in\sum q^{-1}\mathbb{Z}[q^{-1}]K\diamond\iota_+(b'_-\circ b'_+)$$

where the sum is taken over $K\in\mathbf{K}\setminus\mathbf{K}_+$ and $b'_\pm\in\mathbf{B}_{\mathfrak{n}_\pm}$ such that $\deg_{\widehat{\Gamma}}b'_-b'_++\deg_{\widehat{\Gamma}}K=\deg_{\widehat{\Gamma}}b_-b_+$.

We prove this Theorem in Section 2 using the equivariant Lusztig's Lemma (Proposition 2.3).

Corollary 1.6. *The set $\mathbf{B}_{\tilde{\mathfrak{g}}}:=\{K\diamond(b_-\bullet b_+),(b_-,b_+)\in\mathbf{B}_{\mathfrak{n}_-}\times\mathbf{B}_{\mathfrak{n}_+},\,K\in\mathbf{K}\}$ is a $\mathbb{Q}(q^{\frac12})$-basis of $U_q(\tilde{\mathfrak{g}})$.*

We call $\mathbf{B}_{\tilde{\mathfrak{g}}}$ the *double canonical basis* of $U_q(\tilde{\mathfrak{g}})$.

Remark 1.7. Note that $\mathbf{B}_{\tilde{\mathfrak{g}}}$ contains both bases $\mathbf{B}_{\mathfrak{n}^\pm}$ as subsets and therefore has a "dual flavor".

Let $U_q(\tilde{\mathfrak{g}},J)$ (respectively, $U_q(J,\tilde{\mathfrak{g}})$), $J\subset I$ be the subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by the $\mathcal{K}U_q^+$ and F_j, $j\in J$ (respectively, $\mathcal{K}U_q^-$ and E_j, $j\in J$) and let $U_q(J_-,\tilde{\mathfrak{g}},J_+)=U_q(\tilde{\mathfrak{g}},J_+)\cap U_q(J_-,\tilde{\mathfrak{g}})$, $J_\pm\subset I$. The following is immediate.

Theorem 1.8. *For any $J_\pm\subset I$, $\mathbf{B}_{\tilde{\mathfrak{g}}}\cap U_q(J_-,\tilde{\mathfrak{g}},J_+)$ is a basis of $U_q(J_-,\tilde{\mathfrak{g}},J_+)$.*

Remark 1.9. Analogously to the classical ($q=1$) case (cf. e.g. [14]), it is natural to call $U_q(J_-,\tilde{\mathfrak{g}},J_+)$ quantum bi-parabolic (or seaweed) algebras.

As one should expect from a canonical basis, $\mathbf{B}_{\tilde{\mathfrak{g}}}$ is preserved, as a set, by various symmetries of $U_q(\tilde{\mathfrak{g}})$. First, let $x\mapsto x^t$ and $x\to x^*$ be the $\mathbb{Q}(q^{\frac12})$-linear anti-automorphism of $U_q(\tilde{\mathfrak{g}})$ defined by

$$E_i^t=F_i,\quad F_i^t=E_i,\quad (K_{\pm i})^t=K_{\pm i}\qquad\text{and}\qquad E_i^*=E_i,\quad F_i^*=F_i,\quad (K_{\pm i})^*=K_{\mp i}.$$

Then $\mathbf{B}_{\mathfrak{n}_\pm}^t=\mathbf{B}_{\mathfrak{n}_\mp}$ while * preserves both $\mathbf{B}_{\mathfrak{n}_\pm}$ as sets.

Theorem 1.10. $\mathbf{B}_{\tilde{\mathfrak{g}}}^t=\mathbf{B}_{\tilde{\mathfrak{g}}}$. *More precisely, for all $b_\pm\in\mathbf{B}_{\mathfrak{n}_\pm}$, $K\in\mathbf{K}$ be have $(K\diamond(b_-\bullet b_+))^t=K\diamond(b_+)^t\bullet(b_-)^t$.*

We prove this Theorem in Section 2.

Conjecture 1.11. $\mathbf{B}_{\tilde{\mathfrak{g}}}^*=\mathbf{B}_{\tilde{\mathfrak{g}}}$. *More precisely, for all $b_\pm\in\mathbf{B}_{\mathfrak{n}_\pm}$, $K\in\mathbf{K}$ be have $(K\diamond(b_-\bullet b_+))^*=K^*\diamond(b_-)^*\bullet(b_+)^*$.*

Remark 1.12. It is easy to see that this conjecture implies that $\mathbf{B}_{\tilde{\mathfrak{g}}}$ can also be obtained by replacing $\mathcal{H}_q^+(\mathfrak{g})$ with $\mathcal{H}_q^-(\mathfrak{g})$ and interchanging q and q^{-1} in Theorems 1.3 and 1.5.

It turns out that $\mathbf{B}_{\mathfrak{g}}$ and $\mathbf{B}_{\tilde{\mathfrak{g}}}$ are preserved by appropriately modified Lusztig's symmetries. First of all, set $\widehat{U}_q(\tilde{\mathfrak{g}}) = U_q(\tilde{\mathfrak{g}})[\mathbf{K}^{-1}]$. Clearly, $\bar{\cdot}$, ^{t} and * extend naturally to that algebra.

Theorem 1.13. (a) *For each $i \in I$ there exists a unique automorphism T_i of $\widehat{U}_q(\tilde{\mathfrak{g}})$ which satisfies $T_i(K_{\pm j}) = K_{\pm j} K_{\pm i}^{-a_{ij}}$ and*

$$T_i(E_j) = \begin{cases} q_i^{-1} K_{+i}^{-1} F_i, & i = j \\ \displaystyle\sum_{r+s=-a_{ij}} (-1)^r q_i^{s+\frac{1}{2}a_{ij}} E_i^{\langle r\rangle} E_j E_i^{\langle s\rangle}, & i \neq j \end{cases}$$

$$T_i(F_j) = \begin{cases} q_i^{-1} K_{-i}^{-1} E_i, & i = j \\ \displaystyle\sum_{r+s=-a_{ij}} (-1)^r q_i^{s+\frac{1}{2}a_{ij}} F_i^{\langle r\rangle} F_j F_i^{\langle s\rangle}, & i \neq j \end{cases}$$

(b) *For all $x \in \widehat{U}_q(\tilde{\mathfrak{g}})$, $\overline{T_i(x)} = T_i(\overline{x})$, $(T_i(x))^* = T_i^{-1}(x^*)$ and $(T_i(x))^t = T_i^{-1}(x^t)$.*

(c) *The T_i, $i \in I$ satisfy the braid relations on $\widehat{U}_q(\tilde{\mathfrak{g}})$, that is, they define a representation of the Artin braid group $\mathrm{Br}_{\mathfrak{g}}$ of $\mathfrak{g}$ on $\widehat{U}_q(\tilde{\mathfrak{g}})$.*

We prove this Theorem in Section 5.

Remark 1.14. Since for each $i \in I$, T_i preserves the ideal $\mathfrak{J} = \langle K_{+j}K_{-j} - 1 \,:\, j \in I\rangle$, T_i factors through to an automorphism of $U_q(\mathfrak{g}) = U_q(\tilde{\mathfrak{g}})/\mathfrak{J}$ which, for $x \in U_q(\mathfrak{g})$ homogeneous, equals $q_i^{\frac{1}{2}\alpha_i^\vee(\deg x)} T''_{i,-1}(x)$ where $T''_{i,-1}$ is one of Lusztig's symmetries defined in [19, §37.1] (see Lemma 5.2).

Clearly, $\diamond$ extends to the group generated by $\mathbf{K}$ acting on $\widehat{U}_q(\tilde{\mathfrak{g}})$. Then the set $\widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}} := \mathbf{K}^{-1} \diamond \mathbf{B}_{\tilde{\mathfrak{g}}}$ is a $\bar{\cdot}$-invariant basis of $\widehat{U}_q(\tilde{\mathfrak{g}}) = U_q(\tilde{\mathfrak{g}})[\mathbf{K}^{-1}]$.

Conjecture 1.15. *Let $\mathfrak{g}$ be semisimple. Then for all $i \in I$, $T_i(\widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}}) = \widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}}$. In other words, $\mathrm{Br}_{\mathfrak{g}}$ acts on $\widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}}$ by permutations.*

We prove supporting evidence for this conjecture in Section 5. In view of Remark 1.14, the conjecture implies that $T_i(\mathbf{B}_{\mathfrak{g}}) = \mathbf{B}_{\mathfrak{g}}$.

If $\mathfrak{g}$ is infinite dimensional, this does not hold for all elements of $\widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}}$ (see Example 5.6). To amend this conjecture we introduce the following notion. We say that $\mathbf{b} \in \widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}}$ is *tame* if $T_i(\mathbf{b}) \in \widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}}$ for all $i \in I$. We prove (Theorem 5.13) that all elements of $\mathbf{B}_{\mathfrak{n}_\pm}$ are tame.

Conjecture 1.16. *If $\mathbf{b} \in \widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}}$ is tame then $T(\mathbf{b}) \in \widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}}$ for all $T \in \mathrm{Br}_{\mathfrak{g}}$.*

We provide supporting evidence for this conjecture in Section 5. We show some of it below for which more notation is necessary. Let W be the Weyl group of $\mathfrak{g}$. Following [19, §39.4.4], for each $w \in W$ define $T_w \in \mathrm{Br}_{\mathfrak{g}}$ recursively as $T_{s_i} = T_i$ and $T_w = T_{w'}T_{w''}$ for any non-trivial reduced factorization $w = w'w''$, $w', w'' \in W$ (see §5.1 for the details). Define the *quantum Schubert cells* $U_q^+(w)$ and $U_q^-(w)$, $w \in W$ by $U_q^+(w) := T_w(\mathcal{K}U_q^-) \cap U_q^+$ and $U_q^-(w) := U_q^- \cap T_w^{-1}(\mathcal{K}U_q^+)$. Clearly, these are subalgebras of $U_q^\pm$. For $\mathfrak{g}$ semisimple we provide an elementary proof (Proposition 5.4) that $U_q^+(w)$ coincides with the subspace $U_q^+(w,1)$ of U_q^+ defined by Lusztig ([19, §40.2]) via a choice of a reduced decomposition of w, and conjectured that this is the case for all Kac-Moody $\mathfrak{g}$ (Conjecture 5.3[1]). Let $\mathbf{B}_{\mathfrak{n}_\pm}(w) = \mathbf{B}_{\mathfrak{n}_\pm} \cap U_q^\pm(w)$ (since, conjecturally, $U_q^+(w,1) = U_q^+(w)$, by [16, Theorem 4.22] $\mathbf{B}_{\mathfrak{n}_+}(w)$ is a basis of $U_q^+(w)$). The following refines Conjecture 1.15.

[1]While preparing the final version of the present paper we learned that Conjecture 5.3 was proved by Tanisaki in [25]; shortly after an alternative proof was provided by Kimura ([17])

Conjecture 1.17. *$T_w^{-1}(\mathbf{B}_{\mathfrak{n}_+}(ww')) \subset \mathbf{K}^{-1} \diamond \mathbf{B}_{\mathfrak{n}_-}(w) \bullet \mathbf{B}_{\mathfrak{n}_+}(w')$ for all $w, w' \in W$ such that the factorization ww' is reduced.*

Remark 1.18. Note also that this conjecture implies that $\mathbf{K}^{-1} \diamond \mathbf{B}_{\mathfrak{n}_-}(w) \bullet \mathbf{B}_{\mathfrak{n}_+}(w')$ is a basis in the *double Schubert cell* $\mathcal{K}U_q^-(w)U_q^+(w') = T_w^{-1}(U_q^+(ww'))$.

Another application of our construction is a double canonical basis in each *quantum Weyl algebra* $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$. Given a function $\boldsymbol{\epsilon} : I \to \{+,-\}$, $\boldsymbol{\epsilon}(i) = \epsilon_i$, let $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ be a $\Bbbk$-algebra generated by the x_i, $y_i \in I$ subject to the following relations

$$\begin{gathered} x_i y_i - y_i x_i = \epsilon_i (q_i^{-1} - q_i), \qquad x_i y_j = q_i^{\epsilon_i \delta_{\epsilon_i,\epsilon_j} a_{ij}} y_j x_i, \\ \sum_{r+s=1-a_{ij}} (-1)^r q_i^{r\epsilon_j \delta_{\epsilon_i,-\epsilon_j} a_{ij}} x_i^{\langle s\rangle} x_j x_i^{\langle r\rangle} = 0 = \sum_{r+s=1-a_{ij}} (-1)^r q_i^{-r\epsilon_j \delta_{\epsilon_i,-\epsilon_j} a_{ij}} y_i^{\langle s\rangle} y_j y_i^{\langle r\rangle}, \qquad i \neq j. \end{gathered} \tag{1.6}$$

We will show (see Proposition 3.17) that each $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ is naturally a subalgebra of a Heisenberg algebra $\mathcal{H}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ which "interpolates" between $\mathcal{H}_q^+(\mathfrak{g})$ and $\mathcal{H}_q^-(\mathfrak{g})$ (see §3.4 for the details) and obtain the following result.

Theorem 1.19. *Each quantum Weyl algebra $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ has a double canonical basis $\mathbf{B}_{\mathfrak{n}_-} \circ_{\boldsymbol{\epsilon}} \mathbf{B}_{\mathfrak{n}_+}$.*

We prove this Theorem in §3.4.

Remark 1.20. In fact, the $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ are closely related to braided Weyl algebras (see e.g. [15]). Note that algebras $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ and $\mathcal{A}_q^{\boldsymbol{\epsilon}'}(\mathfrak{g})$ are not (anti)isomorphic if $\boldsymbol{\epsilon} \neq -\boldsymbol{\epsilon}'$. Thus, the resulting bases $\mathbf{B}_{\mathfrak{n}_-} \circ_{\boldsymbol{\epsilon}} \mathbf{B}_{\mathfrak{n}_+}$ and $\mathbf{B}_{\mathfrak{n}_-} \circ_{\boldsymbol{\epsilon}'} \mathbf{B}_{\mathfrak{n}_+}$ are rather different. To the best of our knowledge, these bases admit an alternative description similar to that in Theorem 1.3 *only* when $\boldsymbol{\epsilon}$ is a constant function, i.e. $\epsilon_i = +$ (respectively, $\epsilon_i = -$) for all $i \in I$.

Next we discuss the properties of the decomposition of elements of the natural basis of $U_q(\tilde{\mathfrak{g}})$ with respect to $\mathbf{B}_{\tilde{\mathfrak{g}}}$. Define $C_{b'_-,b'_+,K}^{b_-,b_+} \in \Bbbk$ for all $b_\pm, b'_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}$ and $K \in \mathbf{K}$ by

$$\mathbf{d}_{b_-,b_+} b_- b_+ = \sum_{b'_\pm, K} C_{b'_-,b'_+,K}^{b_-,b_+} K \diamond b'_- \bullet b'_+.$$

Then Main Theorem 1.5 immediately implies that $C_{b'_-,b'_+,K}^{b_-,b_+} \in \mathbb{Z}[q, q^{-1}]$. These Laurent polynomials play the role similar to that of Kazhdan-Lusztig polynomials due to the following conjectural result.

Conjecture 1.21. *If $\mathfrak{g}$ is semisimple then $C_{b'_-,b'_+,K}^{b_-,b_+} \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$ for all $b_\pm, b'_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}$, $K \in \mathbf{K}$.*

We provide some examples in Section 4.

Remark 1.22. It is well-known (cf. [19]) that if the Cartan matrix of $\mathfrak{g}$ is symmetric then the structure constants of $\mathbf{B}_{\mathfrak{n}_\pm}$ belong to $\mathbb{Z}_{\geq 0}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. However, we expect that Conjecture 1.21 holds even for those $\mathfrak{g}$ (with non-symmetric Cartan matrix) for which such positivity fails.

Next we discuss the relation between the adjoint action of $\widehat{U}_q(\tilde{\mathfrak{g}})$ on itself and the double canonical basis. We expect that the basis $\widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}}$ is perfect in the sense of the following extension of Definition 5.30 from [5].

Definition 1.23. Let $\mathscr{V}$ be a $\Bbbk$-vector space with linear endomorphisms e_i, $i \in I$ and functions $\varepsilon_i : \mathscr{V} \setminus \{0\} \to \mathbb{Z}$ such that $\varepsilon_i(e_i(v)) = \varepsilon_i(v) - 1$ for all $v \notin \ker e_i$. We say that a basis $\mathbf{B}$ of $\mathscr{V}$ is perfect if for all $i \in I$ and $\mathbf{b} \in \mathbf{B}$ either $e_i(\mathbf{b}) = 0$ or there exists a unique $\mathbf{b}' \in \mathbf{B}$ with $\varepsilon_i(\mathbf{b}') = \varepsilon_i(\mathbf{b}) - 1$ such that

$$e_i(\mathbf{b}) \in \Bbbk^\times \mathbf{b}' + \sum_{\mathbf{b}'' \in \mathbf{B} \,:\, \varepsilon_i(\mathbf{b}'') < \varepsilon_i(\mathbf{b}')} \Bbbk \mathbf{b}''.$$

Consider the adjoint action of $\widehat{U}_q(\tilde{\mathfrak{g}})$ on itself which factors through to an action of $U_q(\mathfrak{g})$ via

$$F_i(x) := F_i x - K_{-i} x K_{-i}^{-1} F_i, \quad E_i(x) := [E_i, x] K_{+i}^{-1}, \quad K_i(x) := K_{+i} x K_{+i}^{-1} \tag{1.7}$$

for all $i \in I$, $x \in \widehat{U}_q(\tilde{\mathfrak{g}})$; here K_i denotes the canonical image of K_{+i} in $U_q(\mathfrak{g})$. It is cuirous that this action preserves the subalgebra $U_q(\tilde{\mathfrak{g}})[\mathbf{K}_+^{-1}] \subset \widehat{U}_q(\tilde{\mathfrak{g}})$ and its ideal generated by the K_{-i}, $i \in I$, hence descends to $\mathcal{H}_q^+(\mathfrak{g})[\mathbf{K}_+^{-1}]$.

Conjecture 1.24. *For any symmetrizable Kac-Moody algebra $\mathfrak{g}$, the bases $\mathbf{K}_+^{-1} \diamond \mathbf{B}_{\mathfrak{n}_-} \circ \mathbf{B}_{\mathfrak{n}_+}$ and $\widehat{\mathbf{B}}_{\tilde{\mathfrak{g}}}$ are perfect with respect to the action (1.7) of $U_q(\mathfrak{g})$ on $\mathcal{H}_q^+(\mathfrak{g})[\mathbf{K}_+^{-1}]$ and $\widehat{U}_q(\tilde{\mathfrak{g}})$, respectively.*

We prove this conjecture for $\mathfrak{g} = \mathfrak{sl}_2$ in §4.2.

We now discuss further the behavior of the double basis with respec to the action (1.7) by using an extension of the remarkable $U_q(\mathfrak{g})$-equivarint map map $U_q(\mathfrak{g})^* \to U_q(\mathfrak{g})$ defined in [10, 23]. In particular, this map yields Joseph's decomposition of the locally finite part of $U_q(\mathfrak{g})$ and, in the finite dimensional case, the center of $U_q(\mathfrak{g})$ (see [4] and Proposition 1.27 below).

Let V be a lowest weight $U_q(\mathfrak{g})$-module (e.g., a Verma module or its unique simple quotient) of lowest weight $-\mu \in \Lambda$ where Λ is an integral weight lattice for $\mathfrak{g}$. Then V inherits a Γ-grading from U_q^+, and we denote $|v|$ the degree of a homogeneous element $v \in V$. Following a remarkable construction of Reshetikhin and Semenov-Tian-Shansky from [23] (see also [9, 13]) we define a $\Bbbk$-linear map $\Xi : V \otimes V \to \check{U}_q(\tilde{\mathfrak{g}})$ by

$$\Xi(u \otimes v) := q^{\frac{1}{2}\underline{\gamma}(|v|) - \frac{1}{2}\underline{\gamma}(|u|)} \sum_{b_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}} q^{\eta(\deg_{\widehat{\Gamma}} b_+)} \langle u \,|\, \check{b}_- \check{b}_+(v) \rangle_V (K_{|\check{b}_+(v)|,0} \diamond b_-)(K_{0,2\mu - |v|} \diamond b_+) \tag{1.8}$$

for any $u, v \in V$ homogeneous, where $\langle \cdot \,|\, \cdot \rangle_V : V \otimes V \to \Bbbk$ is the Shapovalov paring, $\check{b}_\pm \in U_q^\mp$ are elements of Lusztig's canonical basis corresponding to $b_\pm$ and $\check{U}_q(\tilde{\mathfrak{g}})$ is $\widehat{U}_q(\tilde{\mathfrak{g}})$ extended by adjoining elements of the form $K_{0,2\mu}$, $\mu \in \Lambda$ (see §4.4 for the details; the functions $\underline{\gamma} : \Gamma \to \mathbb{Z}$ and $\eta : \widehat{\Gamma} \to \mathbb{Z}$ are defined in §3.1). Let $\underline{\Xi}$ be the composition of Ξ with the canonical projection $\check{U}_q(\tilde{\mathfrak{g}}) \twoheadrightarrow \check{U}_q(\mathfrak{g}) := \check{U}_q(\tilde{\mathfrak{g}})/\langle K_{+i}K_{-i} - 1 \rangle$. In fact, we chose $\mathbf{B}_{\mathfrak{n}_\pm}$ for convenience but the right hand side of (1.8) is independent of the choice of bases of $U_q^\pm$.

Theorem 1.25. *For any symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ we have*

(a) *For any lowest weight module V, Ξ is a homomorphism of $U_q(\mathfrak{g})$-modules $V \otimes V \to \check{U}_q(\tilde{\mathfrak{g}})$ where the action of $U_q(\mathfrak{g})$ on $V \otimes V$ (respectively, $\check{U}_q(\tilde{\mathfrak{g}})$) is defined by $K_i(v \otimes v') = K_i^{-1}(v) \otimes K_i(v')$,*
$$E_i(v \otimes v') = v \otimes E_i(v') - K_i^{-1} F_i(v) \otimes K_i(v'), \;\; F_i(v \otimes v') = K_i(v) \otimes F_i(v') - E_i K_i(v) \otimes v'$$
for all $i \in I$, $v, v' \in V$ while the $U_q(\mathfrak{g})$-action on $\check{U}_q(\tilde{\mathfrak{g}})$ is defined by (1.7).

(b) *If V is simple integrable of lowest weight $-\mu$ then $V \otimes V$ is integrable, Ξ and $\underline{\Xi}$ are injective and $J_V := \underline{\Xi}(V \otimes V)$ is the corresponding Joseph's component* $\operatorname{ad} U_q(\mathfrak{g}) K_{0,2\mu}$ *([13]).*

Our proof of Theorem 1.25 (see §4.4) relies on results of [4].

It is very tempting to relate some known bases in $V \otimes V$ with our basis $\mathbf{B}_{\tilde{\mathfrak{g}}}$. The relation is not immediate. However, as all interesting bases contain the cannonical $U_q(\mathfrak{g})$-invariant element $1_V \in V \otimes V$ (cf. §4.4), we suggest the following Conjecture

Conjecture 1.26. *Let $\mathfrak{g}$ be semisimple and let V be the simple finite dimensional $U_q(\mathfrak{g})$-module of lowest weight $-\mu$. Then $(-1)^{2\rho^\vee(\mu)} C_V$, where $C_V := \Xi(1_V)$ and $2\rho^\vee$ is the sum of all positive coroots of $\mathfrak{g}$ belongs to the double canonical basis of $\check{U}_q(\tilde{\mathfrak{g}})$.*

We prove this conjecture for $\mathfrak{g} = \mathfrak{sl}_2$ and provide other supporting evidence for $\mathfrak{sl}_n$ and $\mathfrak{sp}_4$ in §4.4.

Theorem 1.25 implies that the C_V and $\underline{C}_V := \underline{\Xi}(1_V)$ are central. Their importance for the representation theory of $\check{U}_q(\mathfrak{g})$ is due to following result (see e.g. [4, Theorem 1.11] which in turn was inspired by Drinfeld's construction from [10]).

Theorem 1.27. *For any semisimle Lie algebra $\mathfrak{g}$ the map assinging to a simple $U_q(\mathfrak{g})$-module V the element $\underline{C}_V := \Xi(1_V)$ defines an isomorphisms between the Grothendieck ring $\Bbbk\otimes_{\mathbb{Z}} K_0(\mathfrak{g})$ of the category of finite dimensional $U_q(\mathfrak{g})$-modules and the center of $\check{U}_q(\mathfrak{g})$.*

Thus, the canonical basis of the Grothendieck ring of the category of finite dimensional $\mathfrak{g}$-modules identifies with a subset of the double canonical basis $\mathbf{B}_{\mathfrak{g}}$ and so $\mathbf{B}_{\mathfrak{sl}_n}$ contains (the canonical basis of) all Schur polynomials s_λ. Namely, Conjecture 1.26 and Theorem 1.27 imply that the map assigning to the simple lowest weight module $V(-\mu)$ of lowest map $-\mu$ the element $C_\mu := (-1)^{2\rho^\vee(\mu)}\underline{C}_{V(-\mu)}$ defines a homomorphism of rings $K_0(\mathfrak{g})\to\check{U}_q(\mathfrak{g})$ and that the C_μ belong to the double canonical basis of $\check{U}_q(\mathfrak{g})$. Furthermore, it would be interesting to extend these observations to the case when V is simple infinite dimensional. In that case 1_V is a well-defined element of a certain completion $V\widehat{\otimes}V$ of $V\otimes V$ and is image C_V under Ξ belongs to a completion of $\check{U}_q(\tilde{\mathfrak{g}})$. It would be interesting to relate these elements with the quantum Casimir defined in [12]. We believe that these elements C_V should be important for physical applications, for instance when $\mathfrak{g}$ is affine and V is its basic module.

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2. Equivariant Lusztig's Lemma and bases of Heisenberg and Drinfeld doubles

2.1. **An equivariant Lusztig's Lemma.** Let Γ be an abelian monoid and let R be a unital Γ-graded ring $R=\bigoplus_{\gamma\in\Gamma}R_\gamma$ where R_0 is central in R. Suppose that $\bar{\cdot}$ is an involution of abelian groups $R\to R$ satisfying

$$\overline{r\cdot r'}=\overline{r'}\cdot\overline{r},\qquad r,r'\in R$$

and $\overline{R_\gamma}=R_\gamma$, $\gamma\in\Gamma$. Let $R_+=\bigoplus_{\gamma\in\Gamma\setminus\{0\}}R_\gamma$ and $\varepsilon:R\to R/R_+\cong R_0$ be the canonical projection. Note that ε commutes with $\bar{\cdot}$.

Let $\hat{E}=\bigoplus_{\gamma\in\Gamma}\hat{E}_\gamma$ be a Γ-graded left R-module where each $\hat{E}_\gamma$ is assumed to be free as an R_0-module. Suppose that $\bar{\cdot}$ is an involution of abelian groups on $\hat{E}$ satisfying $\overline{xe}=\overline{x}\cdot\overline{e}$ for all $x\in R_0$, $e\in\hat{E}$ and $\overline{\hat{E}_\gamma}=\hat{E}_\gamma$, $\gamma\in\Gamma$. Assume also that $\overline{R_\gamma\hat{E}}\subset\sum_{\alpha\in\Gamma}R_{\alpha+\gamma}\hat{E}$ for all $\gamma\in\Gamma\setminus\{0\}$, or, equivalently, $\overline{R_+\hat{E}}\subset R_+\hat{E}$. Then $E=\hat{E}/R_+\hat{E}$ is naturally a Γ-graded R_0-module and $\bar{\cdot}$ factors through to an involution of abelian groups on E which also satisfies $\overline{x\cdot e}=\overline{x}\cdot\overline{e}$, $x\in R_0$, $e\in E$.

Suppose now that E is also free as an R_0-module. Since $\hat{E}$ and E are free as R_0-modules and the canonical projection $\pi:\hat{E}\to E$ is a morphism of Γ-graded R_0-modules, it admits a homogeneous splitting $\iota:E\to\hat{E}$.

Define a relation $\prec$ on Γ by $\alpha\prec\beta$ if there exists $\gamma\in\Gamma\setminus\{0\}$ such that $\alpha+\gamma=\beta$. Assume that there exists a function $\ell:\Gamma\to\mathbb{Z}_{\ge0}$ such that for all $\gamma\in\Gamma$, $\gamma_s\prec\gamma_{s-1}\prec\cdots\prec\gamma_1\prec\gamma$ implies that $s\le\ell(\gamma)$. For example, this assumption holds for every monoid Γ which admits a character $\chi:\Gamma\to\mathbb{Z}_{\ge0}$ with $\chi(\gamma)>0$ if $\gamma\ne0$, which is the case for $\Gamma=\mathbb{Z}^I_{\ge0}$ where I is finite. We will call such a monoid Γ bounded. If Γ is bounded then, in particular, $\preceq$ is a partial order and 0 is the unique minimal element of Γ.

Lemma 2.1. *Let Γ be a bounded monoid. Then $\iota(E)$ generates $\hat{E}$ as an R-module.*

Proof. We have $\hat{E} = \iota(E) \oplus R_+\hat{E}$ as Γ-graded R_0-modules, hence $\hat{E}_\gamma = \iota(E_\gamma) \oplus (R_+\hat{E})_\gamma$ for all $\gamma \in \Gamma$. We prove by induction on $(\Gamma, \prec)$ that $\hat{E}_\gamma \subset R\iota(E)$. Since $0 \in \Gamma$ is minimal, $\hat{E}_0 \cap R_+\hat{E} = 0$ hence $\hat{E}_0 \subset \iota(E)$ and the induction begins. For the inductive step, let $\gamma \in \Gamma \setminus \{0\}$ and assume that $\bigoplus_{\alpha \prec \gamma} \hat{E}_\alpha \subset R\iota(E)$. Then $\hat{E}_\gamma \cap R_+\hat{E} \subset \sum_{\alpha+\beta=\gamma\,:\,\alpha \in \Gamma\setminus\{0\}} R_\alpha \hat{E}_\beta \subset R\iota(E)$ by the induction hypothesis. Thus, $\hat{E}_\gamma = \iota(E_\gamma) \oplus (R_+\hat{E})_\gamma \subset R\iota(E)$. □

From now on we will assume that Γ is bounded.

Let $\mathcal{E}$ be a homogeneous basis of E satisfying $\bar{e} = e$ for all $e \in \mathcal{E}$. Clearly

$$\overline{\iota(e)} - \iota(e) \in R_+\hat{E}.$$

The following Lemma is obvious.

Lemma 2.2. *Let $\mathcal{R} \subset R$, $1 \in \mathcal{R}$. The following are equivalent:*

(i) $\{r\iota(e) : (r, e) \in \mathcal{R} \times \mathcal{E}\}$ *is an R_0-basis of $\hat{E}$*

(ii) *As an R_0-module, $R = \operatorname{Ann}_R \iota(E) \oplus \bigoplus_{r \in \mathcal{R}} R_0 r$.*

Given a homogeneous element x of R, $\hat{E}$ or E we denote its degree by $|x|$.

Proposition 2.3. *Suppose that $R_0 = \mathbb{Z}[\nu, \nu^{-1}]$ and that $\bar{\cdot} : R_0 \to R_0$ is the unique ring automorphism satisfying $\bar{\nu} = \nu^{-1}$. Fix an R_0-module splitting $\iota : E \to \hat{E}$ of the canonical projection $\hat{E} \twoheadrightarrow \hat{E}/R_+\hat{E} \cong E$. Suppose that there exists a subset $\mathcal{R} \subset R$ of homogeneous elements containing 1 such that*

(i) *As an R_0-module, $R = \operatorname{Ann}_R \iota(E) \oplus \bigoplus_{r \in \mathcal{R}} R_0 r$;*

(ii) *For all $r \in \mathcal{R}$, $e \in \mathcal{E}$*

$$\overline{r\iota(e)} - r\iota(e) \in \sum_{(r',e') \in \mathcal{R}\times\mathcal{E}\,:\,|r|+|e|=|r'|+|e'|,\,|e'|\prec|e|} R_0 r'\iota(e') \tag{2.1}$$

Then for each $(r, e) \in \mathcal{R} \times \mathcal{E}$ there exists a unique $C_{r,e} \in \hat{E}$ such that $\overline{C_{r,e}} = C_{r,e}$ and

$$C_{r,e} - r\iota(e) \in \sum_{(r',e') \in \mathcal{R}\times\mathcal{E}\,:\,|r'|+|e'|=|r|+|e|,\,|e'|\prec|e|} \nu\mathbb{Z}[\nu] r'\iota(e') \tag{2.2}$$

In particular, the set $\mathbf{B}_{\mathcal{R},\mathcal{E}} := \{C_{r,e} : (r, e) \in \mathcal{R} \times \mathcal{E}\}$ is an R_0-basis of $\hat{E}$.

Proof. Define a relation $<$ on $\mathcal{R}\times\mathcal{E}$ by $(r', e') < (r, e)$ if $|e'| \prec |e|$ and $|r'|+|e'| = |r|+|e|$. It is easy to see that $(r', e') < (r, e)$ implies that $0 \prec |r'|$ (otherwise $|e'| = |r| + |e|$ hence $|e| \preceq |e'| \prec |e|$). Then $\le$ is a partial order and all assumptions of [6, Theorem 1.1] for $L := \mathcal{R} \times \mathcal{E}$, $\mathcal{A} = \hat{E}$ and $E_{(r,e)} = r\iota(e)$, $(r, e) \in L$ are satisfied. Thus, the assertion follows from the aforementioned result. □

We conclude this section with a discussion of some symmetries of the $\mathbf{B}_{\mathcal{R},\mathcal{E}}$ constructed in Proposition 2.3. Consider the data $(R, \mathcal{R}, \hat{E}, E, \mathcal{E}, \iota)$ satisfying the assumptions of Proposition 2.3.

Definition 2.4. We say that a homogeneous $\bar{\cdot}$-equivariant R_0-module automorphism ψ of $\hat{E}$ is *triangular* if there exists a permutation ϕ of $\mathcal{R}$ with $\phi(1) = 1$ and a permutation $\underline{\psi}$ of $\mathcal{E}$ such that

$$\psi(r\iota(e)) - \phi(r)\iota(\underline{\psi}(e)) \in \sum_{(r',e') \in \mathcal{R}'\times\mathcal{E}'\,:\,|r'|+|e'|=|r|+|e|,\,|e'|\prec|e|} \nu\mathbb{Z}[\nu] r'\iota(e'), \qquad (r, e) \in \mathcal{R} \times \mathcal{E}. \tag{2.3}$$

Using the same argument as in the proof of Proposition 2.3, we conclude that in all non-zero terms in the right-hand side we have $0 \prec |r'|$.

Lemma 2.5. *Suppose that $\psi : \hat{E} \to \hat{E}$ is triangular. Then*

$$\psi(C_{r,e}) = C_{\phi(r),\underline{\psi}(e)}, \qquad r \in \mathcal{R},\ e \in \mathcal{E}.$$

Proof. Since ψ commutes with $\bar{\cdot}$, $\overline{\psi(C_{r,e})} = \psi(C_{r,e})$. Applying ψ to (2.2) we obtain

$$\psi(C_{r,e}) - \psi(r\iota(e)) \in \sum_{(r',e')\in\mathcal{R}\times\mathcal{E}\,:\,|r'|+|e'|=|r|+|e|,\,|e'|\prec|e|} \nu\mathbb{Z}[\nu]\psi(r'\iota(e'))$$

Applying (2.3) to the left and the right hand side we conclude that

$$\psi(C_{r,e}) - \phi(r)\iota(\underline{\psi}(e)) \in \sum_{(r',e')\in\mathcal{R}\times\mathcal{E}\,:\,|r'|+|e'|=|r|+|e|,\,|e'|\prec|e|} \nu\mathbb{Z}[\nu]\psi(r'\iota(e'))$$

Proposition 2.3 then implies that $\psi(C_{r,e}) = C_{\phi(r),\underline{\psi}(e)}$. □

2.2. **Double bases of Heisenberg and Drinfeld doubles.** In this section we will use the notation and the setup of §§A.8,A.9.

Let Γ be a bounded abelian monoid as defined in §2.1. Let $\Bbbk = \mathbb{Q}(\nu)$, $R_0 = \mathbb{Z}[\nu,\nu^{-1}]$. Let $H = \Bbbk[\widehat{\Gamma}]$ be the monoidal algebra of $\widehat{\Gamma} = \Gamma\oplus\Gamma$ with a basis $\{K_{\alpha_-,\alpha_+} : \alpha_\pm\in\Gamma\}$ and let $R = \bigoplus_{\alpha_\pm\in\Gamma} R_0 K_{\alpha_-,\alpha_+}$.

Let $V^\pm = \bigoplus V^\pm_\alpha$ be Γ-graded vector spaces. We regard V^+ (respectively, V^-) as a right (respectively, left) Yetter-Drinfeld module over the localization $\widehat{H}$ of H with respect to the monoid $\{K_{\alpha_-,\alpha_+} : \alpha_\pm\in\Gamma\}$ (see §A.9 for the details). Let $\langle\cdot,\cdot\rangle : V^-\otimes V^+\to\Bbbk$ be a pairing such that $\langle V^-_\beta, V^+_\alpha\rangle = 0$, $\alpha\ne\beta$ and $\langle\cdot,\cdot\rangle|_{V^-_\alpha\otimes V^+_\alpha}$ is non-degenerate. Set $\Gamma_0 = \{\alpha\in\Gamma : V^\pm_\alpha\ne 0\}$ and assume that Γ is generated by Γ_0. Let $\chi:\Gamma\times\Gamma\to R_0^\times = \pm\nu^{\mathbb{Z}}$ be a symmetric bicharacter.

Given $t_+,t_-\in\Bbbk$, let $\mathcal{U}_{\chi,t_+,t_-}(V^-,V^+)$ be the algebra $\mathcal{U}_\chi(V^-,V^+)$ defined in §A.9 with $\langle\cdot,\cdot\rangle_\pm = t_\pm\langle\cdot,\cdot\rangle$. We have in $\mathcal{U}_{\chi,t_+,t_-}(V^-,V^+)$

$$K_{\alpha_-,\alpha_+}v^+ = \frac{\chi(\alpha_+,\deg v^+)}{\chi(\alpha_-,\deg v^+)}v^+K_{\alpha_-,\alpha_+},\quad K_{\alpha_-,\alpha_+}v^- = \frac{\chi(\alpha_-,\deg v^-)}{\chi(\alpha_+,\deg v^-)}v^-K_{\alpha_-,\alpha_+}, \tag{2.4}$$

and

$$[v^+,v^-] = t_-K_{\deg v^-,0}\langle v^-,v^+\rangle - t_+K_{0,\deg v^+}\langle v^-,v^+\rangle, \tag{2.5}$$

for all $v^\pm\in\mathcal{B}(V^\pm)$ homogeneous and $\alpha_\pm\in\Gamma$. We regard $\mathcal{U}_{\chi,t_-,t_+}(V^-,V^+)$ as graded by $\widehat{\Gamma}$ with $\deg_{\widehat{\Gamma}} v^+ = (0,\deg v^+)$, $\deg_{\widehat{\Gamma}} v^- = (\deg v^-,0)$ and $\deg_{\widehat{\Gamma}} K_{\alpha_-,\alpha_+} = (\alpha_-+\alpha_+,\alpha_-+\alpha_+)$, where $v^\pm\in V^\pm$ are homogeneous and $\alpha_\pm\in\Gamma$.

Denote

$$\begin{gathered}\mathcal{H}^0_\chi(V^-,V^+) := \mathcal{U}_{\chi,0,0}(V^-,V^+),\\ \mathcal{H}^+_\chi(V^-,V^+) = \mathcal{U}_{\chi,1,0}(V^-,V^+),\quad \mathcal{H}^-_\chi(V^-,V^+) := \mathcal{U}_{\chi,0,1}(V^-,V^+),\\ \mathcal{U}_\chi(V^-,V^+) = \mathcal{U}_{\chi,1,1}(V^-,V^+).\end{gathered}$$

Thus, all these algebras have the same underlying vector space, namely $\mathcal{B}(V^-)\otimes H\otimes\mathcal{B}(V^+)$ and differ only in the cross relations between $\mathcal{B}(V^-)$ and $\mathcal{B}(V^+)$.

Let $\bar{\cdot}:\Bbbk\to\Bbbk$ be the unique field involution defined by $\bar\nu = \nu^{-1}$. Fix its extension to $V^\pm$ preserving the grading and assume that $\overline{\langle\overline{v^-},\overline{v^+}\rangle} = -\langle v^-,v^+\rangle$, $v^\pm\in V^\pm$. Assume also that χ satisfies $\overline{\chi(\alpha,\alpha')} = \chi(\alpha,\alpha')^{-1}$ for all $\alpha,\alpha'\in\Gamma$. Then all algebras described above admit an anti-linear $\bar{\cdot}$-anti-involution extending $\bar{\cdot}: V^\pm\to V^\pm$ and satisfying $\bar K_{\alpha,\alpha'} = K_{\alpha,\alpha'}$, $\alpha,\alpha'\in\Gamma$.

Assume that $\chi(\alpha,\alpha')\in\nu^{2\mathbb{Z}}$ for all $\alpha,\alpha'\in\Gamma$ and let $\chi^{\frac12}:\Gamma\times\Gamma\to\pm\nu^{\mathbb{Z}}$ be a bicharacter satisfying $(\chi^{\frac12}(\alpha,\alpha'))^2 = \chi(\alpha,\alpha')$, $\alpha,\alpha'\in\Gamma$. Extend $\chi^{\frac12}$ to a bicharacter of $\widehat{\Gamma}$ via

$$\chi^{\frac12}((\alpha_-,\alpha_+),(\beta_-,\beta_+)) = \frac{\chi^{\frac12}(\alpha_+,\beta_+)\chi^{\frac12}(\alpha_-,\beta_-)}{\chi^{\frac12}(\alpha_+,\beta_-)\chi^{\frac12}(\alpha_-,\beta_+)},\qquad \alpha_\pm,\beta_\pm\in\Gamma.$$

Then we set for all $\alpha_\pm\in\Gamma$ and for all $x\in\mathcal{U}_{\chi,t_-,t_+}(V^-,V^+)$ homogeneous with respect to $\widehat{\Gamma}$

$$\begin{aligned}K_{\alpha_-,\alpha_+}\diamond x&=(\chi^{\frac{1}{2}}((\alpha_-,\alpha_+),\deg_{\widehat{\Gamma}}x))^{-1}K_{\alpha_-,\alpha_+}x\\&=(\chi^{\frac{1}{2}}((\alpha_-,\alpha_+),(\deg_{\widehat{\Gamma}}x)^t))^{-1}xK_{\alpha_-,\alpha_+},\end{aligned}\tag{2.6}$$

where ${}^t:\widehat{\Gamma}\to\widehat{\Gamma}$ is defined by $(\alpha_-,\alpha_+)^t=(\alpha_+,\alpha_-)$, $\alpha_\pm\in\Gamma$. The following Lemma is obvious.

Lemma 2.6. *For all $t_\pm\in\Bbbk$, (2.6) defines a structure of a left H-module on $\mathcal{U}_{\chi,t_-,t_+}(V^-,V^+)$ satisfying*

$$\overline{K_{\alpha_-,\alpha_+}\diamond x}=K_{\alpha_-,\alpha_+}\diamond\overline{x},\qquad x\in\mathcal{U}_{\chi,t_-,t_+}(V^-,V^+),\ \alpha_\pm\in\Gamma.$$

It should be noted, however, that $\mathcal{U}_{\chi,t_-,t_+}(V^-,V^+)$ is not an H-module algebra with respect to the $\diamond$ action.

We will now use Proposition 2.3 to construct a basis in $\mathcal{H}^+_\chi(V^-,V^+)$ starting from a natural basis in $\mathcal{H}^0_\chi(V^-,V^+)$ and then use the resulting basis to obtain a basis in $\mathcal{U}_\chi(V^-,V^+)$. First we need to construct a suitable "initial basis".

Proposition 2.7. *Let $\mathbf{B}_+$ (respectively, $\mathbf{B}_-$) be a Γ-homogeneous basis of $\mathcal{B}(V^+)$ (respectively, of $\mathcal{B}(V^-)$). Then*

(a) *There exists a unique $\mathbf{d}:\mathbf{B}_-\times\mathbf{B}_+\to\mathbb{Z}[\nu+\nu^{-1}]$, $(b_-,b_+)\mapsto\mathbf{d}_{b_-,b_+}$ such that for all $b_\pm\in\mathbf{B}_\pm$ we have in $\mathcal{U}_{\chi,t_+,t_-}(V^-,V^+)$*

$$\mathbf{d}_{b_-,b_+}(b_+b_--b_-b_+)\in\sum_{(\alpha_-,\alpha_+)\in\widehat{\Gamma}\setminus\{(0,0)\},\,b'_\pm\in\mathbf{B}_\pm}R_0\mathbf{d}_{b'_-,b'_+}K_{\alpha_-,\alpha_+}\diamond(b'_-b'_+),$$

and the degree of $\mathbf{d}_{b_-,b_+}$ in $\nu+\nu^{-1}$ is minimal and the highest coefficient is positive and minimal.

(b) *If $\underline{\Delta}(\mathbf{B}_\pm)\in R_0\mathbf{B}_\pm\otimes\mathbf{B}_\pm$ and $\langle\mathbf{B}_-,\mathbf{B}_+\rangle,\langle\mathbf{B}_-,\underline{S}^{-1}(\mathbf{B}_+)\rangle\subset R_0$ then $\mathbf{d}_{b_-,b_+}=1$ for all $(b_-,b_+)\in\mathbf{B}_-\times\mathbf{B}_+$.*

Proof. We may assume, without loss of generality, that the element of $\mathbf{B}_\pm$ of degree 0 is 1. We have

$$(\underline{\Delta}\otimes 1)\underline{\Delta}(b_\pm)=\sum_{b'_\pm,b''_\pm,b'''_\pm\in\mathbf{B}_\pm}C_{b_\pm}^{b'_\pm,b''_\pm,b''_\pm}b'_\pm\otimes b''_\pm\otimes b'''_\pm,\qquad C_{b_\pm}^{b'_\pm,b''_\pm,b''_\pm}\in\Bbbk,\qquad C_{b_\pm}^{1,b_\pm,1}=1,$$

and $C_{b_\pm}^{b'_\pm,b''_\pm,b''_\pm}=0$ unless $\deg b'_\pm+\deg b''_\pm+\deg b'''_\pm=\deg b_\pm$. Given a homogeneous element $u^\pm\in\mathcal{B}(V^\pm)$, denote its $\mathbb{Z}_{\ge 0}$-degree by $|u^\pm|$. Then (A.43) implies that

$$\begin{aligned}b_+\cdot b_-&=\sum_{b'_\pm,b''_\pm,b'''_\pm\in\mathbf{B}_\pm}(\chi(b''_-,b'_-)\chi(b''_-,b'''_-)\chi(b'''_-,b'_-))^{-1}\times\\&\qquad C_{b_+}^{b'_+,b''_+,b''_+}C_{b_-}^{b'_-,b''_-,b''_-}t_-^{|b'-|}t_+^{|b'_+|}\langle b'_-,\underline{S}^{-1}(b'''_+)\rangle\langle b'''_-,b'_+\rangle K_{\deg b'''_-,0}b''_-b''_+K_{0,\deg b'''_+}\\&=\sum_{b''_\pm\in\mathbf{B}_\pm,\,\alpha_\pm\in\Gamma\,:\,\deg b''_\pm+\alpha_++\alpha_-=\deg b_\pm}F^{b_-,b_+}_{b''_-,b''_+,\alpha_-,\alpha_+}K_{\alpha_-,\alpha_+}b''_-b''_+\end{aligned}\tag{2.7}$$

where

$$\begin{aligned}F^{b_-,b_+}_{b''_-,b''_+,\alpha_-,\alpha_+}&=\sum_{b'_\pm,b'''_\pm\in\mathbf{B}_\pm\,:\,\deg b'''_\pm=\alpha_\pm}(\chi(b''_+,b'''_+)\chi(b''_-,b'''_-)\chi(b'''_-,b'_-))^{-1}\times\\&\qquad C_{b_+}^{b'_+,b''_+,b''_+}C_{b_-}^{b'_-,b''_-,b''_-}t_-^{|b'_-|}t_+^{|b'_+|}\langle b'_-,\underline{S}^{-1}(b'''_+)\rangle\langle b'''_-,b'_+\rangle.\end{aligned}\tag{2.8}$$

Since Γ and hence $\widehat{\Gamma}$ is bounded, we can now construct $\mathbf{d}$ inductively. For $\deg b_-=\deg b_+=0$ we set $\mathbf{d}_{b_-,b_+}=1$. We need the following

Lemma 2.8. *For any finite subset $\mathcal{F} \subset \mathbb{Q}(\nu)$ there exists a unique $\mathbf{d}(\mathcal{F}) \in \mathbb{Z}[\nu+\nu^{-1}]$ such that $\mathbf{d}(\mathcal{F})\mathcal{F} \subset \mathbb{Z}[\nu,\nu^{-1}]$, the degree of $\mathbf{d}(\mathcal{F})$ in $\nu+\nu^{-1}$ is minimal and the highest coefficient of $\mathbf{d}(\mathcal{F})$ is positive and minimal. Moreover, if all poles of elements of $\mathcal{F}$ are roots of unity then $\mathbf{d}(\mathcal{F}) = c(\nu+\nu^{-1}-2)^{m_1}(\nu+\nu^{-1}+2)^{m_2}\prod_{k\ge 3}(\nu^{-\frac{1}{2}\varphi(k)}\Phi_k(\nu))^{m_k}$ with $c, m_k \in \mathbb{Z}_{\ge 0}$, $c \ne 0$, where Φ_k is the kth cyclotomic polynomial and $\varphi(k) = \deg \Phi_k$ is the Euler function.*

Proof. Let $\mathcal{F} = \{f_1,\dots,f_r\}$, $f_i = a_i/b_i$ where $a_i, b_i \in \mathbb{Z}[\nu]$ and are coprime. Then there exists a unique $f \in \mathbb{Z}[\nu]$ of minimal degree such that $ff_i \in \mathbb{Z}[\nu]$ for all $1 \le i \le r$, namely, f is the least common factor of the b_i. Write $f = c\prod_{j=1}^t p_j^{m_j}$, where $c \in \mathbb{Z}$, each $p_j \in \mathbb{Z}[\nu]$ is irreducible and $m_j \in \mathbb{Z}_{>0}$. We may assume without generality that c as well as the highest coefficient of each of the p_j is positive. Given an irreducible $p \in \mathbb{Z}[\nu]$ of positive degree, define

$$\widetilde{p} = \begin{cases} q^{-\frac{1}{2}\deg p}p, & \overline{p} = \nu^{-\deg p}p \text{ and } \deg p \text{ is even} \\ p\overline{p}, & \text{otherwise.} \end{cases}$$

Then $\widetilde{p} \in \mathbb{Z}[\nu+\nu^{-1}]$ and is irreducible in that ring. It follows that $\mathbf{d}(\mathcal{F}) := c\prod_{j=1}^t \widetilde{p}_j{}^{m_j}$ has the desired properties. This proves the first assertion.

If the only zeroes of all the b_i, $1 \le i \le r$ are roots of unity then the only non-constant irreducible factors of f are cyclotomic polynomials. Clearly, $\widetilde{\Phi}_1 = \nu+\nu^{-1}-2$ and $\widetilde{\Phi}_2 = \nu+\nu^{-1}+2$. Since $\varphi(k)$ is even for all $k \ge 3$, it follows that $\widetilde{\Phi}_k = \nu^{-\frac{1}{2}\varphi(k)}\Phi_k$. $\square$

Denote $\mathcal{F}_{b_-,b_+} := \{\mathbf{d}^{-1}_{b'_-,b'_+}F^{b_-,b_+}_{b'_-,b'_+,\alpha_-,\alpha_+} : b'_\pm \in \mathbf{B}_\pm,\, (\alpha_-,\alpha_+) \in \widehat{\Gamma} \setminus \{(0,0)\}\}$. Then $\mathcal{F}_{b_-,b_+}$ is finite and we set $\mathbf{d}_{b_-,b_+} = \mathbf{d}(\mathcal{F}_{b_-,b_+})$. Then by the above computation

$$\mathbf{d}_{b_-,b_+}(b_+b_- - b_-b_+) \in \sum_{(\alpha_-,\alpha_+)\in\widehat{\Gamma}\setminus\{(0,0)\},\, b'_\pm\in\mathbf{B}_\pm} R_0\mathbf{d}_{b'_-,b'_+}K_{\alpha_-,\alpha_+}b'_-b'_+.$$

It remains to observe that $R_0K_{\alpha_-,\alpha_+}b'_-b'_+ = R_0K_{\alpha_-,\alpha_+} \diamond b'_-b'_+$. The uniqueness of $\mathbf{d}$ is obvious.

Part (b) is immediate from (2.8). $\square$

Theorem 2.9. *Suppose that $\mathbf{B}_\pm$ are Γ-homogeneous bases of $\mathcal{B}(V^\pm)$ and $\overline{b_\pm} = b_\pm$ for all $b_\pm \in \mathbf{B}_\pm$. Then for each $(b_-,b_+) \in \mathbf{B}_- \times \mathbf{B}_+$ there exist*

(a) *a unique element $b_- \circ b_+ \in \mathcal{H}^+_\chi(V^-,V^+)$ such that $\overline{b_- \circ b_+} = b_- \circ b_+$ and*

$$b_- \circ b_+ - \mathbf{d}_{b_-,b_+}b_-b_+ \in \sum_{\alpha\in\Gamma\setminus\{0\},\, b'_\pm\in\mathbf{B}_\pm\,:\,\deg b'_\pm+\alpha=\deg b_\pm} \nu\mathbb{Z}[\nu]\mathbf{d}_{b'_-,b'_+}K_{0,\alpha} \diamond b'_-b'_+;$$

The elements $\{K_{\alpha_-,\alpha_+} \diamond (b_- \circ b_+) : \alpha_\pm \in \Gamma,\, b_\pm \in \mathbf{B}_\pm\}$ form a $\bar{\cdot}$-invariant basis of $\mathcal{H}^+_\chi(V^-,V^+)$.

(b) *a unique element $b_- \bullet b_+ \in \mathcal{U}_\chi(V^-,V^+)$ such that $\overline{b_- \bullet b_+} = b_- \bullet b_+$ and*

$$b_- \bullet b_+ - b_- \circ b_+ \in \sum_{\alpha_-,\alpha_+\in\Gamma\, b'_\pm\in\mathbf{B}_\pm\,:\,\deg b'_\pm+\alpha_-+\alpha_+=\deg b_\pm,\,\alpha_-\ne 0} \nu^{-1}\mathbb{Z}[\nu^{-1}]K_{\alpha_-,\alpha_+} \diamond b'_- \circ b'_+.$$

The elements $\{K_{\alpha_-,\alpha_+} \diamond (b_- \bullet b_+) : \alpha_\pm \in \Gamma,\, b_\pm \in \mathbf{B}_\pm\}$ form a $\bar{\cdot}$-invariant basis of $\mathcal{U}_\chi(V^-,V^+)$.

Proof. To prove (a), we apply Proposition 2.3 with the following data. Let $\hat{E}$ be the free R_0-module generated by $\{\mathbf{d}_{b_-,b_+}K_{\alpha_-,\alpha_+} \diamond (b_-b_+) : b_\pm \in \mathbf{B}_\pm,\, \alpha_\pm \in \Gamma\}$, which is clearly a $\widehat{\Gamma}$-graded R-module via the $\diamond$ action. Then E identifies with the R_0-submodule *of the algebra* $\mathcal{H}^0_\chi(V^-,V^+)$ generated by $\mathcal{E} := \{\mathbf{d}_{b_-,b_+}b_-b_+ : b_\pm \in \mathbf{B}_\pm\}$. Let ι be the identity map of vector spaces $\mathcal{H}^0_\chi(V^-,V^+) \to \mathcal{H}^+_\chi(V^-,V^+)$. Let $\mathcal{R} = \{K_{\alpha_-,\alpha_+} : \alpha_\pm \in \Gamma\}$. Then in $\mathcal{H}^+_\chi(V^-,V^+)$ we have by Proposition 2.7

$$\overline{\mathbf{d}_{b_-,b_+}K_{\alpha_-,\alpha_+} \diamond (b_-b_+)} - \mathbf{d}_{b_-,b_+}K_{\alpha_-,\alpha_+} \diamond (b_-b_+) = \mathbf{d}_{b_-,b_+}K_{\alpha_-,\alpha_+} \diamond (b_+b_- - b_-b_+)$$

$$\in \sum_{\alpha\in\Gamma\setminus\{0\},\, b'_\pm\in\mathbf{B}_\pm\,:\,\deg b'_\pm+\alpha=\deg b_\pm} R_0\mathbf{d}_{b'_-,b'_+}K_{\alpha_-,\alpha_++\alpha}\diamond(b'_-b'_+)$$
$$=\sum_{\substack{\alpha'_\pm\in\Gamma,\, b'_\pm\in\mathbf{B}_\pm\\ \deg_{\widehat\Gamma} b'_-b'_++(\alpha'_++\alpha'_-,\alpha'_++\alpha'_-)=\deg_{\widehat\Gamma} b_-b_+}} R_0\mathbf{d}_{b'_-,b'_+}K_{\alpha'_-,\alpha'_+}\diamond(b'_-b'_+).$$

Thus, all assumptions of Proposition 2.3 are satisfied and hence for each $(\alpha_-,\alpha_+)\in\Gamma\oplus\Gamma$, $(b_-,b_+)\in\mathbf{B}_-\times\mathbf{B}_+$ there exists a unique element $C_{\alpha_-,\alpha_+,b_-,b_+}\in\mathcal{H}^+_\chi(V^-,V^+)$ such that $\overline{C_{\alpha_-,\alpha_+,b_-,b_+}}=C_{\alpha_-,\alpha_+,b_-,b_+}$ and

$$C_{\alpha_-,\alpha_+,b_-,b_+}-\mathbf{d}_{b_-,b_+}K_{\alpha_-,\alpha_+}\diamond b_-b_+\in \sum_{\substack{(\alpha'_-,\alpha'_+)\in\widehat\Gamma,\, b'_\pm\in\mathbf{B}_\pm\\ \deg_{\widehat\Gamma} b'_-b'_++(\alpha'_-+\alpha'_+,\alpha'_-+\alpha'_+)=\deg_{\widehat\Gamma} b_-b_+}} \nu\mathbb{Z}[\nu]\mathbf{d}_{b'_-,b'_+}K_{\alpha'_-,\alpha'_+}\diamond b'_-b'_+$$

Set $b_-\circ b_+=C_{0,0,b_-,b_+}$. Then $K_{\alpha_-,\alpha_+}\diamond b_-\circ b_+$ has the same properties as $C_{\alpha_-,\alpha_+,b_-,b_+}$ hence they coincide. This completes the proof of part (a).

To prove part (b), we again employ Proposition 2.3. Let $\hat E$ be the free R_0-submodule of $\mathcal{U}_\chi(V^-,V^+)$ generated by $\{K_{\alpha_-,\alpha_+}\diamond b_-\circ b_+ : \alpha_\pm\in\Gamma,\, b_\pm\in\mathbf{B}_\pm\}$, which is clearly a $\widehat\Gamma$-graded R-module. Then E identifies with the free R_0-submodule of $\mathcal{H}^+_\chi(V^-,V^+)$ generated by $\{K_{0,\alpha}\diamond b_-\circ b_+ : \alpha\in\Gamma,\, b_\pm\in\mathbf{B}_\pm\}$. Let $\mathcal{R}=\{K_{\alpha,0} : \alpha\in\Gamma\}$. By part (a) we have

$$\mathbf{d}_{b_-,b_+}b_-b_+-b_-\circ b_+\in \sum_{\substack{b'_\pm\in\mathbf{B}_\pm,\,\alpha\in\Gamma\setminus\{0\}\\ \deg b_\pm=\deg b'_\pm+\alpha}} \nu\mathbb{Z}[\nu]K_{0,\alpha}\diamond b'_-\circ b'_+. \tag{2.9}$$

Together with Proposition 2.7 this implies that in $\mathcal{U}_\chi(V^-,V^+)$

$$\overline{\mathbf{d}_{b_-,b_+}b_-b_+}\in\mathbf{d}_{b_-,b_+}b_-b_++\sum_{\substack{b'_\pm\in\mathbf{B}_\pm,\,(\alpha_-,\alpha_+)\in\widehat\Gamma\setminus\{(0,0)\}\\ \deg_{\widehat\Gamma} b_-b_+=\deg_{\widehat\Gamma} K_{\alpha_-,\alpha_+}\diamond b'_-b'_+}} R_0\mathbf{d}_{b'_-,b'_+}K_{\alpha_-,\alpha_+}\diamond b'_-b'_+$$
$$=b_-\circ b_++\sum_{\substack{b'_\pm\in\mathbf{B}_\pm,\,(\alpha_-,\alpha_+)\in\widehat\Gamma\setminus\{(0,0)\}\\ \deg_{\widehat\Gamma} b_-b_+=\deg_{\widehat\Gamma} K_{\alpha_-,\alpha_+}\diamond b'_-b'_+}} R_0K_{\alpha_-,\alpha_+}\diamond b'_-\circ b'_+,$$

which together with (a) yields

$$\overline{b_-\circ b_+}\in\mathbf{d}_{b_-,b_+}\overline{b_-b_+}+\sum_{\substack{b'_\pm\in\mathbf{B}_\pm,\,\alpha\in\Gamma\setminus\{0\}\\ \deg b_\pm=\deg b'_\pm+\alpha}} R_0\mathbf{d}_{b'_-,b'_+}K_{0,\alpha}\diamond\overline{b'_-b'_+}$$
$$=b_-\circ b_++\sum_{\substack{b'_\pm\in\mathbf{B}_\pm,\,\alpha_\pm\in\Gamma,\alpha_-\ne0\\ \deg_{\widehat\Gamma} b_-b_+=\deg_{\widehat\Gamma} K_{\alpha_-,\alpha_+}\diamond b'_-b'_+}} R_0K_{\alpha_-,0}\diamond(K_{0,\alpha_+}\diamond b'_-\circ b'_+).$$

Note that in the last sum only terms with $\alpha_-\ne0$ may occur with non-zero coefficients, since $b_-\circ b_+$ is $\bar{\cdot}$-invariant in $\mathcal{H}^+_\chi(V^-,V^+)$. Thus, all assumptions of Proposition 2.3 are satisfied, and, using it with ν replaced by ν^{-1} we obtain the desired basis. The rest of the argument is essentially the same as in part (a) and is omitted. □

Remark 2.10. In view of Remark 1.12, it would be interesting to compare our elements $b_-\bullet b_+$ with those obtained by interchanging ν and ν^{-1} in and/or $\mathcal{H}^+_\chi(V^-,V^+)$ with $\mathcal{H}^-_\chi(V^-,V^+)$ in Theorem 2.9.

Choose bases $\mathbf{B}_+^0 = \{E_i\}_{i\in I}$ of V^+ and $\mathbf{B}_{0,-} = \{F_i\}_{i\in I}$ of V^- such that $\deg E_i = \deg F_i$, $i \in I$; thus, $\Gamma_0 = \{\deg E_i\}_{i\in I}$. Assume that $\overline{E_i} = E_i$ and $\overline{F_i} = F_i$. Let ^{t} be the unique anti-involution ξ, as defined in Lemma A.37(c), such that $E_i{}^t = F_i$, $F_i{}^t = E_i$.

Proposition 2.11. *Let $\mathbf{B}_+$ be a Γ-homogeneous basis of $\mathcal{B}(V^+)$ consisting of $\bar{\cdot}$-invariant and containing $\mathbf{B}_+^0$ and let $\mathbf{B}_- = \mathbf{B}_+{}^t$. Then for all $b_\pm \in \mathbf{B}_\pm$, $\alpha_\pm \in \Gamma$ we have*

$$(K_{\alpha_-,\alpha_+} \diamond b_- \circ b_+)^t = K_{\alpha_-,\alpha_+} \diamond b_+{}^t \circ b_-{}^t, \qquad (K_{\alpha_-,\alpha_+} \diamond b_- \bullet b_+)^t = K_{\alpha_-,\alpha_+} \diamond b_+{}^t \bullet b_-{}^t.$$

Proof. Since $\deg b_\pm{}^t = \deg b_\pm$, we have in $\mathcal{H}_\chi^+(V^-, V^+)$

$$\begin{aligned}(K_{\alpha_-,\alpha_+} \diamond b_- b_+)^t &= (\chi^{\frac{1}{2}}((\alpha_-, \alpha_+), (\deg b_-, \deg b_+)))^{-1} b_+{}^t b_-{}^t K_{\alpha_-,\alpha_+}\\ &= K_{\alpha_-,\alpha_+} \diamond b_+{}^t b_-{}^t.\end{aligned}$$

Thus, the anti-automorphism ^{t} of $\mathcal{H}_\chi^+(V^-, V^+)$ is triangular in the sense of Definition 2.4, hence $(K_{\alpha_-,\alpha_+} \diamond b_- \circ b_+)^t = K_{\alpha_-,\alpha_+} \diamond b_+{}^t \circ b_-{}^t$. This implies that the anti-automorphism ^{t} of $\mathcal{U}_\chi(V^-, V^+)$ is also triangular in the sense of Definition 2.4 with ν replaced by ν^{-1}, and the second assertion follows. □

3. Dual canonical bases and proofs of Theorems 1.3, 1.5, 1.10 and 1.19

We fix some notation which will be used repeatedly throughout the rest of the paper. Define in $\mathbb{Q}(\nu)$

$$[a]_\nu = \frac{\nu^a - 1}{\nu - 1}, \quad [a]_\nu! = \prod_{j=1}^{a} [j]_\nu, \quad \begin{bmatrix} a \\ n \end{bmatrix}_\nu = \frac{[a]_\nu [a-1]_\nu \cdots [a-n+1]_\nu}{[n]_\nu!} \tag{3.1}$$

$$(a)_\nu = \frac{\nu^a - \nu^{-a}}{\nu - \nu^{-1}}, \quad (a)_\nu! = \prod_{j=1}^{a} (j)_\nu, \quad \binom{a}{n}_\nu = \frac{(a)_\nu (a-1)_\nu \cdots (a-n+1)_\nu}{(n)_\nu!} \tag{3.2}$$

and

$$\langle a \rangle_\nu = \nu^a - \nu^{-a}, \quad \langle a \rangle_\nu! = \prod_{j=1}^{a} \langle j \rangle_\nu. \tag{3.3}$$

We always use the convention that $\begin{bmatrix} a \\ n \end{bmatrix}_\nu = 0 = \binom{a}{n}_\nu$ if $n < 0$. If a, n are non-negative integers, then all expressions in (3.1) lie in $1 + \nu\mathbb{Z}_{\ge 0}[\nu]$ while all expressions in (3.2) are in $\mathbb{Z}_{\ge 0}[\nu + \nu^{-1}]$. Clearly,

$$[a]_{\nu^2} = \nu^{a-1}(a)_\nu = \nu^{a-1}(\nu - \nu^{-1})^{-1} \langle a \rangle_\nu,$$

hence

$$[a]_{\nu^2}! = \nu^{\binom{a}{2}}(a)_\nu! = \nu^{\binom{a}{2}}(\nu - \nu^{-1})^{-a} \langle a \rangle_\nu!, \qquad \begin{bmatrix} a \\ n \end{bmatrix}_{\nu^2} = \nu^{(a-n)n} \binom{a}{n}_\nu = \nu^{(a-n)n} \frac{\langle a \rangle_\nu \cdots \langle a-n+1 \rangle_\nu}{\langle a \rangle_\nu!}$$

(thus, there is no need to introduce "angular" ν-binomial coefficients). Finally,

$$\begin{bmatrix} a \\ n \end{bmatrix}_{\nu^{-2}} = \nu^{n(n-a)} \binom{a}{n}_\nu = \nu^{2n(n-a)} \begin{bmatrix} a \\ n \end{bmatrix}_{\nu^2}. \tag{3.4}$$

For every symbol X_i, $i \in I$ such that X_i^n is defined we set $X_i^{(n)} = X_i^n/(n)_{q_i}! = (q_i - q_i^{-1})^n X_i^{\langle n \rangle}$.

3.1. **Bicharacters, pairings, lattices and inner products.** Let $\Bbbk = \mathbb{Q}(q^{\frac{1}{2}})$ and let $R_0 = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra and let $A = (a_{ij})_{i,j\in I}$ be its Cartan matrix. Fix positive integers d_i, $i \in I$ such that $d_i a_{ij} = a_{ji} d_j$, $i, j \in I$. Let $\mathcal{K}$ be the monoidal algebra of $\widehat{\Gamma}$ with the basis $\{K_{\alpha_-,\alpha_+} : \alpha_\pm \in \Gamma\}$ and denote $K_{\pm i} := K_{\alpha_{\pm i}}$. The monoid Γ (and hence $\widehat{\Gamma}$) clearly affords a sign character (cf. §A.8).

Define a symmetric bicharacter $\cdot : \Gamma\times\Gamma \to \mathbb{Z}$ by $\alpha_i \cdot \alpha_j = d_i a_{ij}$ and set $\chi(\alpha, \alpha') = q^{\alpha\cdot\alpha'}$, $\alpha, \alpha' \in \Gamma$. It is easy to see that $\alpha\cdot\alpha \in 2\mathbb{Z}$ for all $\alpha \in \Gamma$. Furthermore, let $\eta : \Gamma \to \mathbb{Z}_{\ge 0}$ be the character defined by $\eta(\alpha_i) = d_i$, $i \in I$. We extend $\cdot$ to a bicharacter of $\widehat{\Gamma}$ via $\alpha_{\pm i} \cdot \alpha_{\mp j} = -d_i a_{ij}$, $i, j \in I$ and η to a character of $\widehat{\Gamma}$ via $\eta(\alpha_{\pm i}) = \eta(\alpha_i)$, $i \in I$. Define $\underline{\gamma} : \Gamma \to \mathbb{Z}$ by $\underline{\gamma}(\alpha) = \frac{1}{2}\alpha\cdot\alpha - \eta(\alpha)$, $\alpha \in \Gamma$. Then

$$\underline{\gamma}(\alpha_i) = 0, \qquad \underline{\gamma}(\alpha+\alpha') = \underline{\gamma}(\alpha) + \underline{\gamma}(\alpha') + \alpha\cdot\alpha', \qquad i \in I,\ \alpha, \alpha' \in \Gamma. \tag{3.5}$$

This implies that $\gamma : \Gamma \to \Bbbk^\times$ defined by $\gamma(\alpha) = q^{\underline{\gamma}(\alpha)}$, $\alpha \in \Gamma$ is the function discussed in §A.8.

Let $V^+ = \bigoplus_{i\in I} \Bbbk E_i$, $V^- = \bigoplus_{i\in I} \Bbbk F_i$. We regard $V^\pm$ as Γ-graded with $\deg E_i = \deg F_i = \alpha_i$. It is well-known (cf. [19, Chapter 1] and §§A.1, A.8) that $U_q^\pm$ is the Nichols algebra $\mathcal{B}(V^\pm, \Psi^\pm)$ where the braiding $\Psi^\pm$ is defined via the bicharacter χ as in §A.8.

Define a pairing $\langle\cdot,\cdot\rangle : V^- \otimes V^+ \to \Bbbk$ by $\langle F_i, E_j\rangle = \delta_{ij}(q_i - q_i^{-1})$. Then $\langle\cdot,\cdot\rangle$ extends to a pairing of braided Hopf algebras $U_q^- \otimes U_q^+ \to \Bbbk$ (see §§A.3,A.8). The algebra $\mathcal{U}_{\chi,t_-,t_+}(V^-, V^+)$, $t_\pm \in \Bbbk$, is then $\widehat{\Gamma}$-graded as in §2.2.

Proposition 3.1. *The algebra $U_q(\tilde{\mathfrak{g}})$ is isomorphic to $\mathcal{U}_\chi(V^-, V^+) = \mathcal{U}_{\chi,1,1}(V^-, V^+)$ while $\mathcal{H}_q^\pm(\mathfrak{g})$ identify with the subalgebra of $\mathcal{H}_\chi^\pm(V^-, V^+)$ generated by the K_{+i} (respectively, K_{-i}), E_i and F_i, $i \in I$, in the notation of §2.2.*

Proof. After [19, Proposition 1.4.3], (1.3) hold in $\mathcal{B}(V^\pm)$, while (A.38) yield (1.2). Thus, $\mathcal{U}_\chi(V^-, V^+)$ is a $\widehat{\Gamma}$-graded quotient of $U_q(\tilde{\mathfrak{g}})$, and it remains to observe that their homogeneous subspaces have the same dimensions. The assertion about $\mathcal{H}_q^\pm(\mathfrak{g})$ is proved similarly. ☐

Define $\bar{\cdot} : V^\pm \to V^\pm$ as the unique anti-linear map satisfying $\overline{E_i} = E_i$, $\overline{F_i} = F_i$, $i \in I$. Then $\overline{\langle \overline{v^-}, \overline{v^+}\rangle} = -\langle v^-, v^+\rangle$, hence by Lemma A.37 $U_q(\tilde{\mathfrak{g}})$ admits an anti-linear anti-involution $\bar{\cdot}$ preserving the generators, an anti-involution * preserving the E_i and the F_i, $i \in I$ and satisfying ${K_{\pm i}}^* = K_{\mp i}$, $i \in I$, and an anti-involution ^{t} which restricts to anti-isomorphisms $U_q^\pm \to U_q^\mp$ such that ${E_i}^t = F_i$, ${F_i}^t = E_i$, and preserves the $K_{\pm i}$, $i \in I$. In particular, ^{*t} is an involution which restricts to isomorphisms $U_q^\pm \to U_q^\mp$.

Let ${}_{\mathbb{Z}}U^+$ (respectively, ${}_{\mathbb{Z}}U^-$) be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of U_q^+ (respectively, U_q^-) generated by the $E_i^{\langle n\rangle}$ (respectively, $F_i^{\langle n\rangle}$), $i \in I$, $n \in \mathbb{Z}_{\ge 0}$; thus, ${}_{\mathbb{Z}}U^\pm$ is the preimage under $\psi_\pm$ of the subalgebra of $U_q^\pm$ generated by the usual divided powers ([19, §1.4.7]). Define

$$U_{\mathbb{Z}}^+ = \{x \in U_q^+ : \langle {}_{\mathbb{Z}}U^-, x\rangle \subset \mathbb{Z}[q, q^{-1}]\}, \qquad U_{\mathbb{Z}}^- = \{x \in U_q^- : \langle x, {}_{\mathbb{Z}}U^+\rangle \subset \mathbb{Z}[q, q^{-1}]\}$$

Proposition 3.2. *$U_{\mathbb{Z}}^\pm$ is a $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q^\pm$ satisfying $\underline{\Delta}(U_{\mathbb{Z}}^\pm) \subset U_{\mathbb{Z}}^\pm \otimes_{\mathbb{Z}[q,q^{-1}]} U_{\mathbb{Z}}^\pm$.*

Proof. We prove the statements for $U_{\mathbb{Z}}^+$ only, the argument for $U_{\mathbb{Z}}^-$ being similar. Let $R = \mathbb{Z}[q, q^{-1}]$. The following result is immediate from [19, Lemma 1.4.1]

Lemma 3.3. *${}_{\mathbb{Z}}U^\pm$ is an R-subalgebra of $U_q^\pm$ satisfying $\underline{\Delta}({}_{\mathbb{Z}}U^\pm) \subset {}_{\mathbb{Z}}U^\pm \otimes_R {}_{\mathbb{Z}}U^\pm$.*

Since $U_{\mathbb{Z}}^+$ is a direct sum of free R-modules of finite length, $U_{\mathbb{Z}}^+$ is canonically isomorphic to the graded $\operatorname{Hom}_R({}_{\mathbb{Z}}U^-, R)$, which immediately implies the proposition. ☐

3.2. **Dual canonical bases.** Let $\psi : U_q(\tilde{\mathfrak{g}}) \to U_q(\tilde{\mathfrak{g}})$ be the homomorphism defined by $E_i \mapsto (q_i^{-1} - q_i)^{-1} E_i$, $F_i \mapsto (q_i - q_i^{-1})^{-1} F_i$, $K_{\pm i} \mapsto K_{\pm i}$. Denote by $\psi_\pm$ its restrictions to $U_q^\pm$. Clearly, the images of generators of $U_q(\tilde{\mathfrak{g}})$ under ψ satisfy the relations of the "standard" presentation of $U_q(\tilde{\mathfrak{g}})$; for example

$$[\psi(E_i), \psi(F_j)] = \delta_{ij} \frac{K_{+i} - K_{-i}}{q_i - q_i^{-1}}.$$

Let $\mathbf{B}^{\mathrm{can}}$ be the preimage under ψ^- of Lusztig's canonical basis of U_q^- ([19, Chapter 14]). By [19, Theorem 14.4.3], $\mathbf{B}^{\mathrm{can}}$ is a $\mathbb{Z}[q, q^{-1}]$-basis of ${}_{\mathbb{Z}}U^-$. If $\mathfrak{g} = \mathfrak{sl}_2$, $\mathbf{B}^{\mathrm{can}} = \{F^{\langle r \rangle} : r \in \mathbb{Z}_{\ge 0}\}$.

Let $(\!| \cdot, \cdot |\!) : U_q^- \otimes U_q^- \to \Bbbk$ be the pairing defined in §A.9 with ξ being the anti-involution ^{t} described above. Since $(\!| \cdot, \cdot |\!)$ is non-degenerate and restricts to non-degenerate bilinear forms on finite dimensional graded components of U_q^-, for each $b \in \mathbf{B}^{\mathrm{can}}$ there exists a unique $\delta_b \in U_q^-$ such that $(\!| \delta_b, b' |\!) = \delta_{b,b'}$ for all $b' \in \mathbf{B}^{\mathrm{can}}$.

Definition 3.4. The *dual canonical basis* $\mathbf{B}_{\mathfrak{n}_-}$ of U_q^- is the set $\{\delta_b : b \in \mathbf{B}^{\mathrm{can}}\}$. The dual canonical basis $\mathbf{B}_{\mathfrak{n}_+}$ of U_q^+ is defined as $\mathbf{B}_{\mathfrak{n}_+} = \mathbf{B}_{\mathfrak{n}_-}^t$.

This definition is justified by the following Lemma.

Lemma 3.5. *For all $b_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}$, $\bar{b}_\pm = b_\pm$ and $b_\pm^* \in \mathbf{B}_{\mathfrak{n}_\pm}$.*

Proof. Note that for all $b \in \mathbf{B}^{\mathrm{can}}$, $\overline{\psi(b)}^* = \psi(b)$, hence $\bar{b}^* = \operatorname{sgn}(b) b$. Moreover, $b^* \in \mathbf{B}^{\mathrm{can}}$ by [19, §14.4]. It remains to apply (A.46). □

Proposition 3.6. *The set $\{q^{-\frac{1}{2}\underline{\gamma}(\deg b_\pm)} b_\pm : b_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}\}$ is a $\mathbb{Z}[q, q^{-1}]$-basis of $U_{\mathbb{Z}}^\pm$.*

Proof. It suffices to prove that $\{q^{-\frac{1}{2}\underline{\gamma}(\deg b)} \delta_b{}^{*t} : b \in \mathbf{B}^{\mathrm{can}}\}$ generates $U_{\mathbb{Z}}^+$ as a $\mathbb{Z}[q, q^{-1}]$-module. Let $b, b' \in \mathbf{B}^{\mathrm{can}}$. Then

$$q^{-\frac{1}{2}\underline{\gamma}(\deg b)} \langle b', \delta_b{}^{*t} \rangle = (\!| \delta_b, b' |\!) = \delta_{b,b'}.$$

Therefore, $q^{-\frac{1}{2}\underline{\gamma}(\deg b)} \delta_b^{*t} \in U_{\mathbb{Z}}^+$. Let $x \in U_{\mathbb{Z}}^+$ and write $x = \sum_{b \in \mathbf{B}^{\mathrm{can}}} q^{-\frac{1}{2}\underline{\gamma}(\deg b)} c_b \delta_b{}^{t*}$, $c_b \in \Bbbk$. Then for all $b \in \mathbf{B}^{\mathrm{can}}$,

$$\langle b, x \rangle = \sum_{b' \in \mathbf{B}^{\mathrm{can}}} q^{-\frac{1}{2}\underline{\gamma}(\deg b')} c_{b'} \langle b, \delta_{b'}{}^{t*} \rangle = c_b.$$

Thus, $c_b \in \mathbb{Z}[q, q^{-1}]$ for all $b \in \mathbf{B}^{\mathrm{can}}$. □

Then Propositions 3.2, 3.6 and (3.5) imply the following.

Corollary 3.7. *The structure constants $\tilde{C}_{b_\pm b'_\pm}^{b''_\pm}$, $\tilde{C}_{b''_\pm}^{b_\pm b'_\pm}$, $b_\pm, b'_\pm, b''_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}$ defined by*

$$b_\pm b'_\pm = q^{-\frac{1}{2} \deg b_\pm \cdot \deg b'_\pm} \sum_{b''_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}} \tilde{C}_{b_\pm b'_\pm}^{b''_\pm} b''_\pm, \qquad \underline{\Delta}(b_\pm) = \sum_{b'_\pm, b''_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}} q^{\frac{1}{2} \deg b'_\pm \cdot \deg b''_\pm} \tilde{C}_{b_\pm}^{b'_\pm b''_\pm} b'_\pm \otimes b''_\pm$$

belong to $\mathbb{Z}[q, q^{\pm 1}]$.

It follows immediately from the above Corollary that for any $b_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}$

$$\underline{\Delta}(b_\pm) = \sum_{b'_\pm, b''_\pm, b'''_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}} q^{\frac{1}{2}(\deg b'_\pm \cdot \deg b''_\pm + \deg b'_\pm \cdot \deg b'''_\pm + \deg b''_\pm \cdot \deg b'''_\pm)} \tilde{C}_{b_\pm}^{b'_\pm, b''_\pm, b'''_\pm} b'_\pm \otimes b''_\pm \otimes b'''_\pm, \tag{3.6}$$

where $\tilde{C}_{b_\pm}^{b'_\pm, b''_\pm, b'''_\pm} = \sum_{\check{b}_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}} \tilde{C}_{b_\pm}^{\check{b}_\pm b'''_\pm} \tilde{C}_{\check{b}_\pm}^{b'_\pm b''_\pm} \in \mathbb{Z}[q, q^{\pm 1}]$.

Remark 3.8. It is easy to check that for any $b, b', b'' \in \mathbf{B}^{\mathrm{can}}$ we have

$$bb' = \sum_{b'' \in \mathbf{B}^{\mathrm{can}}} \tilde{C}^{\delta_b \delta_{b'}}_{\delta_{b''}} b'', \qquad \underline{\Delta}(b'') = \sum_{b,b' \in \mathbf{B}^{\mathrm{can}}} \tilde{C}^{\delta_{b''}}_{\delta_b \delta_{b'}} b \otimes b'.$$

After [19, §14.4.14], these structure constants are Laurent polynomials in q.

Proposition 3.9. *$\langle \mathbf{B}_{\mathfrak{n}_-}, \mathbf{B}_{\mathfrak{n}_+} \rangle \subset \langle U^-_{\mathbb{Z}}, U^+_{\mathbb{Z}} \rangle \subset \mathbb{Z}[q, q^{-1}, \Phi_k^{-1} : k > 0]$, where $\Phi_k \in \mathbb{Z}[q]$ is the kth cyclotomic polynomial.*

Proof. Indeed, it is immediate from properties of $\langle \cdot, \cdot \rangle$ that $\langle F_{j_1}^{b_1} \cdots F_{j_s}^{b_s}, E_{i_1}^{a_1} \cdots E_{i_r}^{a_r} \rangle \in \mathbb{Z}[q, q^{-1}]$ for any $(i_1, \dots, i_r) \in I^r$, $(j_1, \dots, j_s) \in I^s$ and for any $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r_{\ge 0}$, $\mathbf{b} = (b_1, \dots, b_s) \in \mathbb{Z}^s_{\ge 0}$. Therefore,

$$\langle F_{j_1}^{\langle b_1 \rangle} \cdots F_{j_s}^{\langle b_s \rangle}, E_{i_1}^{\langle a_1 \rangle} \cdots E_{i_r}^{\langle a_r \rangle} \rangle \in R' := \mathbb{Z}[q, q^{-1}, \Phi_k^{-1},\ k > 0].$$

This implies that $\langle {}_{\mathbb{Z}}U^-, {}_{\mathbb{Z}}U^+ \rangle \subset R'$. We need the following, apparently well-known result.

Lemma 3.10. *Let $\alpha \in \Gamma$. Let $\mathbf{B}^-_\alpha$ be any basis of ${}_{\mathbb{Z}}U^-_\alpha = \{u \in {}_{\mathbb{Z}}U^- : \deg u = \alpha\}$ and set $G_\alpha = (\langle b', b^{*t} \rangle)_{b,b' \in \mathbf{B}^-_\alpha}$ be the corresponding Gram matrix. Then $\det G_\alpha = q^n \prod_k \Phi_k(q)^{a_k}$ where $a_k \in \mathbb{Z}$ and $n \in \mathbb{Z}$.*

Proof. It well-known ([19]) that the specialization of the form $\langle \cdot, \cdot \rangle$ for any $q = \zeta$, where $\zeta \in \mathbb{C}^\times$ is not a root of unity, is well defined and non-degenerate. Thus, $\det G_\alpha$ is a rational function of q whose zeroes and poles are roots of unity and zero. This implies $\det G_\alpha = cq^n \prod_k \Phi_k(q)^{a_k}$ where $c \in \mathbb{Q}$ and $n, a_k \in \mathbb{Z}$. It remains to prove that $c = 1$. To prove this claim, note that by [19, Theorem 14.2.3] and properties of $\langle \cdot, \cdot \rangle$, for any $b \in \mathbf{B}^{\mathrm{can}}$, there exists $\tilde{b} \in q^{\mathbb{Z}} b$ such that for all $b, b' \in \mathbf{B}^{\mathrm{can}}$, $\langle \tilde{b}', \tilde{b}^{*t} \rangle \in \delta_{b,b'} + q^{-1}\mathbb{Z}[[q^{-1}]]$. This in turn implies that for $\mathbf{B}^-_\alpha = \{\tilde{b} : b \in \mathbf{B}^{\mathrm{can}}, \deg b = \alpha\}$, $\det G_\alpha \in 1 + q^{-1}\mathbb{Z}[[q^{-1}]]$. Since $\det G_\alpha$ is, up to a power of q, independent of the choice of basis $\mathbf{B}^-_\alpha$, it follows that $c = 1$. □

Now, let $\mathbf{B}_{+,\alpha}$ be any basis of $(U^+_{\mathbb{Z}})_\alpha = \{u \in U^+_{\mathbb{Z}} : \deg u = \alpha\}$ and let $\mathbf{B}^-_\alpha$ be the dual basis of $\mathbf{B}_{+,\alpha}$ with respect to $\langle \cdot, \cdot \rangle$. Then the Gram matrix $G^\vee_\alpha = (\langle b_+^{*t}, b'_+ \rangle)_{b_+, b'_+ \in \mathbf{B}_{+,\alpha}}$ satisfies $G^\vee_\alpha = G_\alpha^{-1}$ over $\mathbb{Q}(q)$. As $\langle {}_{\mathbb{Z}}U^-, {}_{\mathbb{Z}}U^+ \rangle \subset R'$, all entries of G_α are in R', while $(\det G_\alpha)^{-1} \in R'$ by Lemma 3.10. Therefore, all entries of $G^\vee_\alpha$ are in R'.

To prove the second inclusion note that by Proposition 3.6, we have for all $b_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}$

$$\langle q^{-\frac{1}{2}\underline{\gamma}(\deg b_-)} b_-, q^{-\frac{1}{2}\underline{\gamma}(\deg b_+)} b_+ \rangle = q^{-\frac{1}{2}(\underline{\gamma}(\deg b_+) + \underline{\gamma}(\deg b_-))} \langle b_-, b_+ \rangle \in R'$$

since $\langle b_-, b_+ \rangle \neq 0$ implies that $\deg b_+ = \deg b_-$. □

For $\mathfrak{g}$ semisimple we can strengthen Proposition 3.9 as follows

Theorem 3.11. *If $\mathfrak{g}$ is semisimple then $\langle \mathbf{B}_{\mathfrak{n}_-}, \mathbf{B}_{\mathfrak{n}_+} \rangle \subset \langle U^-_{\mathbb{Z}}, U^+_{\mathbb{Z}} \rangle = \mathbb{Z}[q, q^{-1}]$.*

We prove this Theorem in Section 5. We expect that the converse is also true: if $\langle U^-_{\mathbb{Z}}, U^+_{\mathbb{Z}} \rangle \subset \mathbb{Z}[q, q^{-1}]$ then $\mathfrak{g}$ is semisimple (see Lemma 4.14 and Example 4.16).

Remark 3.12. We can conjecture that $\langle U^-_{\mathbb{Z}}, U^+_{\mathbb{Z}}(w) \rangle \subset \mathbb{Z}[q, q^{-1}]$ where $w \in W$ and $U^+_{\mathbb{Z}}(w)$ is the corresponding Schubert cell.

3.3. **Proofs of Theorems 1.3, 1.5 and 1.10.** First, we need a stronger version of Proposition 2.7.

Proposition 3.13. *Let $\mathbf{d} : \mathbf{B}_{\mathfrak{n}_-} \times \mathbf{B}_{\mathfrak{n}_+} \to \mathbb{Z}[\nu + \nu^{-1}]$, $\nu = q^{\frac{1}{2}}$, be defined as in Proposition 2.7. Then for all $b_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}$, $\mathbf{d}_{b_-, b_+} = \prod_{k \ge 3} (q^{-\frac{1}{2}\varphi(k)} \Phi_k(q))^{a_k}$, $a_k \in \mathbb{Z}_{\ge 0}$, and, in particular, is monic. Moreover, in $U_q(\tilde{\mathfrak{g}})$ we have*

$$\mathbf{d}_{b_-, b_+}(b_+ b_- - b_- b_+) \in \sum_{\substack{(\alpha_-, \alpha_+) \in \Gamma \oplus \Gamma \setminus \{(0,0)\} \\ b'_\pm \in \mathbf{B}_\pm}} \mathbb{Z}[q, q^{-1}] \mathbf{d}_{b'_-, b'_+} K_{\alpha_-, \alpha_+} \diamond b'_- b'_+.$$

Proof. By (2.7)

$$b_+b_- = \sum_{b''_\pm\in\mathbf{B}_\pm,\,\alpha_\pm\in\Gamma\,:\,\deg b''_\pm+\alpha_++\alpha_-=\deg b_\pm} F^{b_-,b_+}_{b''_-,b''_+,\alpha_-,\alpha_+} K_{\alpha_-,\alpha_+} b''_- b''_+ \tag{3.7}$$

where by (2.8), (3.6), Lemma A.30 and (A.37)

$$\begin{aligned}
F^{b_-,b_+}_{b''_-,b''_+,\alpha_-,\alpha_+} &= \sum_{\substack{b'_\pm,b'''_\pm\in\mathbf{B}_\pm\\ \deg b'''_\pm=\alpha_\pm}} \frac{\chi^{\frac12}(b'_+,b''_+)\chi^{\frac12}(b'_+,b'''_+)\chi^{\frac12}(b'_-,b''_-)}{\chi^{\frac12}(b''_+,b'''_+)\chi^{\frac12}(b''_-,b'''_-)\chi^{\frac12}(b'_-,b'''_-)}\tilde C^{b'_+,b''_+,b''_+}_{b_+}\tilde C^{b'_-,b''_-,b''_-}_{b_-}\langle b'_-,\underline{S}^{-1}(b'''_+)\rangle\langle b'''_-,b'_+\rangle\\
&= \operatorname{sgn}(\alpha_+)q^{-\underline{\gamma}(\alpha_+)}\frac{\chi^{\frac12}(\alpha_-,\deg b''_+)\chi^{\frac12}(\alpha_+,\deg b''_-)}{\chi^{\frac12}(\deg b''_+,\alpha_+)\chi^{\frac12}(\deg b''_-,\alpha_-)}\sum_{\substack{b'_\pm,b'''_\pm\in\mathbf{B}_\pm\\ \deg b'''_\pm=\alpha_\pm}}\tilde C^{b'_+,b''_+,b''_+}_{b_+}\tilde C^{b'_-,b''_-,b''_-}_{b_-}\langle b'_-,b'''^{*}_+\rangle\langle b'''_-,b'_+\rangle\\
&= \chi^{\frac12}((\alpha_-,\alpha_+),\deg_{\widehat\Gamma}(b''_-b''_+))^{-1}\tilde F^{b_-,b_+}_{b''_-,b''_+,\alpha_-,\alpha_+}.
\end{aligned}$$

Since $\tilde C^{b'_\pm,b''_\pm,b'''_\pm}_{b_\pm}\in\mathbb{Z}[q,q^{-1}]$, by Proposition 3.9 we have $\tilde F^{b_-,b_+}_{b''_-,b''_+,\alpha_-,\alpha_+}\in\mathbb{Z}[q,q^{-1},\Phi_k^{-1}\,:\,k>2]$. Thus, by Lemma 2.8 we can choose $\mathbf{d}_{b_-,b_+}$ to satisfy the first assertion. Since by (3.7)

$$b_+b_- = \sum_{b''_\pm\in\mathbf{B}_\pm,\,\alpha_\pm\in\Gamma\,:\,\deg b''_\pm+\alpha_++\alpha_-=\deg b_\pm} \tilde F^{b_-,b_+}_{b''_-,b''_+,\alpha_-,\alpha_+} K_{\alpha_-,\alpha_+}\diamond b''_- b''_+,$$

the second assertion is now immediate. □

Proofs of Theorems 1.3, 1.5. We apply Theorem 2.9 with the data from §3.1:
- $\Bbbk=\mathbb{Q}(\nu)$, $R_0=\mathbb{Z}[\nu,\nu^{-1}]$, $\nu=q^{\frac12}$
- $\Gamma=\widehat\Gamma=\bigoplus_{i\in I}\mathbb{Z}_{\ge0}(\alpha_{-i}+\alpha_{+i})$;
- $K_{0,\alpha_i}=K_{+i}$, $K_{\alpha_i,0}=K_{-i}$, $i\in I$;
- $V^+=\bigoplus_{i\in I}\Bbbk E_i$, $V^-=\bigoplus_{i\in I}\Bbbk F_i$;
- $\bar{\cdot}$ is determined by $\bar E_i=E_i$, $\bar F_i=F_i$, $i\in I$;
- $\chi(\alpha_i,\alpha_j)=q_i^{a_{ij}}$, $\langle F_j,E_i\rangle=\delta_{ij}(q_i-q_i^{-1})$, $i,j\in I$.

Then $U_q(\tilde{\mathfrak g})=\mathcal U_\chi(V^-,V^+)$ while $\mathcal H_q^+(\tilde{\mathfrak g})$ identifies with the subalgebra of $\mathcal H_\chi^+(V^-,V^+)$ generated by the K_{+i}, E_i, F_i, $i\in I$. Applying Theorem 2.9(a), we obtain elements $b_-\circ b_+\in\mathcal H_q^+(\tilde{\mathfrak g})$ which proves Theorem 1.3. Theorem 1.5 then follows from Theorem 2.9(b).

It remains to prove that all coefficients in the decompositions of invariant bases with respect to the initial ones in Theorems 1.3 and 1.5 are polynomials in q or q^{-1} and not just in $q^{\pm\frac12}$. But this is immediate from Proposition 3.13. □

Proof of Theorem 1.10. This is immediate from Proposition 2.11 since the anti-involution ^{t} of $U_q(\tilde{\mathfrak g})$ satisfies $\mathbf{B}_{\mathfrak n_\pm}{}^t=\mathbf{B}_{\mathfrak n_\mp}$. □

3.4. **Colored Heisenberg and quantum Weyl algebras and their bases.** Let $\widehat{\mathcal H}_q^{\boldsymbol\epsilon}(\mathfrak g)$ be the $\Bbbk$-algebra generated by $U_q^\pm$ and $L_i^{\pm1}$, $i\in I$ where

$$L_iE_i=q_i^{\frac12\epsilon_ia_{ij}}E_iL_i,\qquad L_iF_i=q_i^{-\frac12\epsilon_ia_{ij}}F_iL_i,\qquad [E_i,F_j]=\delta_{ij}\epsilon_iL_i^2(q_i^{-1}-q_i).$$

Note that $\widehat{\mathcal H}_q^{\boldsymbol\epsilon}(\mathfrak g)$ admits a $\Bbbk$-anti-linear anti-involution $\bar{\cdot}$ extending $\bar{\cdot}:U_q^\pm\to U_q^\pm$ and satisfying $\overline{L_i^{\pm1}}=L_i^{\pm1}$ and an anti-involution ^{t} extending the anti-isomorphisms $U_q^\pm\to U_q^\mp$ discussed above and preserving the $L_i^{\pm1}$, $i\in I$. The following is obvious.

Lemma 3.14. (a) *The assignments $E_i\mapsto E_i$, $F_i\mapsto F_i$, $K_{\epsilon_ii}\mapsto L_i^2$, $K_{-\epsilon_ii}\mapsto 0$, $i\in I$ define a homomorphism of algebras $\psi^{\boldsymbol\epsilon}:U_q(\tilde{\mathfrak g})\to\widehat{\mathcal H}_q^{\boldsymbol\epsilon}(\mathfrak g)$.*

(b) $\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ *is generated by* $\operatorname{Im}\psi^{\boldsymbol{\epsilon}}$ *and* L_i^{-1}, $i\in I$ *and has the triangular decomposition* $\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})=U_q^-\otimes\mathcal{L}\otimes U_q^+$, *where* $\mathcal{L}$ *is the subalgebra generated by* $L_i^{\pm 1}$, $i\in I$.
(c) $\psi^{\boldsymbol{\epsilon}}$ *commutes with* $\bar{\cdot}$ *and* ^{t}.
(d) *The set* $\mathbf{B}_{\mathfrak{n}_-}\bullet_{\boldsymbol{\epsilon}}\mathbf{B}_{\mathfrak{n}_+}:=\psi^{\boldsymbol{\epsilon}}(\mathbf{B}_{\mathfrak{n}_-}\bullet\mathbf{B}_{\mathfrak{n}_+})$ *is linearly independent and* $\mathbf{L}\cdot\mathbf{B}_{\mathfrak{n}_-}\bullet_{\boldsymbol{\epsilon}}\mathbf{B}_{\mathfrak{n}_+}$, *where* $\mathbf{L}$ *is the multiplicative subgroup of* $\mathcal{L}$ *generated by the* $L_i^{\pm1}$, $i\in I$, *is a basis of* $\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$.

Note that $\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ is graded by the group $Q:=\mathbb{Z}^I$ with $\deg_Q E_i=\deg_Q F_i=\deg_Q L_i=\alpha_i=-\deg_Q L_i^{-1}$, where $\{\alpha_i\}_{i\in I}$ is the standard basis of $\mathbb{Z}^I$. Let $\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})_0$ be the subalgebra of elements of degree 0.

Lemma 3.15. *There exists a unique projection* $\tau:\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})\to\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})_0$ *commuting with* $\bar{\cdot}$ *such that* $\tau(x)\in q^{\frac12\mathbb{Z}}\prod_{i\in I}L_i^{-n_i}x$ *for* x *homogeneous with* $\deg_Q x=\sum_{i\in I}n_i\alpha_i$.

Let $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ be the $\Bbbk$-algebra with presentation (1.6). The following Lemma is easily checked.

Lemma 3.16. *The algebra* $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ *admits an anti-linear anti-involution* $\bar{\cdot}$ *defined on generators by* $\bar{x}_i=x_i$, $\bar{y}_i=y_i$, *and an anti-involution* ^{t} *defined by* ${x_i}^t=y_i$, ${y_i}^t=x_i$.

Proposition 3.17. *The assignments* $x_i\mapsto q_i^{\frac12\epsilon_i}L_i^{-1}E_i$, $y_i\mapsto q_i^{\frac12\epsilon_i}F_iL_i^{-1}$ *define an isomorphism of algebras* $j_{\boldsymbol{\epsilon}}:\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})\to\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})_0$ *which commutes with* $\bar{\cdot}$ *and* ^{t}. *Moreover,* $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ *has a triangular decomposition* $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})=U_q^{\boldsymbol{\epsilon},+}\otimes U_q^{\boldsymbol{\epsilon},-}$ *where* $U_q^{\boldsymbol{\epsilon},+}$ *(respectively,* $U_q^{\boldsymbol{\epsilon},-}$*) is the subalgebra of* $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ *generated by the* x_i *(respectively,* y_i*),* $i\in I$.

Proof. Let $X_i=L_i^{-1}E_i$, $Y_i=F_iL_i^{-1}$. Then in $\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ we have

$$\begin{aligned}0&=\sum_{r+s=1-a_{ij}}(-1)^rE_i^{\langle s\rangle}E_jE_i^{\langle r\rangle}=\sum_{r+s=1-a_{ij}}(-1)^r(L_iX_i)^{\langle s\rangle}L_jX_j(L_iX_i)^{\langle r\rangle}\\&=\sum_{r+s=1-a_{ij}}(-1)^rq_i^{-\epsilon_i(\binom{s}{2}+\binom{r}{2})}L_i^sX_i^{\langle s\rangle}L_jX_jL_i^rX_i^{\langle r\rangle}\\&=L_i^{1-a_{ij}}L_j\sum_{r+s=1-a_{ij}}(-1)^rq_i^{-\epsilon_i(\binom{s}{2}+\binom{r}{2}+rs)-\frac{r}{2}\epsilon_ia_{ij}-\frac{s}{2}\epsilon_ja_{ij}}X_i^{\langle s\rangle}X_jX_i^{\langle r\rangle}\\&=q_i^{-\frac12\epsilon_i(1-a_{ij})^2}L_i^{1-a_{ij}}L_j\sum_{r+s=1-a_{ij}}(-1)^rq_i^{-\frac12(r\epsilon_i+s\epsilon_j)a_{ij}}X_i^{\langle s\rangle}X_jX_i^{\langle r\rangle}.\end{aligned}$$

This implies that

$$\sum_{r+s=1-a_{ij}}(-1)^rq_i^{r\epsilon_ja_{ij}\delta_{\epsilon_i,-\epsilon_j}}X_i^{\langle s\rangle}X_jX_i^{\langle r\rangle}=0.$$

Thus, the X_i satisfy the defining identity of $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$. Since $Y_i=X_i^t$, the identity for the Y_i is now immediate. The remaining identities are trivial. Thus, $j_{\boldsymbol{\epsilon}}$ is a well-defined homomorphism of algebras $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})\to\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$ and its image clearly lies in $\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})_0$. Since the defining relations of $U_q^{\boldsymbol{\epsilon},+}$ are the only relations in the subalgebra of $\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})_0$ generated by the $\{X_i\}_{i\in I}$, it follows that the restrictions of $j_{\boldsymbol{\epsilon}}$ to $U_q^{\boldsymbol{\epsilon},\pm}$ are injective. Since the corresponding subalgebras quasi-commute, the assertion follows. □

Now we have all necessary ingredients to prove Theorem 1.19.

Proof of Theorem 1.19. It follows from Lemma 3.15 and Proposition 3.17 that $\tau(\mathbf{B}_{\mathfrak{n}_-}\bullet_{\boldsymbol{\epsilon}}\mathbf{B}_{\mathfrak{n}_+})$ is a basis of $\widehat{\mathcal{H}}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})_0$. Then $\mathbf{B}_{\mathfrak{n}_-}\circ_{\boldsymbol{\epsilon}}\mathbf{B}_{\mathfrak{n}_+}:=j_{\boldsymbol{\epsilon}}^{-1}\tau(\mathbf{B}_{\mathfrak{n}_-}\bullet_{\boldsymbol{\epsilon}}\mathbf{B}_{\mathfrak{n}_+})$ is the desired basis of $\mathcal{A}_q^{\boldsymbol{\epsilon}}(\mathfrak{g})$. □

3.5. **Invariant quasi-derivations.** Following Lemma A.34 and also [19, Proposition 3.1.6], define $\Bbbk$-linear endomorphisms $\partial_i, \partial_i^{op}$, $i \in I$ of U_q^+ by

$$[F_i, x^+] = (q_i - q_i^{-1})(K_{+i} \diamond \partial_i(x^+) - K_{-i} \diamond \partial_i^{op}(x^+)), \qquad x^+ \in U_q^+. \tag{3.8}$$

Then

$$[x^-, E_i] = (q_i - q_i^{-1})(K_{+i} \diamond \partial_i(x^{-t})^t - K_{-i} \diamond \partial_i^{op}(x^{-t})^t), \qquad x^- \in U_q^-.$$

Lemma 3.18. *For all $x^+, y^+ \in U_q^+$, $i \in I$ we have*

(a) $\overline{\partial_i(x^+)} = \partial_i(\overline{x^+})$, $\overline{\partial_i^{op}(x^+)} = \partial_i^{op}(\overline{x^+})$

(b) $\partial_i(x^{+*}) = (\partial_i^{op}(x^+))^*$

(c) $\partial_{F_i}(x^+) = (q_i - q_i^{-1}) q_i^{\frac{1}{2}\alpha_i^\vee(\deg x^+ - \alpha_i)} \partial_i(x^+)$, $\partial_{F_i}^{op}(x^+) = (q_i - q_i^{-1}) q_i^{\frac{1}{2}\alpha_i^\vee(\deg x^+ - \alpha_i)} \partial_i^{op}(x^+)$.

(d) ∂_i, ∂_i^{op} *are quasi-derivations. Namely, for $x^+, y^+ \in U_q^+$ homogeneous we have*

$$\begin{aligned} \partial_i(x^+y^+) &= q_i^{\frac{1}{2}\alpha_i^\vee(\deg y^+)} \partial_i(x^+) y^+ + q_i^{-\frac{1}{2}\alpha_i^\vee(\deg x^+)} x^+ \partial_i(y^+), \\ \partial_i^{op}(x^+y^+) &= q_i^{-\frac{1}{2}\alpha_i^\vee(\deg y^+)} \partial_i^{op}(x^+) y^+ + q_i^{\frac{1}{2}\alpha_i^\vee(\deg x^+)} x^+ \partial_i^{op}(y^+). \end{aligned} \tag{3.9}$$

(e) $\partial_i \partial_j^{op} = \partial_j^{op} \partial_i$ *for all $i, j \in I$*

Proof. Parts (a)–(b) are immediate consequences of (3.8). Part (c) follows from (3.8) and (A.41). Then (d) is a consequence of (A.33) while (e) is immediate from (c) and Lemma A.9(c). □

In particular, the operators ∂_i, ∂_i^{op} are locally nilpotent hence we can define a function $\ell_i : U_q^+ \to \mathbb{Z}_{\ge 0}$ by

$$\ell_i(x^+) = \max\{k \in \mathbb{Z}_{\ge 0} \,:\, \partial_i^k(x^+) \ne 0\}, \qquad x^+ \in U_q^+.$$

Corollary 3.19. *If $x^+, y^+ \in U_q^+$ are homogeneous then for all $n \ge 0$*

$$\begin{aligned} \partial_i^{(n)}(x^+y^+) &= \sum_{a+b=n} q_i^{\frac{1}{2}\alpha_i^\vee(a \deg y^+ - b \deg x^+)} \partial_i^{(a)}(x^+) \cdot \partial_i^{(b)}(y^+) \\ (\partial_i^{op})^{(n)}(x^+y^+) &= \sum_{a+b=n} q_i^{\frac{1}{2}\alpha_i^\vee(-a \deg y^+ + b \deg x^+)} (\partial_i^{op})^{(a)}(x^+) \cdot (\partial_i^{op})^{(b)}(y^+) \end{aligned} \tag{3.10}$$

In particular,

$$\partial_i^{(top)}(x^+y^+) = q_i^{\frac{1}{2}\alpha_i^\vee(\ell_i(x^+) \deg y^+ - \ell_i(y^+) \deg x^+)} \partial_i^{(top)}(x^+) \partial_i^{(top)}(y^+),$$

where $\partial_i^{(top)}(x^+) = \partial_i^{(\ell_i(x^+))}(x^+)$.

Define $\partial_i^-, \partial_i^{-\,op} : U_q^- \to U_q^-$ by $\partial_i^-(x) = \partial_i(x^t)^t$ and $\partial_i^{-\,op}(x) = \partial_i^{op}(x^t)^t$, $x \in U_q^-$. Then $\ell_i : U_q^- \to \mathbb{Z}_{\ge 0}$ and $(\partial_i^-)^{(top)}$ are defined accordingly. We will sometimes use the notation ∂_i^+, $\partial_i^{+\,op}$ for ∂_i, ∂_i^{op}.

Lemma 3.20. *For all $x, y \in U_q^-$ and $k \in \mathbb{Z}_{\ge 0}$*

$$\langle\!| (\partial_i^-)^{(k)}(x), y |\!\rangle = \langle\!| x, F_i^{\langle k \rangle} y |\!\rangle, \qquad \langle\!| (\partial_i^{-\,op})^{(k)}(x), y |\!\rangle = \langle\!| x, y F_i^{\langle k \rangle} |\!\rangle. \tag{3.11}$$

Proof. It is sufficient to show that $(q_i - q_i^{-1}) \langle\!| \partial_i(x^t)^t, y |\!\rangle = \langle\!| x, F_i y |\!\rangle$. Then an obvious induction yields $(q_i - q_i^{-1})^n \langle\!| (\partial_i^-)^n(x), y |\!\rangle = \langle\!| x, F_i^n y |\!\rangle$ and the assertion follows. We have

$$\begin{aligned} (q_i - q_i^{-1}) \langle\!| \partial_i(x^t)^t, y |\!\rangle &= (q_i - q_i^{-1}) q^{-\frac{1}{2}\underline{\gamma}(\deg y)} \langle y, \partial_i(x^t)^* \rangle = (q_i - q_i^{-1}) q^{-\frac{1}{2}\underline{\gamma}(\deg y)} \langle y, \partial_i^{op}(x^{*t}) \rangle \\ &= q^{-\frac{1}{2}\underline{\gamma}(\deg y) - \frac{1}{2}\alpha_i \cdot (\deg x^{*t} - \alpha_i)} \langle y, \partial_{F_i}^{op}(x^{*t}) \rangle \\ &= q^{-\frac{1}{2}\underline{\gamma}(\deg y) - \frac{1}{2}\alpha_i \cdot \deg y} \langle F_i y, x^{*t} \rangle = q^{-\frac{1}{2}\underline{\gamma}(\deg y + \alpha_i)} \langle F_i y, x^{*t} \rangle = \langle\!| x, F_i y |\!\rangle. \end{aligned}$$

The second identity follows from the first since $\langle\!| x^*, y |\!\rangle = \langle\!| x, y^* |\!\rangle$. □

Example 3.21. Recall ([19, 14.5.3]) that $F_i^{\langle n\rangle} \in \mathbf{B}^{\mathrm{can}}$ for all $i \in I$, $n \in \mathbb{Z}_{\geq 0}$. Clearly, $(\!| x, F_i^{\langle n\rangle} |\!) = 0$ unless $x \in \Bbbk F_i^n$. Since $\partial_i(E_i^n) = (n)_{q_i} E_i^{n-1}$, it follows from Lemma 3.20 that $(\!| F_i^n, F_i^{\langle n\rangle} |\!) = 1$, hence $F_i^n = \delta_{F_i^{\langle n\rangle}} \in \mathbf{B}_{\mathfrak{n}_-}$.

We will need some properties of $\mathbf{B}_{\mathfrak{n}_\pm}$ with respect to $\partial_i^\pm$ which we gather in the following proposition

Proposition 3.22. *Let $b_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}$. Then*

(a) *For all $r \in \mathbb{Z}_{\geq 0}$,*

$$(\partial_i^-)^{(r)}(b_-) = \sum_{b'_- \in \mathbf{B}_{\mathfrak{n}_-}} \tilde{C}_{b'_-}^{F_i^r, b_-} b'_-, \qquad (\partial_i^{-\,op})^{(r)}(b_-) = \sum_{b'_- \in \mathbf{B}_{\mathfrak{n}_-}} \tilde{C}_{b'_-}^{b_-, F_i^r} b'_-,$$
$$(\partial_i^+)^{(r)}(b_+) = \sum_{b'_+ \in \mathbf{B}_{\mathfrak{n}_+}} \tilde{C}_{b'_+}^{E_i^r, b_+} b'_+, \qquad (\partial_i^{+\,op})^{(r)}(b_+) = \sum_{b'_+ \in \mathbf{B}_{\mathfrak{n}_+}} \tilde{C}_{b'_+}^{b_+, E_i^r} b'_+,$$

where $\tilde{C}_{b_\pm}^{b'_\pm, b''_\pm} \in \mathbb{Z}[q, q^{-1}]$ are defined in Corollary 3.7. Thus, in particular, $(\partial_i^\pm)^{(r)}(\mathbf{B}_{\mathfrak{n}_\pm}) \subset \mathbb{Z}[q, q^{-1}]\mathbf{B}_{\mathfrak{n}_\pm}$.

(b) $(\partial_i^\pm)^{(top)}(b_\pm), (\partial_i^{\pm\,op})^{(top)}(b_\pm) \in \mathbf{B}_{\mathfrak{n}_\pm}$. *Moreover, for each $b_\pm \in \mathbf{B}_{\mathfrak{n}_\pm} \cap \ker \partial_i^\pm$ and for each $n \in \mathbb{Z}_{\geq 0}$ there exists a unique $\hat{b}_\pm \in \mathbf{B}_{\mathfrak{n}_\pm}$ such that $\partial_i^{\pm(top)}(\hat{b}_\pm) = b_\pm$ and $\ell_i(\hat{b}_+) = n$.*

Proof. To prove (a), note that by Lemma 3.20, Remark 3.8 and Example 3.21 we have for any $b, b' \in \mathbf{B}^{\mathrm{can}}$

$$(\partial_i^-)^{(r)}(\delta_b) = \sum_{b' \in \mathbf{B}^{\mathrm{can}}} (\!| (\partial_i^-)^{(r)}(\delta_b), b' |\!) \delta_{b'} = \sum_{b' \in \mathbf{B}^{\mathrm{can}}} (\!| \delta_b, F_i^{\langle r\rangle} b' |\!) \delta_{b'}$$
$$= \sum_{b', b'' \in \mathbf{B}^{\mathrm{can}}} \tilde{C}_{\delta_{b''}}^{\delta_{F_i^{\langle r\rangle}}, \delta_{b'}} (\!| \delta_b, b'' |\!) \delta_{b'} = \sum_{b' \in \mathbf{B}^{\mathrm{can}}} \tilde{C}_{\delta_b}^{F_i^r, \delta_{b'}} \delta_{b'}.$$

The remaining identities are proved similarly.

To prove (b), note that since $\mathbf{B}_{\mathfrak{n}_+} = \mathbf{B}_{\mathfrak{n}_-}^t$ and $\mathbf{B}_{\mathfrak{n}_\pm}{}^* = \mathbf{B}_{\mathfrak{n}_\pm}$, it suffices to prove that $(\partial_i^-)^{(top)}(b_-) \in \mathbf{B}_{\mathfrak{n}_-}$. Following [19, §14.3], denote $\mathbf{B}_{i;\geq r}^{\mathrm{can}} = \mathbf{B}^{\mathrm{can}} \cap F_i^r U_q^-$ and $\mathbf{B}_{i;r}^{\mathrm{can}} = \mathbf{B}_{i;\geq r}^{\mathrm{can}} \setminus \mathbf{B}_{i;\geq r+1}^{\mathrm{can}}$. It follows from [19, §14.3] that for all $i \in I$, $\mathbf{B}^{\mathrm{can}} = \bigsqcup_{r \geq 0} \mathbf{B}_{i;r}^{\mathrm{can}}$. Let $b \in \mathbf{B}^{\mathrm{can}}$ and let $n = \ell_i(\delta_b)$, $u = (\partial_i^-)^{(top)}(\delta_b) = (\partial_i^-)^{(n)}(\delta_b)$. Then $u \in \ker \partial_i^-$ which, by Lemma 3.20, is orthogonal to $\mathbf{B}_{i;s}^{\mathrm{can}}$, $s > 0$. Thus, we can write

$$u = \sum_{b' \in \mathbf{B}_{i;0}^{\mathrm{can}}} (\!| u, b' |\!) \delta_{b'} = \sum_{b' \in \mathbf{B}_{i;0}^{\mathrm{can}}} (\!| \delta_b, F_i^{\langle n\rangle} b' |\!) \delta_{b'}.$$

By [19, Theorem 14.3.2], for each $b' \in \mathbf{B}_{i;0}^{\mathrm{can}}$ there exists a unique $\pi_{i,n}(b') \in \mathbf{B}_{i;n}^{\mathrm{can}}$ such that $F_i^{\langle n\rangle} b' - \pi_{i;n}(b') \in \sum_{r > n} \mathbb{Z}[q, q^{-1}] \mathbf{B}_{i;r}^{\mathrm{can}}$. Using Lemma 3.20, we conclude that for any $b'' \in \mathbf{B}_{i;r}^{\mathrm{can}}$ with $r > n$, $(\!| \delta_b, b'' |\!) \in (\!| \delta_b, F_i^{\langle r\rangle} U_q^- |\!) = (\!| (\partial_i^-)^{(r)}(\delta_b), U_q^- |\!) = 0$. Thus,

$$u = \sum_{b' \in \mathbf{B}_{i;0}^{\mathrm{can}}} (\!| \delta_b, \pi_{i;n}(b') |\!) \delta_{b'}.$$

Note that, since $u \neq 0$, we cannot have $(\!| \delta_b, \pi_{i;n}(b') |\!) = 0$ for all $b' \in \mathbf{B}_{i;0}^{\mathrm{can}}$. Since $(\!| \delta_b, b'' |\!) = \delta_{b,b''}$, we conclude that there exists a unique $b' \in \mathbf{B}_{i;0}^{\mathrm{can}}$ such that $\pi_{i;n}(b') = b$ and then $u = (\partial_i^-)^{(top)}(\delta_b) = \delta_{b'}$. Since $\pi_{i;n} : \mathbf{B}_{i;0}^{\mathrm{can}} \to \mathbf{B}_{i;n}^{\mathrm{can}}$ is a bijection by [19, Theorem 14.3.2], the assertion follows. $\square$

Let $b_+\in\mathbf{B}_{\mathfrak{n}_+}$ and let $r\le \ell_i(b_+)$. By the above Proposition, there exists a unique b'_+ such that $\ell_i(b'_+)=r$ and $\partial_i^{(r)}(b'_+)=\partial_i^{(top)}(b_+)$. This implies that for each $b_+\in\mathbf{B}_{\mathfrak{n}_+}$ and each $0\le r\le \ell_i(b_+)$ there exists a unique element of $\mathbf{B}_{\mathfrak{n}_+}$, denoted $\tilde\partial_i^r(b_+)$, such that $\ell_i(\tilde\partial_i^r(b_+))=\ell_i(b_+)-r$ and

$$\partial_i^{(r)}(b_+)-\binom{\ell_i(b_+)}{r}_{q_i}\tilde\partial_i^r(b_+)\in\sum_{b'_+\in\mathbf{B}_{\mathfrak{n}_+}\,:\,\ell(b'_+)<\ell_i(b_+)-r}\mathbb{Z}[q,q^{-1}]b'_+. \tag{3.12}$$

The correspondence $b_+\mapsto\tilde\partial_i^r(b_+)$ is a bijection. In particular, $\partial_i^{(top)}(b_+)=\tilde\partial_i^{\ell_i(b_+)}(b_+)$. Moreover, using [18, 5.3.8-5.3.10] we obtain

$$\partial_i^{(r)}(b_+)-\binom{\ell_i(b_+)}{r}_{q_i}\tilde\partial_i^r(b_+)\in q_i^{\binom{r+1}{2}-r\ell_i(b_+)}\sum_{b'_+\in\mathbf{B}_{\mathfrak{n}_+}\,:\,\ell(b'_+)<\ell_i(b_+)-r}q\mathbb{Z}[q]b'_+. \tag{3.13}$$

Example 3.23. We now discuss the construction of elements of the form $F_i^r\bullet b_+$, $i\in I$, $r\in\mathbb{Z}_{\ge0}$, $b_+\in\mathbf{B}_{\mathfrak{n}_+}$. We need the following

Lemma 3.24. *For all $x_+\in U_q^+$, $i\in I$ and $r\in\mathbb{Z}_{\ge0}$*

$$x^+F_i^r=\sum_{\substack{r',r''\ge0\\ r'+r''\le r}}(-1)^{r'}q_i^{-\binom{r'}{2}+\binom{r''}{2}}\langle r'+r''\rangle_{q_i}!\binom{r}{r'+r''}_{q_i}K_{+i}^{r'}K_{-i}^{r''}\diamond(F_i^{r-r'-r''}\partial_i^{(r')}\partial_i^{op\,(r'')}(x^+)).$$

Proof. Since by (A.36)

$$\begin{aligned}\underline{\Delta}(1\otimes\underline{\Delta})(F_i^r)&=\sum_{r'+r''+r'''=r}\begin{bmatrix}r\\ r'+r''\end{bmatrix}_{q_i^2}\begin{bmatrix}r'+r''\\ r'\end{bmatrix}_{q_i^2}F_i^{r'}\otimes F_i^{r''}\otimes F_i^{r'''}\\ &=\sum_{r'+r''+r'''=r}q_i^{r'r''+r'r'''+r''r'''}\frac{(r)_{q_i}!}{(r')_{q_i}!(r'')_{q_i}!(r''')_{q_i}!}F_i^{r'}\otimes F_i^{r''}\otimes F_i^{r'''},\end{aligned}$$

we have by Proposition A.36, Lemma A.30, (A.6) and Lemma 3.18(c)

$$\begin{aligned}x^+F_i^r=&\sum_{r'+r''+r'''=r}(-1)^{r'}q_i^{-r'(r'-1)-r'r''-r'r'''-r''r'''}\frac{(r)_{q_i}!}{(r')_{q_i}!(r'')_{q_i}!(r''')_{q_i}!}\times\\ &\qquad\langle F_i^{r'},\underline{x}^+_{(3)}\rangle\langle F_i^{r'''},\underline{x}^+_{(1)}\rangle K_{-i}^{r'''}F_i^{r''}\underline{x}^+_{(2)}K_{+i}^{r'}\\ =&\sum_{r'+r''+r'''=r}(-1)^{r'}q_i^{-r'(r'-1)-r'r''-r'r'''-r''r'''}\frac{(r)_{q_i}!}{(r')_{q_i}!(r'')_{q_i}!(r''')_{q_i}!}K_{-i}^{r'''}F_i^{r''}\partial_{F_i}^{r'}((\partial_{F_i}^{op})^{r'''}(x^+))K_{+i}^{r'}\\ =&\sum_{r'+r''+r'''=r}(-1)^{r'}q_i^{-r'(r'-1)-r'r''-2r'r'''-r''r'''+\frac12(r'+r''')\alpha_i^\vee(\deg x_+)-\binom{r'+1}{2}-\binom{r'''+1}{2}}\times\\ &\qquad\frac{(r)_{q_i}!(q_i-q_i^{-1})^{r'+r'''}}{(r'')_{q_i}!}K_{-i}^{r'''}F_i^{r''}\partial_i^{(r')}((\partial_i^{op})^{(r''')}(x^+))K_{+i}^{r'}\\ =&\sum_{r',r''\ge0}(-1)^{r'}q_i^{\frac12(r'+r'')\alpha_i^\vee(\deg x_+-r\alpha_i)-\binom{r'}{2}+\binom{r''}{2}}\langle r'+r''\rangle_{q_i}\binom{r}{r'+r''}_{q_i}\times\\ &\qquad K_{-i}^{r''}F_i^{r-r'-r''}\partial_i^{(r')}((\partial_i^{op})^{(r'')}(x^+))K_{+i}^{r'}\\ =&\sum_{r',r''\ge0}(-1)^{r'}q_i^{-\binom{r'}{2}+\binom{r''}{2}}\langle r'+r''\rangle_{q_i}\binom{r}{r'+r''}_{q_i}K_{+i}^{r'}K_{-i}^{r''}\diamond F_i^{r-r'-r''}\partial_i^{(r')}((\partial_i^{op})^{(r'')}(x^+)).\end{aligned}$$ □

In particular, if $b_+ \in \mathbf{B}_{\mathfrak{n}_+} \cap \ker \partial_i^{op}$ then we have

$$\overline{F_i^r b_+} = F_i^r b_+ + \sum_{r'=0}^{r} (-1)^{r'} q_i^{-\binom{r'}{2}} \langle r' \rangle_{q_i}! \binom{r}{r'}_{q_i} \sum_{\substack{b'_+ \in \mathbf{B}_{\mathfrak{n}_+} \cap \ker \partial_i^{op} \\ \ell_i(b'_+) \le \ell_i(b_+) - r}} \tilde{C}_{b'_+}^{E_i^r, b_+} K_{+i}^{r'} \diamond (F_i^{r-r'} b'_+)$$

This implies that $F_i^r \bullet b_+ = F_i^r \circ b_+$ is the unique $\bar{\cdot}$-invariant element of $U_q(\tilde{\mathfrak{g}})$ of the form

$$F_i^r b_+ + \sum_{r'=1}^{\min(r, \ell_i(b_+))} \sum_{\substack{b'_+ \in \mathbf{B}_{\mathfrak{n}_+} \cap \ker \partial_i^{op} \\ \ell_i(b'_+) \le \ell_i(b_+) - r'}} C^+_{b_+, b'+; r} K_{+i}^{r'} \diamond (F_i^{r-r'} b'_+), \qquad C^+_{b_+, b'+; r} \in q\mathbb{Z}[q]. \tag{3.14}$$

Similarly, if $b_+ \in \ker \partial_i$ then $F_i^r \circ b_+ = F_i^r b_+$ and $F_i^r \bullet b_+$ is the unique $\bar{\cdot}$-invariant element of $U_q(\tilde{\mathfrak{g}})$ of the form

$$F_i^r b_+ + \sum_{r'=1}^{\min(r, \ell_i(b_+^*))} \sum_{\substack{b'_+ \in \mathbf{B}_{\mathfrak{n}_+} \cap \ker \partial_i \\ \ell_i({b'_+}^*) \le \ell_i(b_+^*) - r'}} C^-_{b_+, b'+; r} K_{-i}^{r'} \diamond (F_i^{r-r'} b'_+), \qquad C^-_{b_+, b'+; r} \in q^{-1}\mathbb{Z}[q^{-1}]. \tag{3.15}$$

The coefficients $C^{\pm}_{b_+, b'_+; r}$ can be expressed inductively, but in general it is not possible to write an explicit formula for them.

4. Examples of double canonical bases

4.1. **Double canonical basis of $U_q(\mathfrak{sl}_2)$.** In this section we explicitly compute the double canonical basis in $\mathcal{H}_q^+(\mathfrak{g})$ and $U_q(\tilde{\mathfrak{g}})$ for $\mathfrak{g} = \mathfrak{sl}_2$.

Lemma 4.1. *In $\mathcal{H}_q^+(\mathfrak{sl}_2)$ we have*

$$F^{m_-} \circ E^{m_+} = \sum_{j \ge 0} (-1)^j q^{j(|m_+ - m_-| + 1)} \begin{bmatrix} \min(m_-, m_+) \\ j \end{bmatrix}_{q^2} K_+^j \diamond F^{m_- - j} E^{m_+ - j}, \quad m_\pm \in \mathbb{Z}_{\ge 0}. \tag{4.1}$$

Proof. For $m_\pm \in \mathbb{Z}_{\ge 0}$ denote the right hand side of (4.1) by $\mathbf{b}_{m_-, m_+}$. Let $C_+ = \mathbf{b}_{1,1} = FE - qK_+$. Observe that C_+ is central in $\mathcal{H}_q^+(\mathfrak{g})$, since $C_+ F = F(FE + (q^{-1} - q)K_+) - qK_+ F = FC_+$ and similarly $[E, C_+] = 0$. Furthermore, $\overline{C}_+ = EF - q^{-1}K_+ = FE + (q^{-1} - q)K_+ - q^{-1}K_+ = C_+$. We have

$$\begin{aligned}
\mathbf{b}_{m,m} C_+ &= \sum_{j \ge 0} (-1)^j q^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} K_+^j F^{m-j} E^{m-j} (FE - qK_+) \\
&= \sum_{j \ge 0} (-q)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} K_+^j (F^{m+1-j} E^{m+1-j} - q^{2(m-j)+1} K_+ F^{m-j} E^{m-j}) \\
&= \sum_{j \ge 0} (-q)^j \Big(\begin{bmatrix} m \\ j \end{bmatrix}_{q^2} + q^{2(m-j+1)} \begin{bmatrix} m \\ j-1 \end{bmatrix}_{q^2} \Big) K_+^j F^{m+1-j} E^{m+1-j} = \mathbf{b}_{m+1, m+1}.
\end{aligned}$$

Therefore, $\mathbf{b}_{m,m} = C_+^m$, $m \in \mathbb{Z}_{\ge 0}$, whence for all $m_\pm \in \mathbb{Z}_{\ge 0}$

$$\mathbf{b}_{m_-, m_+} = \sum_{j \ge 0} (-q)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} F^{[m_- - m_+]_+} (K_+^j \diamond F^{m-j} E^{m-j}) E^{[m_+ - m_-]_+} = F^{[m_- - m_+]_+} C_+^m E^{[m_+ - m_-]_+},$$

where $m = \min(m_+, m_-)$ and $[a]_+ = \max(0, a)$. Since C_+ is $\bar{\cdot}$-invariant and central, it follows that $\overline{\mathbf{b}_{m_-, m_+}} = \mathbf{b}_{m_-, m_+}$. By definition, $\mathbf{b}_{m_-, m_+} - F^{m_-} E^{m_+} \in \sum_{j > 0} q\mathbb{Z}[q] K_+^j \diamond F^{m_- - j} E^{m_+ - j}$, and the assertion follows by Theorem 1.3. □

Thus, the double canonical basis of $\mathcal{H}_q^+(\mathfrak{sl}_2)$ is

$$\mathbf{B}^+_{\mathfrak{sl}_2}=\{K_+^a\diamond F^{m_-}C_+^{m_0}E^{m_+}\,:\,a,m_\pm,m_0\in\mathbb{Z}_{\ge0},\,\min(m_+,m_-)=0\}.$$

Let $C^{(0)}=1$, $C^{(1)}=C=C_+-q^{-1}K_-=FE-qK_+-q^{-1}K_-$ and define inductively

$$C^{(m+1)}=CC^{(m)}-K_+K_-C^{(m-1)},\qquad m\ge1. \tag{4.2}$$

Note that C is central in $U_q(\tilde{\mathfrak{g}})$ and $\bar{\ }$-invariant, hence $\overline{C^{(m)}}=C^{(m)}$. It follows directly by induction on m that

Lemma 4.2. *For all $m,k\ge0$*

$$\begin{aligned}F^kC^{(m)}&=\sum_{a,b\ge0}(-1)^{a+b}q^{(k+1)(a-b)}\begin{bmatrix}m-a\\ b\end{bmatrix}_{q^{-2}}\begin{bmatrix}m-b\\ a\end{bmatrix}_{q^2}K_+^aK_-^b\diamond F^{m+k-a-b}E^{m-a-b}\\ C^{(m)}E^k&=\sum_{a,b\ge0}(-1)^{a+b}q^{(k+1)(a-b)}\begin{bmatrix}m-a\\ b\end{bmatrix}_{q^{-2}}\begin{bmatrix}m-b\\ a\end{bmatrix}_{q^2}K_+^aK_-^b\diamond F^{m-a-b}E^{m+k-a-b}\end{aligned}\tag{4.3}$$

Proposition 4.3. *For all $m\ge0$,*

$$C^{(m)}=\sum_{0\le i\le j,\,i+j\le m}(-1)^jq^{-j-i^2}\begin{bmatrix}m-i\\ j\end{bmatrix}_{q^{-2}}\begin{bmatrix}j\\ i\end{bmatrix}_{q^{-2}}K_+^iK_-^jF^{m-i-j}\circ E^{m-i-j},\qquad m\ge0.\tag{4.4}$$

In particular, $C^{(m)}=F^m\bullet E^m\in\mathbf{B}_{\widetilde{\mathfrak{sl}_2}}$.

Proof. Let $\iota:\mathcal{H}_q^+(\mathfrak{g})\to U_q(\tilde{\mathfrak{g}})$ be the natural inclusion of vector spaces. One can show by induction on k that in $U_q(\tilde{\mathfrak{g}})$

$$\iota(C_+^k)C=\iota(C_+^{k+1})-q^{-2k-1}K_-\iota(C_+^k)+(1-q^{-2k})K_-K_+\iota(C_+^{k-1}).\tag{4.5}$$

Denote by X_m the right hand side of (4.4). It follows from (4.5) that

$$\begin{aligned}&X_mC-K_+K_-X_{m-1}\\ &=\sum_{i,j\ge0}(-1)^jq^{-j-i^2}\begin{bmatrix}m-i\\ j\end{bmatrix}_{q^{-2}}\begin{bmatrix}j\\ i\end{bmatrix}_{q^{-2}}\Big(K_+^iK_-^j\iota(C_+^{m+1-i-j})\\ &\qquad-q^{-2(m-i-j)-1}K_+^iK_-^{j+1}\iota(C_+^{m-i-j})+(1-q^{-2(m-i-j)})K_+^{i+1}K_-^{j+1}\iota(C_+^{m-1-i-j})\Big)\\ &\qquad+\sum_{i,j\ge0}(-1)^{j+1}q^{-j-i^2}\begin{bmatrix}m-1-i\\ j\end{bmatrix}_{q^{-2}}\begin{bmatrix}j\\ i\end{bmatrix}_{q^{-2}}K_+^{i+1}K_-^{j+1}\iota(C_+^{m-1-i-j})\\ &=\sum_{i,j\ge0}(-1)^jq^{-j-i^2}\begin{bmatrix}m-i\\ j\end{bmatrix}_{q^{-2}}\begin{bmatrix}j\\ i\end{bmatrix}_{q^{-2}}\Big(K_+^iK_-^j\iota(C_+^{m+1-i-j})-q^{-2(m-i-j)-1}K_+^iK_-^{j+1}\iota(C_+^{m-i-j})\Big)\\ &\quad+\sum_{i,j\ge0}(-1)^{j+1}q^{-j-i^2}q^{-2(m-i)}\begin{bmatrix}m-1-i\\ j\end{bmatrix}_{q^{-2}}\begin{bmatrix}j\\ i\end{bmatrix}_{q^{-2}}K_+^{i+1}K_-^{j+1}\iota(C_+^{m-1-i-j})\\ &=\sum_{i,j\ge0}(-1)^jq^{-j-i^2}\Big(\begin{bmatrix}m-i\\ j\end{bmatrix}_{q^{-2}}\begin{bmatrix}j\\ i\end{bmatrix}_{q^{-2}}+q^{-2(m+1-i-j)}\begin{bmatrix}m-i\\ j-1\end{bmatrix}_{q^{-2}}\Big(\begin{bmatrix}j-1\\ i\end{bmatrix}_{q^{-2}}\\ &\qquad+q^{-2(j-i)}\begin{bmatrix}j-1\\ i-1\end{bmatrix}_{q^{-2}}\Big)\Big)K_+^iK_-^j\iota(C_+^{m+1-i-j})\\ &=\sum_{i,j\ge0}(-1)^jq^{-j-i^2}\begin{bmatrix}m+1-i\\ j\end{bmatrix}_{q^{-2}}\begin{bmatrix}j\\ i\end{bmatrix}_{q^{-2}}K_+^iK_-^j\iota(C_+^{m+1-i-j})=X_{m+1}\end{aligned}$$

since

$$\begin{bmatrix} j-1 \\ i \end{bmatrix}_{q^{-2}} + q^{-2(j-i)} \begin{bmatrix} j-1 \\ i-1 \end{bmatrix}_{q^{-2}} = \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}},$$
$$q^{-2(m-i-j+1)} \begin{bmatrix} m-i \\ j-1 \end{bmatrix}_{q^{-2}} + \begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} = \begin{bmatrix} m+1-i \\ j \end{bmatrix}_{q^{-2}}.$$

Thus, X_m satisfies the recurrence relation (4.2). Since $X_0 = 1$ and $X_1 = C$, we conclude that $X_m = C^{(m)}$ for all $m \geq 0$. The second assertion is now immediate by Theorem 1.5 since $\overline{C^{(m)}} = C^{(m)}$ and by (4.4),

$$C^{(m)} - F^m \circ E^m \in \sum_{\substack{j>0 \\ 0\leq i\leq \min(j,m-j)}} q^{-1}\mathbb{Z}[q^{-1}] K_-^j K_+^i F^{m-i-j} \circ E^{m-i-j}. \qquad \square$$

Corollary 4.4. *For all $m_-, m_+ \geq 0$,*

$$F^{m_-} \bullet E^{m_+} = \sum_{\substack{0\leq i\leq j \\ i+j\leq m}} (-1)^j q^{-j-i^2-(j-i)|m_+-m_-|} \begin{bmatrix} m-i \\ j \end{bmatrix}_{q^{-2}} \begin{bmatrix} j \\ i \end{bmatrix}_{q^{-2}} K_+^i K_-^j \diamond F^{m_- -i-j} \circ E^{m_+ -i-j}, \tag{4.6}$$

where $m = \min(m_+, m_-)$.

Combining Lemma 4.2 and Proposition 4.3 and using (3.4) we obtain the following identity.

Proposition 4.5. *For all $m, a, b \geq 0$ with $a + b \leq m$ we have in $\mathbb{Z}[\nu]$*

$$\sum_{r=0}^{\min(a,b)} (-1)^r \nu^{\binom{r}{2}} \frac{[m-r]_\nu!}{[a-r]_\nu! [b-r]_\nu! [r]_\nu!} = \nu^{ab} [m-a-b]_\nu! \begin{bmatrix} m-b \\ a \end{bmatrix}_\nu \begin{bmatrix} m-a \\ b \end{bmatrix}_\nu$$

Our preceding computations, together with Theorem 1.5, immediately yield the following

Proposition 4.6. *For all $m_\pm \in \mathbb{Z}_{\geq 0}$,*

$$\begin{aligned} F^{m_-} \bullet E^{m_+} &= F^{m_- - m} C^{(m)} E^{m_+ - m} \\ &= \sum_{0\leq a+b\leq m} (-1)^{a+b} q^{(|m_+-m_-|+1)(a-b)} \begin{bmatrix} m-a \\ b \end{bmatrix}_{q^{-2}} \begin{bmatrix} m-b \\ a \end{bmatrix}_{q^2} K_+^a K_-^b \diamond F^{m_- -a-b} E^{m_+ -a-b} \end{aligned}$$

where $m = \min(m_+, m_-)$. Thus, the double canonical basis in $U_q(\widetilde{\mathfrak{sl}_2})$ is given by

$$\{K_+^{a_+} K_-^{a_-} \diamond F^{m_-} C^{(m_0)} E^{m_+} \,:\, a_\pm, m_\pm, m_0 \in \mathbb{Z}_{\geq 0},\, \min(m_+, m_-) = 0\}.$$

An easy induction shows that

$$C^{(a)} C^{(b)} = \sum_{j=0}^{\min(a,b)} (K_- K_+)^j C^{(a+b-2j)} \tag{4.7}$$

Lemma 4.7. *For all $n \geq 0$ we have*

$$F^n E^n = \sum_{r=0}^{n} \Big(\sum_{j=0}^{r} c_{r,j}^{(n)} K_-^j K_+^{r-j} \Big) C^{(n-r)}, \qquad E^n F^n = \sum_{r=0}^{n} \Big(\sum_{j=0}^{r} c_{r,j}^{(n)} K_-^{r-j} K_+^j \Big) C^{(n-r)}$$

where $c_{0,0}^{(n)} = 1$, $\overline{c_{r,j}^{(n)}} = c_{r,r-j}^{(n)} \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$. In particular, Conjecture 1.21 holds for $\mathfrak{g} = \mathfrak{sl}_2$.

Proof. The induction base is clear since $FE = C+qK_+ + q^{-1}K_-$. Thus, $c^{(1)}_{0,0} = 1$ and $c^{(1)}_{1,0} = q = \overline{c^{(1)}_{1,1}}$.
For the inductive step we have

$$
\begin{aligned}
F^{n+1}E^{n+1} &= \sum_{r=0}^{n} F\Big(\sum_{j=0}^{r} c^{(n)}_{r,j} K_-^j K_+^{r-j}\Big)C^{(n-r)}E = \sum_{r=0}^{n}\Big(\sum_{j=0}^{r} c^{(n)}_{r,j} q^{2(r-2j)} K_-^j K_+^{r-j}\Big)FEC^{(n-r)}\\
&= \sum_{r=0}^{n}\Big(\sum_{j=0}^{r} c^{(n)}_{r,j} q^{2(r-2j)} K_-^j K_+^{r-j}\Big)(C+qK_+ + q^{-1}K_-)C^{(n-r)}\\
&= \sum_{r=0}^{n}\Big(\sum_{j=0}^{r} c^{(n)}_{r,j} q^{2(r-2j)} K_-^j K_+^{r-j}\Big)C^{(n+1-r)} + \sum_{r=0}^{n-1}\Big(\sum_{j=0}^{r} c^{(n)}_{r,j} q^{2(r-2j)} K_-^{j+1} K_+^{r+1-j}\Big)C^{(n-r-1)}\\
&+ \sum_{r=0}^{n}\Big(\sum_{j=0}^{r} c^{(n)}_{r,j} q^{2(r-2j)+1} K_-^j K_+^{r+1-j}\Big)C^{(n-r)} + \sum_{r=0}^{n}\Big(\sum_{j=0}^{r} c^{(n)}_{r,j} q^{2(r-2j)-1} K_-^{j+1} K_+^{r-j}\Big)C^{(n-r)}\\
&= \sum_{r=0}^{n+1}\Big(\sum_{j=0}^{r} q^{2(r-2j)}(c^{(n)}_{r,j} + c^{(n)}_{r-2,j-1} + q^{-1}c^{(n)}_{r-1,j} + qc^{(n)}_{r-1,j-1})\Big)K_-^j K_+^{r-j} C^{(n+1-r)}
\end{aligned}
$$

where we use the convention that $c^{(n)}_{r,j} = 0$ if $r < 0$, $j < 0$, $j > r$ or $r > n$. Set

$$c^{(n+1)}_{r,j} = q^{2(r-2j)}(c^{(n)}_{r,j} + c^{(n)}_{r-2,j-1} + q^{-1}c^{(n)}_{r-1,j} + qc^{(n)}_{r-1,j-1})$$

Then $c^{(n+1)}_{0,0} = 1$ and $c^{(n+1)}_{r,j} \in \mathbb{Z}_{\ge 0}[q, q^{-1}]$. Also

$$\overline{c^{(n+1)}_{r,j}} = q^{2(2j-r)}(c^{(n)}_{r,r-j} + c^{(n)}_{r-2,r-j-1} + qc^{(n)}_{r-1,r-j-1} + q^{-1}c^{(n)}_{r-1,r-j}) = c^{(n+1)}_{r,r-j}.$$

This proves the inductive step. The second formula follows from the first by applying $\bar{\cdot}$. □

Remark 4.8. One can prove, using the above computation, an even stronger statement, namely that for any two elements $\mathbf{b}$, $\mathbf{b}'$ of $\mathbf{B}_{\widetilde{\mathfrak{sl}_2}}$, $\mathbf{b}\mathbf{b}'$ decomposes as a linear combination of elements of the same basis with coefficients being Laurent polynomials in q with positive coefficients. However, this fact is special for $\mathfrak{sl}_2$ and is unlikely to hold in greater generality.

4.2. **Action on a double basis for $\mathfrak{sl}_2$.** We now consider the action of $U_q(\mathfrak{g})$ on the double canonical basis of $U_q(\widetilde{\mathfrak{sl}_2})$. To preserve $\bar{\cdot}$-invariance, it is necessary to consider its twisted version given by

$$F_i(x) := q_i^{\frac{1}{2}(\lambda_i+\alpha_i^\vee(x))}F_i x - q_i^{-\frac{1}{2}(\lambda_i+\alpha_i^\vee(x))} xF_i, \quad E_i(x) := K_{+i}^{-1} \diamond (q_i^{-\frac{\lambda_i}{2}} E_i x - q_i^{\frac{\lambda_i}{2}} xE_i), \tag{4.8}$$

for any $\lambda \in \mathbb{Z}^I$ (cf. [13]). We denote the corresponding operators by E_λ, F_λ.

Lemma 4.9. *Let $\lambda, a_\pm \in \mathbb{Z}$. Then for all $m_+ > m_-$*

$$
\begin{aligned}
F_\lambda(K_-^{a_-}K_+^{a_+} \diamond F^{m_-} \bullet E^{m_+}) &= \langle \tfrac{1}{2}\lambda + a_+ - a_- + 2m_+ - 2m_-\rangle_q K_+^{a_+ +1}K_-^{a_-} \diamond F^{m_-} \bullet E^{m_+ -1}\\
&+ \langle \tfrac{1}{2}\lambda + a_+ - a_- + m_+ - m_-\rangle_q K_-^{a_- +1}K_+^{a_+ +1} \diamond F^{m_- -1} \bullet E^{m_+ -2}\\
&+ \langle \tfrac{1}{2}\lambda + a_+ - a_- + m_+ - m_-\rangle_q K_-^{a_-}K_+^{a_+} \diamond F^{m_- +1} \bullet E^{m_+}\\
&+ \langle \tfrac{1}{2}\lambda + a_+ - a_-\rangle_q K_-^{a_- +1}K_+^{a_+} \diamond F^{m_-} \bullet E^{m_+ -1}
\end{aligned}
\tag{4.9}
$$

where we use the convention that $F^r \bullet E^s = 0$ if $r < 0$ or $s < 0$, while for $m_+ \le m_-$

$$F_\lambda(K_-^{a_-}K_+^{a_+} \diamond F^{m_-} \bullet E^{m_+}) = \langle \tfrac{1}{2}\lambda + a_+ - a_- + m_+ - m_-\rangle_q K_-^{a_-}K_+^{a_+} \diamond F^{m_- +1} \bullet E^{m_+} \tag{4.10}$$

Furthermore, for all $m_+ \geq m_-$

$$E_\lambda(K_-^{a_-}K_+^{a_+} \diamond F^{m_-} \bullet E^{m_+}) = \langle \tfrac{1}{2}\lambda + a_+ - a_- \rangle_q K_+^{a_+-1}K_-^{a_-} \diamond F^{m_-} \bullet E^{m_++1} \tag{4.11}$$

while for all $m_+ < m_-$

$$\begin{aligned} E_\lambda(K_-^{a_-}K_+^{a_+} \diamond F^{m_-} \bullet E^{m_+}) &= \langle \tfrac{1}{2}\lambda + a_+ - a_- + m_- - m_+ \rangle_q K_-^{a_-}K_+^{a_+} \diamond F^{m_--1} \bullet E^{m_+} \\ &+ \langle \tfrac{1}{2}\lambda + a_+ - a_- \rangle_q K_-^{a_-+1}K_+^{a_+} \diamond F^{m_--2} \bullet E^{m_+-1} \\ &+ \langle \tfrac{1}{2}\lambda + a_+ - a_- \rangle_q K_-^{a_-}K_+^{a_+-1} \diamond F^{m_-} \bullet E^{m_++1} \\ &+ \langle \tfrac{1}{2}\lambda + a_+ - a_- + m_+ - m_- \rangle_q K_-^{a_-+1}K_+^{a_+-1} \diamond F^{m_--1} \bullet E^{m_+}. \end{aligned} \tag{4.12}$$

Proof. It is an easy consequence of (4.3) that

$$\begin{aligned} FE^k &= CE^{k-1} + q^k K_+ \diamond E^{k-1} + q^{-k} K_- \diamond E^{k-1}, \\ E^kF &= CE^{k-1} + q^{-k} K_+ \diamond E^{k-1} + q^k K_- \diamond E^{k-1}, \qquad k \geq 1 \end{aligned}$$

We also have

$$F_\lambda(K_+^{a_+}K_-^{a_-} \diamond x) = K_+^{a_+}K_-^{a_-} \diamond F_{\lambda+2a_+-2a_-}(x),$$

so it is sufficient to prove all identities for $a_+ = a_- = 0$. Suppose first that $m_+ > m_-$. Then $F^{m_-} \bullet F^{m_+} = C^{(m_-)}E^{m_+-m_-}$ and

$$\begin{aligned} F_\lambda(C^{(m_-)}E^{m_+-m_-}) &= q^{\frac{1}{2}\lambda+m_+-m_-}FC^{(m_-)}E^{m_+-m_-} - q^{-\frac{1}{2}\lambda-(m_+-m_-)}C^{(m_-)}E^{m_+-m_-}F \\ &= \langle \tfrac{1}{2}\lambda + m_+ - m_- \rangle_q CC^{(m_-)}E^{m_+-m_--1} + \langle \tfrac{1}{2}\lambda + 2(m_+ - m_-) \rangle_q K_+ \diamond C^{(m_-)}E^{m_+-m_--1} \\ &\quad + \langle \tfrac{1}{2}\lambda \rangle_q K_- \diamond C^{(m_-)}E^{m_+-m_--1}, \end{aligned}$$

and it remains to use (4.2). If $m_+ \leq m_-$ then $F^{m_-} \bullet E^{m_+} = F^{m_--m_+}C^{(m_+)}$ and

$$F_\lambda(F^{m_-} \bullet E^{m_+}) = (q^{\frac{\lambda}{2}-(m_--m_+)} - q^{-\frac{\lambda}{2}+m_--m_+})F^{m_--m_++1}C^{(m_+)} = \langle \tfrac{1}{2}\lambda + m_+ - m_- \rangle_q F^{m_-+1} \bullet E^{m_+}.$$

The identities involving E_λ are proved similarly. □

Corollary 4.10. *If $k_+ = \frac{\lambda}{2} + a_+ - a_-\max(0, m_- - m_+)$ is a non-negative integer then $k_+ = \max\{k \geq 0 : E_\lambda^k(K_-^{a_-}K_+^{a_+} \diamond F^{m_-} \bullet E^{m_+}) \neq 0\}$. Similarly, if $k_- = \frac{\lambda}{2} + a_+ - a_- + m_+ - m_- + \max(0, m_+ - m_-)$ is a non-negative integer then $k_- = \max\{k \geq 0 : F_\lambda^k(K_-^{a_-}K_+^{a_+} \diamond F^{m_-} \bullet E^{m_+}) \neq 0\}$.*

Proof. We prove only the first statement, the proof of the second one being similar. If $m_- \leq m_+$ then by an obvious induction we obtain

$$E_\lambda^s(F^{m_-} \bullet E^{m_+}) = \langle \tfrac{\lambda}{2} \rangle_q \cdots \langle \tfrac{\lambda}{2} - s + 1 \rangle_q K_+^{-s} \diamond F^{m_-} \bullet E^{m_++s},$$

which is zero if and only if $\lambda \in 2\mathbb{Z}_{\geq 0}$ and $s \geq \frac{\lambda}{2} + 1$. If $m_- > m_+$ then each term in the right hand side of (4.12) is of the form $K \diamond F^a \bullet E^b$ with $a - b = m_- - m_+ - 1$ and the term with the largest coefficient is $F^{m_--1} \bullet E^{m_+}$. Thus,

$$E_\lambda^{m_--m_+}(F^{m_-} \bullet E^{m_+}) = \langle \tfrac{\lambda}{2} + m_- - m_+ \rangle_q \cdots \langle \tfrac{\lambda}{2} + 1 \rangle_q F^{m_+} \bullet E^{m_+} + \cdots$$

where the remaining terms are of the form $K \diamond F^a \bullet E^a$ with the coefficients being of the form $\prod_{j=0}^{s} \langle \frac{\lambda}{2} + k - j \rangle_q$ with $k < m_- - m_+$. It follows that $E_\lambda^s(F^{m_-} \bullet E^{m_+}) = 0$ only if $\frac{1}{2}\lambda + m_- - m_+ \in \mathbb{Z}_{\geq 0}$ and $s > \frac{1}{2}\lambda + m_- - m_+$. □

Define

$$\varepsilon^\lambda(F^{m_-} \bullet E^{m_+}) = \tfrac{\lambda}{2} + \max(0, m_- - m_+).$$

Then we obtain the following

Corollary 4.11. *For all $\lambda\in\mathbb{Z}$, $m_\pm\in\mathbb{Z}_{\ge 0}$*

$$E_\lambda(F^{m_-}\bullet E^{m_+})=\langle\varepsilon^\lambda(F^{m_-}\bullet E^{m_+})\rangle_q\mathbf{b}+\sum_{\mathbf{b}'\,:\,\varepsilon^\lambda(\mathbf{b}')<\varepsilon^\lambda(F^{m_-}\bullet E^{m_+})}c_{\mathbf{b}'}\bullet\mathbf{b}'$$

where

$$\mathbf{b}=\begin{cases}F^{m_--1}\bullet E^{m_+}, & m_->m_+\\ K_+^{-1}\diamond F^{m_-}\bullet E^{m_++1}, & m_-\le m_+.\end{cases}$$

4.3. **Some elements in double canonical bases in ranks 2 and 3.** We will need explicit formulae for some elements of dual canonical bases for computational purposes. We already listed the most obvious ones in Example 3.21.

Example 4.12. It easy to see, extending [19, §14.5.4], that the elements $F_i^{\langle s\rangle}F_j^{\langle 1\rangle}F_i^{\langle n-s\rangle}$, $0\le s\le n\le -a_{ij}$ are in $\mathbf{B}^{\rm can}$ and form a basis of the homogeneous component of U_q^- of degree $n\alpha_{-i}+\alpha_{-j}$. Let $F_{i^sji^r}=\delta_{F_i^{\langle r\rangle}F_j^{\langle 1\rangle}F_i^{\langle s\rangle}}$, $r,s\ge 0$, $r+s\le -a_{ij}$ and let $E_{i^sji^r}=(F_{i^sji^r})^{*t}$. We summarize their properties in the following Lemma, which is proved by direct computations based on Lemma 3.20.

Lemma 4.13. (a) *For all $k,l\ge 0$, $k+l<-a_{ij}$ we have*

$$F_iF_{i^kji^l}=q^{l+\frac12a_{ij}}F_{i^{k+1}ji^l}+q^{-k-\frac12a_{ij}}F_{i^kji^{l+1}},\quad F_{i^kji^l}F_i=q^{-l-\frac12a_{ij}}F_{i^{k+1}ji^l}+q^{k+\frac12a_{ij}}F_{i^kji^{l+1}}$$

(b) *For all $r,s\ge 0$, $r+s\le -a_{ij}$ we have*

$$\underline{\Delta}(F_{i^sji^r})=\sum_{r'+r''=r}q_i^{r'(r''+s+\frac12a_{ij})}\binom{r}{r'}_{q_i}F_i^{r'}\otimes F_{i^sji^{r''}}+\sum_{s'+s''=s}q_i^{s''(s'+r+\frac12a_{ij})}\binom{s}{s''}_{q_i}F_{i^{s'}ji^r}\otimes F_i^{s''}.$$

(c) *For all $s,r,s',r'\ge 0$, $s+r=s'+r'\le -a_{ij}$, we have*

$$\langle F_{i^{s'}ji^{r'}},E_{i^sji^r}\rangle=(-1)^{r+s'}q_i^{r's'+(r'-r)(a_{ij}+r'-1)}p_{s',r,r'}(q)\frac{\langle 1\rangle_{q_j}\langle s\rangle_{q_i}!\langle r\rangle_{q_i}!}{\prod_{t=0}^{s+r-1}q_i^{a_{ij}+t}-q_i^{-a_{ij}-t}}$$

where

$$p_{s',r,r'}(q)=\sum_{l=0}^{\min(s',r)}q_i^{l(r'+s'+2a_{ij}-2)}\binom{s'}{l}_{q_i}\binom{r'}{r-l}_{q_i}\in\mathbb{Z}_{\ge 0}[q,q^{-1}].$$

The following Lemma provides a partial converse to Theorem 3.11.

Lemma 4.14. *Suppose that $\langle b_-,b_+\rangle\in\mathbb{Z}[q,q^{-1}]$ for all $b_\pm\in\mathbf{B}_{\mathfrak{n}_\pm}$. Then for every $i\ne j$, $a_{ij}a_{ji}<4$.*

Proof. We may assume without generality that $a_{ij},a_{ji}\ne 0$ and $|a_{ij}|\ge|a_{ji}|$ hence $d_i\le d_j$. Then by Lemma 4.13, $\langle F_{ij},E_{ij}\rangle=(q_j-q_j^{-1})(q_i-q_i^{-1})/(q_i^{a_{ij}}-q_i^{-a_{ij}})$ which can only be in $\mathbb{Z}[q,q^{-1}]$ if $d_j|a_{ji}|=d_i|a_{ij}|\le d_i+d_j$. Therefore, $|a_{ji}|\le 1+d_i/d_j<2$, hence $a_{ji}=-1$ and $d_j=-d_ia_{ij}$. Suppose that $|a_{ij}|\ge 4$. Applying Lemma 4.13 again we obtain

$$\langle F_{i^2j},E_{i^2j}\rangle=\frac{(q_j-q_j^{-1})(q_i-q_i^{-1})(q_i^2-q_i^{-2})}{(q_i^{a_{ij}}-q_i^{-a_{ij}})(q_i^{a_{ij}+1}-q_i^{-a_{ij}-1})}=\frac{(q_i-q_i^{-1})(q_i^2-q_i^{-2})}{q_i^{|a_{ij}|-1}-q_i^{-|a_{ij}|+1}}.$$

This cannot be a Laurent polynomial if $|a_{ij}|>4$ by the degree considerations, while for $a_{ij}=-4$ we have $\langle F_{i^2j},E_{i^2j}\rangle=(q^4-1)/(q^4+q^2+1)\notin\mathbb{Z}[q,q^{-1}]$. Thus, $|a_{ij}|\le 3$. □

From now on, given $f=\sum_j a_j\nu^j\in\mathbb{Z}[\nu,\nu^{-1}]$, let $[f]_+=\sum_{j>0}a_j\nu^j$ and $[f]_-=\sum_{j<0}a_j\nu^j$. We will now consider some examples in rank 2.

First, assume that $a_{ji}=-1$ (in particular, this includes all subdiagrams of rank 2 for $\mathfrak g$ semisimple and all affine cases except those of rank 2). Then $d_j=dd_i$, $a_{ij}=-d$ and by Lemma 4.13

$$[E_{i^sj},F_{i^sj}]=[E_{ji^s},F_{ji^s}]=-\frac{\langle s\rangle_{q_i}!}{\langle d-1\rangle_{q_i}\cdots\langle d-s+1\rangle_{q_i}}(K_{+i}^sK_{+j}-K_{-i}^sK_{-j}),$$

hence $\mathbf d_{F_{i^sj},E_{i^sj}}=\mathbf d_{F_{ji^s},E_{ji^s}}$ and $F_{i^sj}\bullet E_{i^sj}-\mathbf d_{F_{i^sj},E_{i^sj}}F_{i^sj}E_{i^sj}=F_{ji^s}\bullet E_{ji^s}-\mathbf d_{F_{ji^s},E_{ji^s}}F_{ji^s}E_{ji^s}$, while

$$\begin{aligned}&F_{ij}\bullet E_{ij}=F_{ij}E_{ij}-q_iK_{+i}K_{+j}-q_i^{-1}K_{-i}K_{-j},\\&F_{i^dj}\bullet E_{i^dj}=F_{i^dj}E_{i^dj}-q_i^dK_{+i}^dK_{+j}-q_i^{-d}K_{-i}^dK_{-j},\end{aligned}$$

and for $d>2$

$$F_{i^2j}\bullet E_{i^2j}=\begin{cases}(d-1)_{q_i}F_{i^2j}E_{i^2j}-q_i^2K_{+i}^2K_{+j}-q_i^{-2}K_{-i}^2K_{-j}, & d \text{ even}\\ (\frac12(d-1))_{q_i^2}F_{i^2j}E_{i^2j}-q_iK_{+i}^2K_{+j}-q_i^{-1}K_{-i}^2K_{-j}, & d\text{ odd}\end{cases}$$

while for $d>3$

$$F_{i^3j}\bullet E_{i^3j}=\begin{cases}\binom{d-1}{2}_{q_i}F_{i^3j}E_{i^3j}-q_i^3K_{+i}^3K_{+j}-q_i^{-3}K_{-i}^3K_{-j}, & d=0\pmod 3\\ (\frac13(d-1))_{q_i^3}(d-2)_{q_i}F_{i^3j}E_{i^3j}-q_i^2K_{+i}^3K_{+j}-q_i^{-2}K_{-i}^3K_{-j}, & d=1\pmod 6\\ (\frac13(d-2))_{q_i^3}(d-1)_{q_i}F_{i^3j}E_{i^3j}-q_i^2K_{+i}^3K_{+j}-q_i^{-2}K_{-i}^3K_{-j}, & d=2\pmod 6\\ (\frac13(d-1))_{q_i^3}(\frac12(d-2))_{q_i^2}F_{i^3j}E_{i^3j}-q_iK_{+i}^3K_{+j}-q_i^{-1}K_{-i}^3K_{-j}, & d=4\pmod 6\\ (\frac13(d-2))_{q_i^3}(\frac12(d-1))_{q_i^2}F_{i^3j}E_{i^3j}-q_iK_{+i}^3K_{+j}-q_i^{-1}K_{-i}^3K_{-j}, & d=5\pmod 6.\end{cases}$$

Note that if $d\le 2$ then $F_{ij}^k\in\mathbf B_{\mathfrak n_-}$; for $d=2$ we also have $F_{i^2j}^k\in\mathbf B_{\mathfrak n_-}$ for all $k\in\mathbb Z_{\ge0}$. Then we can use (4.6) to compute $F_{ij}^{m_-}\bullet E_{ij}^{m_+}$ (respectively, $F_{i^2j}^{m_-}\bullet E_{i^2j}^{m_+}$) for all $m_\pm\in\mathbb Z_{\ge0}$. Similarly, we obtain

$$\begin{aligned}&F_{ij}\circ E_{ji}=F_{ij}E_{ji}-q_i^dK_{+j}F_iE_i+(q_i^{d+1}-[q_i^{d-1}]_+)K_{+i}K_{+j},\\&F_{ij}\bullet E_{ji}=F_{ij}\circ E_{ji}-q_i^{-1}K_{-i}F_j\circ E_j+q_i^{-1-d}K_{-i}K_{-j}\\&F_{ji}\circ E_{ij}=F_{ji}E_{ij}-q_iK_{+i}F_jE_j+q_i^{d+1}K_{+i}K_{+j}\\&F_{ji}\bullet E_{ij}=F_{ji}\circ E_{ij}-q_i^{-d}K_{-j}F_i\circ E_i+([q_i^{1-d}]_- q_i^{-1-d})K_{-i}K_{-j}\end{aligned}$$

If $d_i=d_j$, $a_{ij}=a_{ji}=-a$ we obtain

$$[E_{i^sj},F_{i^sj}]=[E_{ji^s},F_{ji^s}]=-\frac{\langle 1\rangle_{q_i}\langle s\rangle_{q_i}!}{\langle a\rangle_{q_i}\cdots\langle a-s+1\rangle_{q_i}}(K_{+i}^sK_{+j}-K_{-i}^sK_{-j}),$$

and so for all $1\le s\le a$

$$F_{i^sj}\bullet E_{i^sj}-\binom{a}{s}_{q_i}F_{i^sj}E_{i^sj}=-q_iK_{+i}^sK_{+j}-q_i^{-1}K_{-i}^sK_{-j}=F_{ji^s}\bullet E_{ji^s}-\binom{a}{s}_{q_i}F_{ji^s}E_{ji^s}.$$

We also have

$$\begin{aligned}&F_{ij}\circ E_{ji}=(a)_{q_i}F_{ij}F_{ji}-q_i^aK_{+j}F_iE_i+(q_i^{a+1}-[q_i^{a-1}]_+)K_{+i}K_{+j},\\&F_{ij}\bullet E_{ji}=F_{ij}\circ E_{ji}-q_i^{-a}K_{-i}F_j\circ E_j-[q_i^{1-a}]_-(K_{-i}K_{+j}+K_{-i}K_{-j}).\end{aligned}$$

Furthermore, for $a>1$

$$\begin{aligned}F_{ij^2}\circ E_{j^2i}&=\tbinom{a}{2}_{q_i}F_{ij^2}E_{j^2i}-(q_i^a+[q_i^{a-2}]_+)(a)_{q_i}K_{+j}F_{ij}E_{ji}+(q_i^2(1+[q_i^{2a-4}]_+))K_{+j}^2F_iE_i\\&\quad-(q_i^3+(q_i^3-q_i)[q_i^{2a-4}]_+)K_{+i}K_{+j}^2\\F_{ij^2}\bullet E_{j^2i}&=F_{ij^2}\circ E_{j^2i}-q_i^{-1-2\{a\}}[\tfrac{a-\{a\}}{2}]_{q_i^{-4}}K_{-i}F_j^2\circ E_j^2\end{aligned}$$

$$
\begin{aligned}
&+ q_i^{-4}[a-3]_{q_i^{-2}} K_{-i}K_{+j}F_j \circ E_j + (q_i^{-2a} + q^{-2(a-1)} - [q_i^{-2\{a\}}]_-)K_{-i}K_{-j}F_j \circ E_j \\
&+ (q_i^{-2a+1} + q_i^{-2a+3-2\delta_{a,2}} - q_i^{-3} - [q_i^{-\{a\}}]_-)K_{-i}K_{-j}K_{+j} \\
&+ q_i^{-1}(1-\delta_{a,2} - q_i^{-2\{a\}}[\tfrac{a-\{a\}}{2}-1]_{q_i^{-4}})K_{-i}K_{+j}^2 + (q_i^{-2a+3} - q_i^{-2a+1} - [q_i^{-1+\{a\}}]_-)K_{-i}K_{-j}^2
\end{aligned}
$$

$$
\begin{aligned}
F_{i^2j} \circ E_{ji^2} &= \tbinom{a}{2}_{q_i} F_{i^2j}E_{ji^2} - q_i^{1+2\{a\}}[\tfrac{a-\{a\}}{2}]_{q_i^4} K_{+j}F_i^2E_i^2 + (q_i^{2a} + q_i^{2(a-1)} - [q_i^{2\{a\}}]_+)K_{+i}K_{+j}F_iE_i \\
&+ (q_i^{2a-3} - q_i^{2a-1} - [q_i^{1-\{a\}}]_+)K_{+i}^2K_{+j} \\
F_{i^2j} \bullet E_{ji^2} &= F_{i^2j} \circ E_{ji^2} - (q_i^{-a} + [q_i^{-a+2}]_-)K_{-i}F_{ij} \circ E_{ji} + q_i^{-2}(1 + [q^{-2a+4}]_-)K_{-i}^2F_j \circ E_j \\
&+ [q_i^{-2+2\{a\}}]_- K_{-i}K_{+j}F_i \circ E_i - [q_i^{-\{a\}}]_- K_{-i}K_{+i}K_{+j} \\
&+ (-q_i^{-3}[q_i^{-2a+4}]_- + q_i^{-2}([q_i^{-2a+5}]_- - q_i^{-1}))K_{-i}^2K_{-j} + (q_i^{-1}[q_i^{-2a+4}]_- + q_i^{-\{a\}})K_{-j}K_{+i}^2 \\
F_{iji} \circ E_{ji^2} &= \tbinom{a}{2}_{q_i} F_{iji}E_{ji^2} - q_i^{a-1}K_{+i}K_{+j}F_iE_i + (q_i^a - [q_i^{a-2}]_+)K_{+i}^2K_{+j} \\
F_{iji} \bullet E_{ji^2} &= F_{iji} \circ E_{ji^2} - q_i^{1-a}K_{-i}F_{ji} \circ E_{ji} - [q_i^{2-a}]_- K_{-i}K_{+i}K_{+j} + (q_i^{-a} - [q_i^{2-a}]_-)K_{-i}^2K_{-j} \\
F_{iji} \circ E_{iji} &= (a)_{q_i}^2(a-1)_{q_i}F_{iji}E_{iji} - q_i^2(a)_{q_i}[a-1]_{q_i^2}K_{+i}F_{ij}E_{ji} \\
&+ q_i^{a+2}[a-1]_{q_i^2}K_{+i}K_{+j}F_iE_i - q_i^{a-1}(1+q_i^{2a})K_{+i}^2K_{+j} \\
F_{iji} \bullet E_{iji} &= F_{iji} \circ E_{iji} - q_i^{-2}[a-1]_{q_i^{-2}}K_{-i}F_{ji} \circ E_{ij} + q_i^{-a-2}[a-1]_{q_i^{-2}}K_{-j}K_{-i}F_i \circ E_i \\
&- q_i^{1-a}K_{-i}K_{+i}K_{+j} + (q_i^{-1-a}[a-1]_{q_i^{-2}} - q_i^{1-a})K_{-j}K_{-i}K_{+i} - q_i^{1-a}(1+q_i^{-2a})K_{-i}^2K_{-j}
\end{aligned}
$$

where $\{a\} = a \pmod 2$, $\{a\} \in \{0,1\}$. If $a > 2$ we also have

$$
\begin{aligned}
F_{iji^2} \circ E_{ji^3} &= \tbinom{a}{3}_{q_i} F_{iji^2}E_{ji^3} - q_i^{a-2}K_{+i}^2K_{+j}F_iE_i + (q_i^{a-1} - [q_i^{a-3}]_+)K_{+i}^3K_{+j} \\
F_{iji^2} \bullet E_{ji^3} &= F_{iji^2} \circ E_{ji^3} - q_i^{2-a}K_{-i}F_{ji^2} \circ E_{ji^2} - [q_i^{3-a}]_- K_{-i}K_{+i}^2K_{+j} + (q_i^{1-a} - [q_i^{3-a}]_-)K_{-i}^3K_{-j}.
\end{aligned}
$$

Example 4.15. Let $d_i = d_j = 1$ and $a_{ij} = a_{ji} = -2$. After [19, §14.5.5], the elements of degree $2(\alpha_{-i} + \alpha_{-j})$ in $\mathbf{B}^{\mathrm{can}}$ are

$$
F_i^{\langle 2\rangle}F_j^{\langle 2\rangle},\ F_i^{\langle 1\rangle}F_j^{\langle 2\rangle}F_i^{\langle 1\rangle},\ F_i^{\langle 1\rangle}F_j^{\langle 1\rangle}F_i^{\langle 1\rangle}F_j^{\langle 1\rangle} - F_i^{\langle 2\rangle}F_j^{\langle 2\rangle}
$$

as well as three more elements obtained from these by applying the automorphism which interchanges F_i and F_j. The corresponding elements of $\mathbf{B}_{\mathfrak{n}_-}$ are, respectively,

$$
\begin{aligned}
F_{j^2i^2} &= (q^2(2)_qF_i^2F_j^2 - (2q^3 + (2)_q)F_iF_jF_iF_j + (q-q^{-1})(F_iF_j^2F_i + F_jF_i^2F_j) \\
&+ ((2)_q + 2q^{-3})F_jF_iF_jF_i - q^{-2}(2)_qF_j^2F_i^2)/((q-q^{-1})(q^2-q^{-2})(q^4-q^{-4})) \\
F_{ij^2i} &= (F_i^2F_j^2 + F_j^2F_i^2 + F_jF_i^2F_j + (3)_q(F_iF_j^2F_i - F_iF_jF_iF_j - F_jF_iF_jF_i))/((q-q^{-1})(q^4-q^{-4})) \\
F_{jiji} &= (q^{-2}(2)_qF_j^2F_i^2 - q^2(2)_qF_i^2F_j^2 + (q^{-3}-q^3)(F_jF_i^2F_j + F_iF_j^2F_i) \\
&+ q^4(2q^{-3} + (2)_q)F_iF_jF_iF_j - q^{-4}(2q^3 + (2)_q)F_jF_iF_jF_i)/((q-q^{-1})(q^2-q^{-2})(q^4-q^{-4}))
\end{aligned}
$$

Set $E_\alpha = F_\alpha{}^{t*}$. Since $\mathbf{d}_{F_{ji},E_{ji}} = (2)_q$ by the previous example we have

$$
\begin{aligned}
F_{j^2i^2} \circ E_{j^2i^2} &= (2)_q(4)_qF_{j^2i^2}E_{j^2i^2} + (q-q^3)(2)_qK_{+i}K_{+j}F_{ji}E_{ji} - 2q^2K_{+i}^2K_{+j}^2 \\
E_{j^2i^2} \bullet F_{j^2i^2} &= F_{j^2i^2} \circ E_{j^2i^2} + (q^{-1}-q^{-3})K_{-i}K_{-j}F_{ji} \circ E_{ji} - 2q^{-2}K_{-i}^2K_{-j}^2 - q^{-2}K_{-i}K_{-j}K_{+i}K_{+j}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
F_{ij^2i} \circ E_{ij^2i} &= (2)_{q^2}F_{ij^2i}E_{ij^2i} + (q-q^3)K_{+i}F_{ij^2}E_{j^2i} + (q^5-q^3)(2)_qK_{+i}K_{+j}F_{ij}E_{ji} \\
&\qquad + (q^3-q^5)K_{+i}K_{+j}^2F_iE_i + (q^6-q^4-q^2)K_{+i}^2K_{+j}^2 \\
F_{ij^2i} \bullet E_{ij^2i} &= F_{ij^2i} \circ E_{ij^2i} + (q^{-1}-q^{-3})K_{-i}F_{j^2i} \circ E_{ij^2} - q^{-2}K_{-i}K_{+i}F_j^2 \circ E_j^2
\end{aligned}
$$

$$
\begin{aligned}
&+(q^{-5}-q^{-3})K_{-i}K_{-j}F_{ji}\circ E_{ij}+2q^{-3}K_{-i}K_{-j}K_{+i}F_j\circ E_j\\
&+(q^{-3}-q^{-5})K_{-i}K_{-j}^2F_i\circ E_i+(q^{-6}-q^{-4}-q^{-2})K_{-i}^2K_{-j}^2-q^{-4}K_{-i}K_{+i}K_{+j}^2\\
&+q^{-4}K_{-i}K_{-j}K_{+i}K_{+j}
\end{aligned}
$$

$$
F_{jiji}\circ E_{jiji}=(2)_q(4)_qF_{jiji}E_{jiji}+(q-q^3)(2)_qK_{+i}K_{+j}F_{ji}E_{ji}-2q^2K_{+i}^2K_{+j}^2
$$

$$
F_{jiji}\bullet E_{jiji}=F_{jiji}\circ E_{jiji}+(q^{-1}-q^{-3})K_{-i}K_{-j}F_{ji}\circ E_{ji}-2q^{-2}K_{-i}^2K_{-j}^2-q^{-2}K_{-i}K_{-j}K_{+i}K_{+j}.
$$

Example 4.16. Let $\mathfrak{g}=\widehat{\mathfrak{sl}_3}$, that is, $I=\{1,2,3\}$ and $a_{ij}=a_{ji}=-1$ for all $i\neq j$. For $\{i,j,k\}=\{1,2,3\}$ let

$$
F_{ijk}=((q-q^{-1})(q^3-q^{-3}))^{-1}\Big(q^{\frac{3}{2}}((2)_qF_kF_jF_i-F_jF_kF_i-F_kF_iF_j)
+q^{-\frac{3}{2}}((2)_qF_iF_jF_k-F_iF_kF_j-F_jF_iF_k)\Big).
$$

Then $F_{ijk}=\delta_{F_k^{\langle 1\rangle}F_j^{\langle 1\rangle}F_i^{\langle 1\rangle}}$. We have

$$
\begin{aligned}
F_{ijk}\bullet E_{ijk}&=(3)_qF_{ijk}E_{ijk}-q^2K_{+i}K_{+j}K_{+k}-q^{-2}K_{-i}K_{-j}K_{-k}\\
F_{ijk}\bullet E_{ikj}&=(3)_qF_{ijk}E_{ikj}-qK_{+i}K_{+j}K_{+k}-q^{-1}K_{-i}K_{-j}K_{-k}\\
F_{ijk}\bullet E_{jik}&=(3)_qF_{ijk}E_{jik}-qK_{+i}K_{+j}K_{+k}-q^{-1}K_{-i}K_{-j}K_{-k}\\
F_{ijk}\circ E_{jki}&=(3)_qF_{ijk}E_{jki}-q^3K_{+j}K_{+k}F_iE_i+(q^4-q^2)K_{+i}K_{+j}K_{+k}\\
F_{ijk}\bullet E_{jki}&=F_{ijk}\circ E_{jki}-q^{-3}K_{-i}F_{jk}\circ E_{jk}-q^{-2}K_{-i}K_{+j}K_{+k}+(q^{-4}-q^{-2})K_{-i}K_{-j}K_{-k}\\
F_{ijk}\circ E_{kji}&=(3)_qF_{ijk}E_{kji}-q^3K_{+k}F_{ij}E_{ji}+q^4K_{+j}K_{+k}F_iE_i+(q-q^5)K_{+i}K_{+j}K_{+k}\\
F_{ijk}\bullet E_{kji}&=F_{ijk}\circ E_{kji}-q^{-3}K_{+i}F_{jk}\circ E_{kj}-q^{-2}K_{-i}K_{+k}F_j\circ E_j+q^{-4}K_{-i}K_{-j}F_k\circ E_k\\
&\quad+(q^{-1}-q^{-5})K_{+i}K_{+j}K+k-q^{-3}K_{-i}K_{-j}K_{+k}\\
F_{ijk}\circ E_{kij}&=(3)_qF_{ijk}E_{kij}-q^3K_{+k}F_{ij}E_{ij}-q^2K_{+i}K_{+j}K_{+k}\\
F_{ijk}\bullet E_{kij}&=F_{ijk}\circ E_{kij}-q^{-3}K_{-i}K_{-j}F_k\circ E_k+(q^{-4}-q^{-2})K_{-i}K_{-j}K_{-k}-q^{-2}K_{-i}K_{-j}K_{+k}.
\end{aligned}
$$

These examples shows that we can have $\mathbf{d}_{b_-,b_+}\neq 1$ even if all subdiagrams of rank 2 of the Dynkin diagram of $\mathfrak{g}$ are of finite type.

4.4. **Reshetikhin–Semenov-Tian-Shanvksy map.** Define a pairing $\{\cdot,\cdot\}:U_q^-\otimes U_q^+\to\Bbbk$ by $\{u_-,u_+\}=(\!|u_-,u_+^{*t}|\!)$, $u_\pm\in U_q^\pm$. It follows from Proposition 3.20 that

$$
\begin{aligned}
&\{u_-,u_+E_i^{\langle 1\rangle}\}=\{\partial_i^{-\,op}(u^-),u_+\}, &&\{u_-,E_i^{\langle 1\rangle}u_+\}=\{\partial_i^-(u_-),u_+\}\\
&\{F_i^{\langle 1\rangle}u_-,u_+\}=\{u_-,\partial_i(u_+)\}, &&\{u_-F_i^{\langle 1\rangle},u_+\}=\{u_-,\partial_i^{op}(u_+)\}.
\end{aligned}\tag{4.13}
$$

Let Λ be a fixed weight lattice for $\mathfrak{g}$ containing Γ and let $\pi:\widehat{\Gamma}\to\Lambda$ be the homomorphism of monoids defined by $\pi(\alpha_{\pm i})=\pm\alpha_i$. Given $u\in U_q(\tilde{\mathfrak{g}})$ homogeneous, let $\deg_\Gamma u=\pi(\deg_{\widehat{\Gamma}}u)$. Note that $\{u_-,u_+\}\neq 0$ implies that $\deg_\Gamma u_-=-\deg_\Gamma u_+$, $u_\pm\in U_q^\pm$.

Extend the $\alpha_i^\vee$ to elements of $\operatorname{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z})$. Let $\check{U}_q(\tilde{\mathfrak{g}})$ be the algebra $\widehat{U}_q(\mathfrak{g})$ extended by adjoining elements of the form $K_{0,2\mu}$, $\mu\in\Lambda$. Thus, $\check{U}_q(\tilde{\mathfrak{g}})$ is generated by the $U_q^\pm$ and $K_{\alpha_-,2\mu+\alpha_+}$, $\mu\in\Lambda$, $\alpha_\pm\in\mathbb{Z}\Gamma$ such that for all $i\in I$

$$
K_{\alpha_-,2\mu+\alpha_+}E_i=q_i^{\alpha_i^\vee(2\mu+\alpha_+-\alpha_-)}E_iK_{\alpha_-,2\mu+\alpha_+},\qquad K_{\alpha_-,2\mu+\alpha_+}F_i=q_i^{-\alpha_i^\vee(2\mu+\alpha_+-\alpha_-)}F_iK_{\alpha_-,2\mu+\alpha_+}.
$$

It should be noted that $\check{U}_q(\tilde{\mathfrak{g}})=\widehat{U}_q(\tilde{\mathfrak{g}})$ if $2\Lambda=\mathbb{Z}\Gamma$.

Recall that a $U_q(\mathfrak{g})$-module V is called *lowest weight* of lowest weight $-\mu\in\Lambda$ if there exists $v_{-\mu}\in V\setminus\{0\}$ such that $V=U(\mathfrak{g})v_{-\mu}$, $U_q^-v_{-\mu}=0$ and $K_iv_{-\mu}=q_i^{-\alpha_i^\vee(\mu)}v_{-\mu}$, $i\in I$. Clearly, a lowest weight module is graded by Γ and we denote by $|v|$ the degree of a homogeneous element v of V; then $K_i v = q_i^{\alpha_i^\vee(-\mu+|v|)} v$, $i \in I$.

Let V be a lowest weight module of lowest weight $-\mu \in \Lambda$. Let $\langle\cdot\,|\,\cdot\rangle_V$ be a symmetric pairing $V \otimes V \to \Bbbk$ such that $\langle xu\,|\,v\rangle_V = \langle u\,|\,x^t v\rangle_V$ for all $x \in U_q(\mathfrak{g})$, $u, v \in V$. The radical of such a pairing is clearly a submodule of V hence for V simple it is non-degenerate. Since $\langle u\,|\,v\rangle_V \neq 0$ implies that $|u| = |v|$ for $u, v \in V$ homogeneous and homogeneous components of V are finite dimensional, it follows that if $\langle\cdot\,|\,\cdot\rangle_V$ is non-degenerate then any basis of V admits a dual basis with respect to $\langle\cdot\,|\,\cdot\rangle_V$.

Let $\mathbf{B}_\pm$ be a homogeneous bases of $U_q^\pm$. Define a map $\Xi : V \otimes V \to \check{U}_q(\tilde{\mathfrak{g}})$ by

$$\Xi(v \otimes v') := q^{\frac{1}{2}\underline{\gamma}(|v'|)-\frac{1}{2}\underline{\gamma}(|v|)} \sum_{b_\pm \in \mathbf{B}_\pm} q^{\eta(b_+)} \langle \check{b}_+ v'\,|\,\check{b}_-{}^t v\rangle_V (K_{|\check{b}_+ v'|,0} \diamond b_-)(K_{0,2\mu-|v'|} \diamond b_+), \tag{4.14}$$

for all $v, v' \in V$, where $\{\check{b}_\pm\}_{b_\pm \in \mathbf{B}_\pm} \subset U_q^\mp$ denotes the dual basis to $\mathbf{B}_\pm$ with respect to the pairing $\{\cdot,\cdot\}$. Thus, $\{\check{b}_+, b'_+\} = \delta_{b_+,b'_+}$, $\{b'_-, \check{b}_-\} = \delta_{b'_-,b_-}$. Note that the sum in (4.14) is finite since $|xv| = |v| + \deg_\Gamma x$ for any $v \in V$, $x \in U_q^-$ homogeneous, $\deg_\Gamma x \in -\Gamma$, there are finitely many $\nu \in \Gamma$ such that $|v| - \nu \in \Gamma$ and all homogeneous components of U_q^- are finite dimensional.

Proposition 4.17 (Theorem 1.25)**.** *Let $V^\#$ be V with the left action of $U_q(\mathfrak{g})$ defined by $x \triangleright v = S(x)^t v$, $x \in U_q(\mathfrak{g})$, $v \in V$. Then $\Xi : V^\# \otimes V \to \check{U}_q(\tilde{\mathfrak{g}})$ is a homomorphism of left $U_q(\mathfrak{g})$-modules where $V^\# \otimes V$ is endowed with a $U_q(\mathfrak{g})$-module structure via the comultiplication and the $U_q(\mathfrak{g})$ action on $\check{U}_q(\tilde{\mathfrak{g}})$ is the adjoint one.*

Remark 4.18. The formulae in Theorem 1.25 are obtained from the action defined above. The module $V^\#$ is highest weight of highest weight μ.

Proof. Let $v, v' \in V$ be homogeneous and set $\xi = |v|, \xi' = |v'|$. We also abbreviate $\langle\cdot\,|\,\cdot\rangle = \langle\cdot\,|\,\cdot\rangle_V$, $|x| = \deg_\Gamma x$ for $x \in U_q(\tilde{\mathfrak{g}})$ homogeneous and set $\kappa(\xi',\xi) = \frac{1}{2}\underline{\gamma}(\xi') - \frac{1}{2}\underline{\gamma}(\xi)$. Since $\langle \check{b}_+ v'\,|\,\check{b}_-{}^t v\rangle \neq 0$ implies that $|b_-| + |b_+| = \xi' - \xi$, it follows that $K_i \Xi(v \otimes v') = q_i^{\alpha_i^\vee(\xi'-\xi)} \Xi(v \otimes v') = \Xi(K_i^{-1} v \otimes K_i v')$. Furthermore,

$$\begin{aligned}
&q^{-\kappa(\xi',\xi)} \Xi(E_i^{\langle 1\rangle}(v \otimes v')) = q^{-\kappa(\xi',\xi)} \Xi(v \otimes E_i^{\langle 1\rangle}(v')) - q^{-\kappa(\xi',\xi)} \Xi(K_{+i}^{-1} F_i^{\langle 1\rangle}(v) \otimes K_{+i}(v'))\\
&= q^{\kappa(\xi'+\alpha_i,\xi)-\kappa(\xi',\xi)} \sum_{b_\pm \in \mathbf{B}_\pm} q^{\eta(b_+)} \langle \check{b}_+ E_i^{\langle 1\rangle}(v')\,|\,\check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0} \diamond b_-)(K_{0,2\mu-\xi'-\alpha_i} \diamond b_+)\\
&- q^{\kappa(\xi',\xi-\alpha_i)-\kappa(\xi',\xi)} q_i^{\alpha_i^\vee(\xi'-\xi)+2} \sum_{b_\pm \in \mathbf{B}_\pm} q^{\eta(b_+)} \langle \check{b}_+(v')\,|\,(E_i^{\langle 1\rangle} \check{b}_-)^t(v)\rangle (K_{\xi'-|b_+|,0} \diamond b_-)(K_{0,2\mu-\xi'} \diamond b_+)\\
&= q_i^{\frac{1}{2}\alpha_i^\vee(\xi')} \sum_{b_\pm \in \mathbf{B}_\pm} q^{\eta(b_+)} \langle E_i^{\langle 1\rangle} \check{b}_+(v')\,|\,\check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0} \diamond b_-)(K_{0,2\mu-\xi'-\alpha_i} \diamond b_+)\\
&+ q_i^{\frac{1}{2}\alpha_i^\vee(\xi')} \sum_{b_\pm \in \mathbf{B}_\pm} q^{\eta(b_+)} \langle K_{+i} \diamond \partial_i^-(\check{b}_+)(v')\,|\,\check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0} \diamond b_-)(K_{0,2\mu-\xi'-\alpha_i} \diamond b_+)\\
&- q_i^{\frac{1}{2}\alpha_i^\vee(\xi')} \sum_{b_\pm \in \mathbf{B}_\pm} q^{\eta(b_+)} \langle K_{-i} \diamond \partial_i^{-\,op}(\check{b}_+)(v')\,|\,\check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0} \diamond b_-)(K_{0,2\mu-\xi'-\alpha_i} \diamond b_+)\\
&- q_i^{\frac{1}{2}\alpha_i^\vee(2\xi'-\xi)+1} \sum_{b_\pm \in \mathbf{B}_\pm} q^{\eta(b_+)} \langle \check{b}_+(v')\,|\,(E_i^{\langle 1\rangle} \check{b}_-)^t(v)\rangle (K_{\xi'-|b_+|,0} \diamond b_-)(K_{0,2\mu-\xi'} \diamond b_+)\\
&= q_i^{\frac{1}{2}\alpha_i^\vee(\xi')} \sum_{b_\pm \in \mathbf{B}_\pm} q^{\eta(b_+)} \langle \check{b}_+(v')\,|\,(\check{b}_- E_i^{\langle 1\rangle})^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0} \diamond b_-)(K_{0,2\mu-\xi'-\alpha_i} \diamond b_+)
\end{aligned}$$

$$
\begin{aligned}
&- q_i^{\frac{1}{2}\alpha_i^\vee(2\xi'-\xi)+1} \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)} \langle \check{b}_+(v') \,|\, (E_i^{\langle 1\rangle}\check{b}_-)^t(v)\rangle (K_{\xi'-|b_+|,0}\diamond b_-)(K_{0,2\mu-\xi'}\diamond b_+)\\
&+ \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)} q_i^{\frac{1}{2}\alpha_i^\vee(3\xi'-2\mu-|b_+|)+1} \langle \partial_i^-(\check{b}_+)(v') \,|\, \check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&- \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)} q_i^{\frac{1}{2}\alpha_i^\vee(2\mu-\xi'+|b_+|)-1} \langle \partial_i^{-\,op}(\check{b}_+)(v') \,|\, \check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&q^{-\kappa(\xi',\xi)}[E_i^{\langle 1\rangle}, \Xi(v\otimes v')]K_{+i}^{-1}\\
&= q_i^{\frac{1}{2}\alpha_i^\vee(\xi')} \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)} \langle \check{b}_+(v') \,|\, \check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|,0}\diamond E_i^{\langle 1\rangle} b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&- q_i^{\frac{1}{2}\alpha_i^\vee(2\mu-\xi')+1} \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)} q_i^{\frac{1}{2}\alpha_i^\vee(|b_+|)} \langle \check{b}_+(v') \,|\, \check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+E_i^{\langle 1\rangle})\\
&= q_i^{\frac{1}{2}\alpha_i^\vee(\xi')} \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)} \langle \check{b}_+(v') \,|\, \check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|,0}\diamond b_-E_i^{\langle 1\rangle})(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&- q_i^{\frac{1}{2}\alpha_i^\vee(\xi')} \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)} \langle \check{b}_+(v') \,|\, \check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|,0}K_{+i}\diamond \partial_i^-(b_-))(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&+ q_i^{\frac{1}{2}\alpha_i^\vee(\xi')} \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)} \langle \check{b}_+(v') \,|\, \check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond \partial_i^{-\,op}(b_-))(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&- \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)} q_i^{\frac{1}{2}\alpha_i^\vee(2\mu-\xi'+|b_+|)+1} \langle \check{b}_+(v') \,|\, \check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+E_i^{\langle 1\rangle})\\
&= \sum_{b'_+\in\mathbf{B}_+,\, b_-\in\mathbf{B}_-} q^{\eta(b'_+)} q_i^{\frac{1}{2}\alpha_i^\vee(3\xi'-2\mu-|b'_+|)+1} \langle \check{b}'_+(v') \,|\, \check{b}_-{}^t(v)\rangle (K_{\xi'-|b'_+|,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond E_i^{\langle 1\rangle} b'_+)\\
&- \sum_{b'_+\in\mathbf{B}_+,\, b_-\in\mathbf{B}_-} q^{\eta(b'_+)} q_i^{\frac{1}{2}\alpha_i^\vee(2\mu-\xi'+|b'_+|)+1} \langle \check{b}'_+(v') \,|\, \check{b}_-{}^t(v)\rangle (K_{\xi'-|b'_+|,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b'_+E_i^{\langle 1\rangle})\\
&+ q_i^{\frac{1}{2}\alpha_i^\vee(\xi')} \sum_{b_+\in\mathbf{B}_+,\, b'_-\in\mathbf{B}_-} q^{\eta(b_+)} \langle \check{b}_+(v') \,|\, \check{b}'_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond \partial_i^{-\,op}(b'_-))(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&- q_i^{\frac{1}{2}\alpha_i^\vee(\xi')+1} \sum_{b_+\in\mathbf{B}_+,\, b'_-\in\mathbf{B}_-} q^{\eta(b_+)} q_i^{\frac{1}{2}\alpha_i^\vee(|b'_-|+|b_+|)} \langle \check{b}_+(v') \,|\, \check{b}'_-{}^t(v)\rangle (K_{\xi'-|b_+|,0}\diamond \partial_i^-(b'_-))(K_{0,2\mu-\xi'}\diamond b_+).
\end{aligned}
$$

Furthermore, since

$$
u_+ = \sum_{b_+\in\mathbf{B}_+} \{\check{b}_+, u_+\} b_+ = \sum_{b_-\in\mathbf{B}_-} \{b_-, u_+\}\check{b}_-, \quad u_- = \sum_{b_+\in\mathbf{B}_+} \{u_-, b_+\}\check{b}_+ = \sum_{b_-\in\mathbf{B}_-} \{u_-, \check{b}_-\} b_-
$$

for all $u_\pm \in U_q^\pm$, we obtain, using (4.13)

$$
\begin{aligned}
&q^{-\kappa(\xi',\xi)}[E_i^{\langle 1\rangle}, \Xi(v\otimes v')]K_{+i}^{-1}\\
&= \sum_{\substack{b_+, b'_+\in\mathbf{B}_+,\\ b_-\in\mathbf{B}_-}} q^{\eta(b_+)} q_i^{\frac{1}{2}\alpha_i^\vee(3\xi'-2\mu-|b_+|)+1} \{\check{b}_+, E_i^{\langle 1\rangle} b'_+\} \langle \check{b}'_+(v') \,|\, \check{b}_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\substack{b_+,b'_+\in\mathbf{B}_+,\\ b_-\in\mathbf{B}_-}} q^{\eta(b_+)} q_i^{\frac12\alpha_i^\vee(2\mu-\xi'+|b_+|)-1}\{\check b_+, b'_+E_i^{\langle 1\rangle}\}\langle \check b'_+(v')\,|\,\check b_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&+ q_i^{\frac12\alpha_i^\vee(\xi')} \sum_{b_+\in\mathbf{B}_+,\, b_-,b'_-\in\mathbf{B}_-} q^{\eta(b_+)}\{\partial_i^{-\,op}(b'_-),\check b_-\}\langle \check b_+(v')\,|\,\check b'_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&- q_i^{\frac12\alpha_i^\vee(2\xi'-\xi)+1} \sum_{b_+\in\mathbf{B}_+,\, b'_-\in\mathbf{B}_-} q^{\eta(b_+)}\{\partial_i^{-}(b'_-),\check b_-\}\langle \check b_+(v')\,|\,\check b'_-{}^t(v)\rangle (K_{\xi'-|b_+|,0}\diamond b_-)(K_{0,2\mu-\xi'}\diamond b_+)\\
&= \sum_{\substack{b_+,b'_+\in\mathbf{B}_+,\\ b_-\in\mathbf{B}_-}} q^{\eta(b_+)} q_i^{\frac12\alpha_i^\vee(3\xi'-2\mu-|b_+|)+1}\{\partial_i^-(\check b_+), b'_+\}\langle \check b'_+(v')|\check b_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&- \sum_{\substack{b_+,b'_+\in\mathbf{B}_+,\\ b_-\in\mathbf{B}_-}} q^{\eta(b_+)} q_i^{\frac12\alpha_i^\vee(2\mu-\xi'+|b_+|)-1}\{\partial_i^{-\,op}(\check b_+), b'_+\}\langle \check b'_+(v')\,|\,\check b_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&+ q_i^{\frac12\alpha_i^\vee(\xi')} \sum_{b_+\in\mathbf{B}_+,\, b_-,b'_-\in\mathbf{B}_-} q^{\eta(b_+)}\{b'_-,\check b_-E_i^{\langle 1\rangle}\}\langle \check b_+(v')\,|\,\check b'_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&- q_i^{\frac12\alpha_i^\vee(2\xi'-\xi)+1} \sum_{b_+\in\mathbf{B}_+,\, b_-,b'_-\in\mathbf{B}_-} q^{\eta(b_+)}\{b'_-,E_i^{\langle 1\rangle}\check b_-\}\langle \check b_+(v')\,|\,\check b'_-{}^t(v)\rangle (K_{\xi'-|b_+|,0}\diamond b_-)(K_{0,2\mu-\xi'}\diamond b_+)\\
&= \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)} q_i^{\frac12\alpha_i^\vee(3\xi'-2\mu-|b_+|)+1}\langle \partial_i^-(\check b_+)(v')\,|\,\check b_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&- \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)} q_i^{\frac12\alpha_i^\vee(2\mu-\xi'+|b_+|)-1}\langle \partial_i^{-\,op}(\check b_+)(v')\,|\,\check b_-{}^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&+ q_i^{\frac12\alpha_i^\vee(\xi')} \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)}\langle \check b_+(v')\,|\,(\check b_-E_i^{\langle 1\rangle})^t(v)\rangle (K_{\xi'-|b_+|+\alpha_i,0}\diamond b_-)(K_{0,2\mu-\xi'-\alpha_i}\diamond b_+)\\
&- q_i^{\frac12\alpha_i^\vee(2\xi'-\xi)+1} \sum_{b_\pm\in\mathbf{B}_\pm} q^{\eta(b_+)}\langle \check b_+(v')\,|\,(E_i^{\langle 1\rangle}\check b_-)^t(v)\rangle (K_{\xi'-|b_+|,0}\diamond b_-)(K_{0,2\mu-\xi'}\diamond b_+)\\
&= q^{-\kappa(\xi',\xi)}\Xi(E_i^{\langle 1\rangle}(v\otimes v')).
\end{aligned}
$$

The computation for the action of $F_i^{\langle 1\rangle}$ is similar and is omitted. □

Let ρ be an element of Λ satisfying $\alpha_i^\vee(\rho)=1$ for all $i\in I$. If $\mathfrak g$ is finite dimensional then ρ is uniquely defined by this condition. Extend the pairing $\cdot:\Gamma\times\Gamma\to\mathbb Z$ to a pairing $\Lambda\times\Lambda\to\mathbb Q$.

Lemma 4.19. *Let V be a lowest weight module of lowest weight $-\mu\in\Lambda$ and suppose that the pairing $\langle\cdot,\cdot\rangle_V$ is non-degenerate. Then the canonical invariant 1_V in $V^\#\widehat\otimes V$ is given by*

$$1_V = q^{2\rho\cdot\mu}\sum_{v\in\mathscr B_V} q^{-2\eta(|v^j|)} v\otimes\check v$$

where $\mathscr B_V$ is a homogeneous basis of V and $\{\check v\}_{v\in\mathscr B_V}\subset V$ is its dual basis with respect to the pairing $\langle\cdot\,|\,\cdot\rangle_V$.

Proof. Note that for any $u\in V$ we have $u=\sum_{b\in\mathscr B_V}\langle u,\check b\rangle_V b=\sum_{b\in\mathscr B_V}\langle b,u\rangle_V\check b$. This sum is finite since each homogeneous piece of a lowest weight module is finite dimensional. Since $|v|=|\check v|$ for all $v \in \mathscr{B}_V$, $K_i(1_V) = 1_V$. Furthermore,

$$\begin{aligned}E_i(q^{-2\rho\cdot\mu}1_V) &= \sum_{b\in\mathscr{B}_V} q^{-2\eta(|\check{b}|)} b\otimes E_i\check{b} - \sum_{b\in\mathscr{B}_V} q^{2\eta(|\check{b}|)} K_i^{-1}F_i b\otimes K_i\check{b}\\ &= \sum_{b,b'\in\mathscr{B}_V} q^{-2\eta(|\check{b}|)}\langle b', E_i\check{b}\rangle_V b\otimes \check{b}' - \sum_{b\in\mathscr{B}_V} q^{-2\eta(|\check{b}|)} q_i^2 F_i b\otimes\check{b}\\ &= \sum_{b,b'\in\mathscr{B}_V} q^{-2\eta(|\check{b}|)} q_i^2\langle F_i b, \check{b}'\rangle_V b'\otimes\check{b} - \sum_{b\in\mathscr{B}_V} q^{-2\eta(|\check{b}|)} q_i^2 F_i b\otimes\check{b} = 0\end{aligned}$$

and similarly $F_i(1_V) = 0$. □

4.5. **Towards Conjecture 1.26.**

Example 4.20. Let $\mathfrak{g} = \mathfrak{sl}_2$ and let V be the $(m+1)$-dimensional $U_q(\tilde{\mathfrak{g}})$-module with its standard basis $v_a = E^{\langle a\rangle}v_0$, $0\le a\le m$. Then $E^{\langle a\rangle}v_b = \binom{a+b}{a}_q v_{b+a}$, $F^{\langle a\rangle}v_b = (-1)^a\binom{m-b+a}{a}_q v_{b-a}$, $0\le b\le m$ where we set $v_k = 0$ if $k<0$ or $k>m$. Denote $\{v^a\}_{0\le a\le m}$ the dual basis of V with respect to the pairing $\langle\cdot,\cdot\rangle_m := \langle\cdot,\cdot\rangle_V$. Then we have

$$\begin{aligned}\Xi(v^a\otimes v_b) &= q^{\binom{b}{2}-\binom{a}{2}}\sum_{k=\max(0,b-a)}^{b} q^k\langle v^a, E^{\langle a-b+k\rangle}F^{\langle k\rangle}v_b\rangle_m (K_-^{b-k}\diamond F^{a-b+k})(K_+^{m-b}\diamond E^k)\\ &= q^{\binom{b}{2}-\binom{a}{2}}\sum_{k=\max(0,b-a)}^{b}(-1)^k q^k\binom{m-b+k}{k}_q\binom{a}{b-k}_q (K_-^{b-k}\diamond F^{a-b+k})(K_+^{m-b}\diamond E^k),\end{aligned}$$

whence we obtain, using (3.4) and (4.3)

$$\begin{aligned}\Xi(1_V) &= \sum_{a=0}^{m} q^{m-2a}\,\Xi(v^a\otimes v_a)\\ &= \sum_{0\le k\le a\le m}(-1)^k q^{k+m-2a}\binom{m-a+k}{k}_q\binom{a}{k}_q (K_-^{a-k}\diamond F^k)(K_+^{m-a}\diamond E^k)\\ &= \sum_{0\le k\le a\le m}(-1)^k q^{(k+1)(m+k-2a)}\binom{m-a+k}{k}_q\binom{a}{k}_q K_-^{a-k}K_+^{m-a}\diamond F^kE^k\\ &= \sum_{r,s\ge0,\,r+s\le m}(-1)^{m-r-s}q^{(r+s-m-1)(r-s)}\binom{m-r}{s}_q\binom{m-s}{r}_q K_-^rK_+^s\diamond F^{m-r-s}E^{m-r-s} = (-1)^m C^{(m)}.\end{aligned}$$

This proves Conjecture 1.26 for $\mathfrak{g} = \mathfrak{sl}_2$.

Example 4.21. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and let V be the simple lowest weight module of lowest weight $-\omega_1$. It standard basis is v_i, $0\le i\le n$, where $E_i^{\langle 1\rangle}v_j = \delta_{i,j+1}v_{j+1}$ and $F_i^{\langle 1\rangle}v_j = -\delta_{i,j}v_{j-1}$. Denote $\alpha_{i,j} = \sum_{k=i}^{j}\alpha_k$. Then $|v_i| = \alpha_{1,i}$ and $\underline{\gamma}(|v_i|) = -i+1-\delta_{i,0}$, $0\le i\le n$. Let $\{v^j\}$, $0\le j\le n$ be the dual basis of V with respect to the pairing $\langle\cdot,\cdot\rangle_V$. We have

$$\Xi(v^i\otimes v_j) = q^{\frac{i-j+\delta_{i,0}-\delta_{j,0}}{2}}\sum_{\substack{b_-\in\mathbf{B}_-\\ 1\le k\le j+1}} q^{j-k+1}\langle v^i, \check{b}_- F_k^{\langle 1\rangle}\cdots F_j^{\langle 1\rangle}v_j\rangle_V (K_{\alpha_{1,k-1},0}\diamond b_-)(K_{0,2\omega_1-\alpha_{1,j}}\diamond {E_{[k,j]}}^*)$$

$$= q^{\frac{i-j+\delta_{i,0}-\delta_{j,0}}{2}}\sum_{k=1}^{\min(i,j)+1}(-q)^{j-k+1}(K_{\alpha_{1,k-1},0}\diamond F_{[k,i]})(K_{0,2\omega_1-\alpha_{1,j}}\diamond {E_{[k,j]}}^*)$$

$$= q^{\frac{i+j+1-\delta_{i,j}}{2}} \sum_{k=1}^{\min(i,j)+1} (-1)^{j-k+1} q^{1-k} (K_{\alpha_{1,k-1},2\omega_1-\alpha_{1,j}} \diamond F_{[k,i]} E_{[k,j]}{}^*)$$

where $E_{[j+1,j]} = 1$, $E_{[j,j]} = E_j$, $E_{[i,j]} = q^{\frac{1}{2}} E_{[i+1,j]} E_i^{\langle 1 \rangle} - q^{-\frac{1}{2}} E_i^{\langle 1 \rangle} E_{[i+1,j]}$, $1 \le i < j \le n$ and $F_{[i,j]} = E_{[i,j]}{}^{*t}$. It is not hard to check that $E_{[i,j]} = T_i \cdots T_{j-1}(E_j)$ and that $\partial_k(E_{[i,j]}) = \delta_{i,k} E_{[i+1,j]}$, $\partial_k^{op}(E_{[i,j]}) = \delta_{k,j} E_{[i,j-1]}$. Then

$$\Xi(1_V) = K_{0,\omega_1-\omega_n} \sum_{0 \le i \le j \le n} (-1)^{i+j} q^{n-i-j} K_{-1} \cdots K_{-i} K_{+(j+1)} \cdots K_{+n} F_{[i+1,j]} E_{[i+1,j]}{}^*. \tag{4.15}$$

We claim that

$$\Xi(1_V) = (-1)^n K_{0,\omega_1-\omega_n} \diamond F_{[1,n]} \bullet E_{[1,n]}{}^*. \tag{4.16}$$

First, we need to prove that $\Xi(1_V)$ is $\bar{\cdot}$-invariant. For, it is easy to show by induction on $j-i$ that

$$\underline{\Delta}(E_{[i,j]}{}^*) = \sum_{k=i-1}^{j} q^{\frac{1}{2}\alpha_{i,k}\cdot\alpha_{k+1,j}} E_{[i,k]}{}^* \otimes E_{[k+1,j]}{}^*, \quad \underline{\Delta}(F_{[i,j]}) = \sum_{k=i-1}^{j} q^{\frac{1}{2}\alpha_{k+1,j}\cdot\alpha_{i,k}} F_{[k+1,j]} \otimes F_{[i,k]},$$

which in particular implies that

$$\langle F_{[i,j]}, E_{[a,b]} \rangle = \delta_{i,a}\delta_{j,b}(q-q^{-1})^{1-\delta_{i,j+1}}, \qquad \langle F_{[i,j]}, E_{[a,b]}{}^* \rangle = \delta_{i,a}\delta_{j,b}(-q)^{i-j-\delta_{i,j+1}}(q-q^{-1})^{1-\delta_{i,j+1}}.$$

Since

$$(\underline{\Delta} \otimes 1)\underline{\Delta}(E_{[i,j]}{}^*) = \sum_{i-1 \le r \le k \le j} q^{\frac{1}{2}\alpha_{i,k}\cdot\alpha_{k+1,j} + \frac{1}{2}\alpha_{i,r}\cdot\alpha_{r+1,k}} E_{[i,r]}{}^* \otimes E_{[r+1,k]}{}^* \otimes E_{[k+1,j]}{}^*,$$
$$(1 \otimes \underline{\Delta})\underline{\Delta}(F_{[i,j]}) = \sum_{i-1 \le r \le k \le j} q^{\frac{1}{2}\alpha_{k+1,j}\cdot\alpha_{i,k} + \frac{1}{2}\alpha_{i,r}\cdot\alpha_{r+1,k}} F_{[k+1,j]} \otimes F_{[r+1,k]} \otimes F_{[i,r]},$$

by Proposition A.36 we obtain

$$E^*_{[i,j]} F_{[i,j]} = \sum_{i-1 \le r \le k \le j} (-1)^{j-k+i-r-\delta_{i,r+1}} q^{j-k-1-r+i-\delta_{i,r+1}+\delta_{j,k}} (q-q^{-1})^{2-\delta_{i,r+1}-\delta_{k,j}} \times$$
$$K_{-i} \cdots K_{-r} K_{+(k+1)} \cdots K_{+j} F_{[r+1,k]} E_{[r+1,k]}{}^*.$$

Therefore,

$$\begin{aligned}
\overline{K_{0,-\omega_1+\omega_n}\Xi(1_V)} &= \sum_{0 \le i \le j \le n} (-1)^{i+j} q^{-n+i+j} K_{-1} \cdots K_{-i} K_{+(j+1)} \cdots K_{+n} E_{[i+1,j]}{}^* F_{[i+1,j]} \\
&= \sum_{0 \le i \le r \le k \le j \le n} (-1)^{1-k-r-\delta_{i,r}} q^{-n+2(i+j)-k-r-\delta_{i,r}+\delta_{j,k}} (q-q^{-1})^{2-\delta_{i,r}-\delta_{k,j}} \times \\
&\qquad K_{-1} \cdots K_{-r} K_{+(k+1)} \cdots K_{+n} F_{[r+1,k]} E_{[r+1,k]}{}^* \\
&= \sum_{0 \le r \le k \le n} (-1)^{k+r-1} q^{-n-k-r} \Big(\sum_{j=k}^{n} \sum_{i=0}^{r} (-1)^{\delta_{i,r}} q^{2(i+j)-\delta_{i,r}+\delta_{j,k}} (q-q^{-1})^{2-\delta_{i,r}-\delta_{j,k}} \Big) \times \\
&\qquad K_{-1} \cdots K_{-r} K_{+(k+1)} \cdots K_{+n} F_{[r+1,k]} E_{[r+1,k]}{}^* = K_{0,-\omega_1+\omega_n}\Xi(1_V)
\end{aligned}$$

since

$$\Big(\sum_{i=0}^{r} (-1)^{\delta_{i,r}} q^{2i-\delta_{i,r}} (q-q^{-1})^{1-\delta_{i,r}} \Big) \Big(\sum_{j=k}^{n} q^{2j+\delta_{j,k}} (q-q^{-1})^{1-\delta_{j,k}} \Big) = -q^{2n}.$$

This computation also shows that the image of $(-1)^n K_{0,-\omega_1+\omega_n}\Xi(1_V)$ in $\mathcal{H}_q^+(\mathfrak{g})$ is $\bar{\cdot}$-invariant. Together with Theorems 1.3 and 1.8 this implies that

$$F_{[a,b]}\circ {E_{[a,b]}}^* = \sum_{j=a-1}^{b} (-q)^{b-j} K_{+(j+1)}\cdots K_{+b} F_{[a,j]} {E_{[a,j]}}^*. \tag{4.17}$$

Then

$$\begin{aligned}
&\sum_{i=0}^{n} (-q)^{-i} K_{-1}\cdots K_{-i} F_{[i+1,n]}\circ {E_{[i+1,n]}}^*\\
&= \sum_{0\le i\le j\le n} (-1)^{n-i-j} q^{n-i-j} K_{-1}\cdots K_{-i} K_{+(j+1)}\cdots K_{+n} F_{[i+1,j]} {E_{[i+1,j]}}^* = (-1)^n K_{0,-\omega_1+\omega_n}\Xi(1_V),
\end{aligned}$$

and (4.16) follows by Theorem 1.5. In particular, we obtain an explicit formula for $F_{[i,j]}\circ {E_{[i,j]}}^*$ and $F_{[i,j]}\bullet {E_{[i,j]}}^*$, $1\le i\le j\le n$.

Example 4.22. Let $\mathfrak{g}=\mathfrak{sp}_4$ and let $V(-\omega_1)$ be the lowest weight module of the lowest weight $-\omega_1$. Its standard basis is $\{v_i\}_{0\le i\le 3}$ with the non-trivial actions being

$$E_1^{\langle 1\rangle} v_0 = v_1,\quad E_2^{\langle 1\rangle} v_1 = v_2,\quad E_1^{\langle 1\rangle} v_2 = v_3,\quad F_1^{\langle 1\rangle} v_3 = -v_2,\quad F_2^{\langle 1\rangle} v_2 = -v_1,\quad F_1^{\langle 1\rangle} v_1 = -v_0.$$

Denote $\{v^i\}_{0\le i\le 3}$ the dual basis of $V(-\omega_1)$ with respect to the pairing $\langle\cdot,\cdot\rangle_{V(-\omega_1)}$. Then

$$\begin{aligned}
\Xi(1_{V(-\omega_1)}) &= q^4\Xi(v^0\otimes v_0) + q^2\Xi(v^1\otimes v_1) + q^{-2}\Xi(v^2\otimes v_2) + q^{-4}\Xi(v^3\otimes v_3)\\
&= q^4 K_{+1}^2 K_{+2} + q^2(K_{-1}K_{+1}K_{+2} - qK_{+1}K_{+2}\diamond F_1E_1)\\
&\quad + q^{-2}(K_{-1}K_{-2}K_{+1} - q^2K_{-1}K_{+1}\diamond F_2E_2 + q^3K_{+1}\diamond F_{12}E_{21})\\
&\quad + q^{-4}(K_{-1}^2K_{-2} - qK_{-1}K_{-2}\diamond F_1E_1 + q^3K_{-1}\diamond F_{21}E_{12} - q^4F_{121}E_{121}).
\end{aligned}$$

It is easy to check that $\Xi(1_{V(-\omega_1)}) = -F_{121}\bullet E_{121}$ since

$$\begin{aligned}
F_{121}\circ E_{121} &= F_{121}E_{121} - qK_{+1}F_{12}E_{21} + q^3K_{+1}K_{+2}F_1E_1 - q^4K_{+1}^2K_{+2}\\
F_{121}\bullet E_{121} &= F_{121}\circ E_{121} - q^{-1}K_{-1}F_{21}\circ E_{12} - q^{-3}K_{-1}K_{-2}F_1\circ E_1 + q^{-4}K_{-1}^2K_{-2}.
\end{aligned}$$

Similarly, for the lowest weight module $V(-\omega_2)$ of lowest weight $-\omega_2$. Its standard basis $\{v_i\}_{0\le i\le 4}$ satisfies

$$\begin{aligned}
&E_2^{\langle 1\rangle} v_0 = v_1, && E_1^{\langle 1\rangle} v_1 = v_2, && E_1^{\langle 1\rangle} v_2 = (2)_q v_3, && E_2^{\langle 1\rangle} v_3 = v_4\\
&F_2^{\langle 1\rangle} v_1 = -v_0, && F_1^{\langle 1\rangle} v_2 = -(2)_q v_1, && F_1^{\langle 1\rangle} v_3 = -v_2, && F_2^{\langle 1\rangle} v_4 = -v_3
\end{aligned}$$

and

$$\begin{aligned}
\Xi(1_{V(-\omega_2)}) &= q^6\Xi(v^0\otimes v_0) + q^2\Xi(v^1\otimes v_1) + \Xi(v^2\otimes v_2) + q^{-2}\Xi(v^3\otimes v_3) + q^{-6}\Xi(v^4\otimes v_4)\\
&= q^6K_{+1}^2K_{+2}^2 + q^2(K_{-2}K_{+1}^2K_{+2} - q^2K_{+1}^2K_{+2}\diamond F_2E_2)\\
&\quad + (K_{-1}K_{-2}K_{+1}K_{+2} - (q+q^3)K_{-2}K_{+1}K_{+2}\diamond F_1E_1 + (q^3+q^5)_qK_{+1}K_{+2}\diamond F_{21}E_{12})\\
&\quad + q^{-2}(K_{-1}^2K_{-2}K_{+2} - (2)_qK_{-1}K_{-2}K_{+2}\diamond F_1E_1 + q^2K_{-2}K_{+2}\diamond F_1^2E_1^2 - q^4K_{+2}\diamond F_{211}E_{112})\\
&\quad + q^{-6}(K_{-1}^2K_{-2}^2 - q^2K_{-1}^2K_{-2}\diamond F_2E_2 + q^2(2)_qK_{-1}K_{-2}\diamond F_{12}E_{21} + q^4K_{-2}\diamond F_{112}E_{211}\\
&\qquad\qquad + q^6F_{2112}E_{2112})
\end{aligned}$$

where $E_{2112} = E_2E_{112} - q^2E_{12}^2$ and $F_{2112} = {E_{2112}}^{*t}$. Since

$$\begin{aligned}
F_{2112}\circ E_{2112} &= F_{2112}E_{2112} - q^2K_{+2}F_{211}E_{112} + (q^5+q^3)K_{+1}K_{+2}F_{21}E_{12}\\
&\quad - q^4K_{+1}^2K_{+2}F_2E_2 + q^6K_{+1}^2K_{+2}^2
\end{aligned}$$

$$F_{2112} \bullet E_{2112} = F_{2112} \circ E_{2112} - q^{-2} K_{-2} F_{112} \circ E_{211} + (q^{-5} + q^{-3}) K_{-1} K_{-2} F_{12} \circ E_{21}$$
$$- q^{-4} K_{-1}^2 K_{-2} F_2 \circ E_2 + q^{-6} (K_{-1} K_{-2})^2 + q^{-4} K_{-1} K_{+1} K_{-2} K_{+2},$$

it follows that $\Xi(1_{V(-\omega_2)}) = F_{2112} \bullet E_{2112}$.

5. Bar-equivariant braid group actions

5.1. **Invariant braid group action on Drinfeld double.** Denote by $U_q'(\tilde{\mathfrak{g}})$ the quotient of $\Bbbk[z_i^{\pm 1} : i \in I] \otimes_{\Bbbk} U_q(\tilde{\mathfrak{g}})$ by the ideal generated by $z_i^2 \otimes 1 - 1 \otimes K_{+i} K_{-i}$. It is easy to see that $\bar{\cdot}$ extends to an $\mathbb{Q}$-linear anti-involution of $U_q'(\tilde{\mathfrak{g}})$ by $\bar{z}_i = z_i$. Then it is immediate that the set

$$\mathbf{B}'_{\tilde{\mathfrak{g}}} = \{ \big(\prod_{i \in I} z_i^{a_i} \big) \mathbf{b} \,:\, \mathbf{b} \in \mathbf{B}_{\tilde{\mathfrak{g}}},\, a_i \in \mathbb{Z} \}$$

is a $\bar{\cdot}$-invariant basis of $U_q'(\tilde{\mathfrak{g}})$. In the sequel we use the presentation of $U_q(\mathfrak{g})$ obtained from (1.2) and (1.3) by replacing $K_{\pm i}$ with $K_i^{\pm 1}$. The following Lemma is immediate.

Lemma 5.1. (a) *The assignments $E_i \mapsto E_i$, $F_i \mapsto F_i$, $K_{\pm i} \mapsto K_i^{\pm 1}$, $z_i \mapsto 1$ extends to a surjective homomorphism of algebras $\phi : U_q'(\tilde{\mathfrak{g}}) \to U_q(\mathfrak{g})$.*
(b) *The assignments $E_i \mapsto E_i z_i^{-1}$, $F_i \mapsto F_i$, $K_i^{\pm 1} \mapsto K_{\pm i} z_i^{-1}$ extends to an injective homomorphism of algebras $\iota : U_q(\mathfrak{g}) \to U_q'(\tilde{\mathfrak{g}})$ which splits ϕ.*

Clearly, there exists a unique anti-involution $\bar{\cdot}$ on $U_q'(\tilde{\mathfrak{g}})$ which commutes with ι and ϕ. It is also easy to see that there exists a unique basis $\mathbf{B}_{\mathfrak{g}}$ of $U_q(\mathfrak{g})$ such that $\iota(\mathbf{B}_{\mathfrak{g}}) = \mathbf{B}'_{\tilde{\mathfrak{g}}} \cap \iota(U_q(\mathfrak{g}))$. Clearly $\mathbf{B}_{\mathfrak{g}} = \phi(\mathbf{B}'_{\tilde{\mathfrak{g}}})$ and each element of $\mathbf{B}_{\mathfrak{g}}$ is fixed by $\bar{\cdot}$. From now on we refer to $\mathbf{B}_{\mathfrak{g}}$ as the *double canonical basis* of $U_q(\mathfrak{g})$.

Given $\alpha_\pm \in \Gamma$, set $\operatorname{Ad}^{\frac{1}{2}} K_{\alpha_-,\alpha_+}(x) = \chi^{\frac{1}{2}}((\alpha_-,\alpha_+), \deg_{\widehat{\Gamma}} x) x$ for $x \in U_q(\tilde{\mathfrak{g}})$ homogeneous. Let Q be the free abelian group generated by the α_i, $i \in I$ and let $\widehat{Q} = Q \oplus Q$. Then Γ (respectively, $\widehat{\Gamma}$) is a submonoid of Q (respectively, $\widehat{Q}$). Extend $\alpha_i^\vee \in \operatorname{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z})$ to elements of $\operatorname{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})$ in a natural way. The Weyl group W of $\mathfrak{g}$ acts on Q and hence on $\widehat{Q}$ via $s_i(\alpha) = \alpha - \alpha_i^\vee(\alpha)\alpha_i$, $i \in I$.

Lemma 5.2. *In the presentation (1.2)–(1.3) of $U_q'(\tilde{\mathfrak{g}})$, we have $T_i(z_j) = z_j z_i^{-a_{ij}}$, $i, j \in I$*

$$T_i(K_{\pm j}) = \begin{cases} K_{\mp i} z_i^{-2}, & i = j \\ K_{\pm j} K_{\pm i}^{-a_{ij}}, & i \neq j \end{cases}$$

and

$$T_i(E_j) = \begin{cases} K_{-i} z_i^{-2} \diamond F_i, & i = j \\ \displaystyle\sum_{r+s=-a_{ij}} (-1)^r q_i^{s + \frac{1}{2} a_{ij}} E_i^{\langle r \rangle} E_j E_i^{\langle s \rangle}, & i \neq j \end{cases}$$

$$T_i(F_j) = \begin{cases} K_{+i} z_i^{-2} \diamond E_i, & i = j \\ \displaystyle\sum_{r+s=-a_{ij}} (-1)^r q_i^{s + \frac{1}{2} a_{ij}} F_i^{\langle r \rangle} F_j F_i^{\langle s \rangle}, & i \neq j \end{cases}$$

Moreover, the T_i satisfy the braid relations, commute with $\bar{\cdot}$ and satisfy $T_i * = * T_i^{-1}$, $T_i \circ^t = {}^t\circ T_i^{-1}$.

Proof. Recall that our presentation of $U_q(\mathfrak{g})$ is obtained from the standard one by rescaling $E_i \mapsto (q_i^{-1} - q)^{-1} E_i$, $F_i \mapsto (q_i - q_i^{-1})^{-1} F_i$ for all $i \in I$. In this presentation the symmetries $T'_{i,1}$, $T''_{i,-1}$ of $U_q(\mathfrak{g})$ defined in [19, §37.1.3] are given by by $T'_{i,1}(K_j) = K_j K_i^{-a_{ij}} = T''_{i,-1}(K_j)$,

$$T''_{i,-1}(E_j) = \begin{cases} F_i K_i^{-1}, & i = j \\ \displaystyle\sum_{r+s=-a_{ij}} (-1)^r q_i^s E_i^{\langle r \rangle} E_j E_i^{\langle s \rangle}, & i \neq j \end{cases}$$

$$T''_{i,-1}(F_j) = \begin{cases} K_i E_i, & i = j \\ \sum\limits_{r+s=-a_{ij}} (-q_i)^{-r} F_i^{\langle r\rangle} F_j F_i^{\langle s\rangle}, & i \neq j \end{cases}$$

and

$$T'_{i,1}(E_j) = \begin{cases} K_i F_i, & i = j \\ \sum\limits_{r+s=-a_{ij}} (-1)^s q_i^r E_i^{\langle r\rangle} E_j E_i^{\langle s\rangle}, & i \neq j \end{cases}$$

$$T'_{i,1}(F_j) = \begin{cases} E_i K_i^{-1}, & i = j \\ \sum\limits_{r+s=-a_{ij}} (-q_i)^{-r} F_i^{\langle s\rangle} F_j F_i^{\langle r\rangle}, & i \neq j \end{cases}$$

By [19, Proposition 37.1.2] $T'_{i,1}$, $T''_{i,-1}$ are automorphisms of $U_q(\mathfrak{g})$ while by [19, Theorem 39.4.3] they satisfy the braid relations of the braid group of $\mathfrak{g}$. Also, $T''_{i,-1} = (T'_{i,1})^{-1}$. It is easy to see that $T'_{i,-1}(E_j)$, $T''_{i,-1}(E_j)$ and $T'_{i,1}(E_j)$, $T''_{i,-1}(F_j)$, $i \neq j$, are given on $U'_q(\tilde{\mathfrak{g}})$ by the same formula as on $U_q(\mathfrak{g})$. Furthermore we have

$$z_i T''_{i,-1}(E_i) = T''_{i,-1}(E_i z_i^{-1}) = F_i K_{-i} z_i^{-1},$$

$$z_i T'_{i,1}(E_i) = T'_{i,1}(E_i z_i^{-1}) = K_{+i} F_i z_i^{-1}$$

whence $T''_{i,-1}(E_i) = F_i K_{-i} z_i^{-2}$ and $T'_{i,1}(E_i) = K_{+i} z_i^{-2} F_i$. Similarly, $T''_{i,-1}(F_i) = K_{+i} z_i^{-1} E_i z_i^{-1} = K_{+i} z_i^{-2} E_i$ and $T'_{i,1} = F_i K_{-i} z_i^{-2}$. Finally,

$$z_i T''_{i,-1}(K_{\pm i}) = T''_{i,-1}(K_{\pm i} z_i^{-1}) = K_{\mp i} z_i^{-1} = z_i T'_{i,1}(K_{\pm i})$$

whence $T''_{i,-1}(K_{\pm i}) = K_{\mp i} z_i^{-2} = T'_{i,1}(K_{\pm i})$, while $z_j^{-1} z_i^{a_{ij}} T''_{i,-1}(K_{\pm j}) = K_{\pm j} z_j^{-1} (K_{\pm i} z_i^{-1})^{-a_{ij}}$.

Define $T_i(x) = T''_{i,-1}(\operatorname{Ad}^{\frac{1}{2}} K_{+i}(x)) = \operatorname{Ad}^{\frac{1}{2}} K_{-i}(T''_{i,-1}(x))$, $x \in U'_q(\tilde{\mathfrak{g}})$. Then we have $T_i^{-1}(x) = T'_{i,1}(\operatorname{Ad}^{\frac{1}{2}} K_{+i}(x))$. It is easy to see that T_i is given on generators by the formulae from Lemma 5.2. For example, $T_i(E_i) = q_i K_{+i} z_i^{-2} F_i = K_{+i} z_i^{-2} \diamond F_i$. Thus, in particular, T_i is an automorphism of $U'_q(\tilde{\mathfrak{g}})$. Clearly, $\overline{T_i(E_i)} = T_i(E_i)$, while for $j \neq i$

$$\overline{T_i(E_j)} = \sum_{r+s=-a_{ij}} (-1)^s q_i^{-s-\frac{1}{2}a_{ij}} E_i^{\langle s\rangle} E_j E_i^{\langle r\rangle} = \sum_{r+s=-a_{ij}} (-1)^r q_i^{s+\frac{1}{2}a_{ij}} E_i^{\langle r\rangle} E_j E_i^{\langle s\rangle} = T_i(E_j)$$

where we used that $\overline{E^{\langle k\rangle}} = (-1)^k E^{\langle k\rangle}$. The remaining identities are checked similarly. The identities involving $*$ and ^{t} can be checked using the explicit formulae for $T_i^{-1} = T'_{i,1} \circ \operatorname{Ad}^{\frac{1}{2}} K_{+i}$.

It remains to prove that the T_i satisfy the braid relations. For, let w be an element of the Weyl group of $\mathfrak{g}$ and let $w = s_{i_1} \cdots s_{i_r}$ be its reduced decomposition. It is sufficient to prove that $T_{i_1} \circ \cdots \circ T_{i_r}$ depends only on w and not on the reduced decomposition. This holds for Lusztig symmetries $T'_{i,1}$, $T''_{i,-1}$ by [19, §39.4.4], whence for each $w \in W$ one has a well-defined automorphism $T''_{w,-1}$ of $U_q(\mathfrak{g})$ satisfying $T''_{w,-1} = T''_{i_1,-1} \cdots T''_{i_r,-1}$. We have

$$T_{i_1} \circ \cdots \circ T_{i_r}(x) = \operatorname{Ad}^{\frac{1}{2}} K_{\sum_{j=1}^r s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}),0} \circ T''_{i_1,-1} \circ \cdots \circ T''_{i_r,-1} = \operatorname{Ad}^{\frac{1}{2}} K_{\sum_{j=1}^r s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}),0} \circ T''_{w,-1}.$$

It is well-known that $\sum_{j=1}^r s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) = \sum_{\beta \in R_+ \cap w(-R_+)} \beta$, where $R_+ \subset Q$ denotes the set of positive roots of $\mathfrak{g}$, depends only on w and not on its reduced decomposition. Therefore, the right hand side depends only on w. □

Proof of Theorem 1.13. Note that $\widehat{U}_q(\tilde{\mathfrak{g}})$ embeds into $U'_q(\tilde{\mathfrak{g}})$ via $E_i \mapsto E_i$, $F_i \mapsto F_i$, $K_{\pm i} \mapsto K_{\pm i}$, $K_{\pm i}^{-1} \mapsto K_{\mp i} z_i^{-2}$ for all $i \in I$. All assertions of Theorem 1.13 are then immediate consequences of Lemma 5.2. □

In particular, for each $w\in W$, we have a unique automorphism T_w of $U_q(\tilde{\mathfrak g})$ such that $T_{s_i}=T_i$ and $T_w=T_{w'}T_{w''}$ for any reduced decomposition $w=w'w''$, $w',w''\in W$. It follows from Lemma 5.2 that for all $x\in U_q(\tilde{\mathfrak g})$

$$\overline{T}_w(x)=T_w(\overline{x}),\qquad T_w(x^*)=(T_{w^{-1}}^{-1}(x))^*,\qquad T_w(x^t)=(T_{w^{-1}}^{-1}(x))^t. \tag{5.1}$$

Furthermore, we have for $x\in U_q(\tilde{\mathfrak g})$ homogeneous

$$T_w(x)=\chi^{\frac12}(((\langle w\rangle,0),w\deg_{\widehat\Gamma}x)T''_{w,-1}(x)=\chi^{\frac12}((0,\langle w^{-1}\rangle),\deg_{\widehat\Gamma}x)T''_{w,-1}(x). \tag{5.2}$$

where $\langle w\rangle=\sum_{\beta\in R^+\cap w(-R_+)}\beta$ and the action of W on Γ is extended to $\widehat\Gamma$ diagonally.

5.2. **Elements T_w, quantum Schubert cells and their bases.** Let $\mathfrak g$ be any Kac-Moody Lie algebra. Given $w\in W$ define

$$U_q^+(w)=T_w(\mathcal K U_q^-)\cap U_q^+.$$

Let $\mathbf i=(i_1,\dots,i_m)$, $m=\ell(w)$, be such that $w=s_{i_1}\cdots s_{i_m}$ is a reduced decomposition. Then for $\mathbf a=(a_1,\dots,a_m)\in\mathbb Z_{\ge0}^m$ define

$$E_{\mathbf i}^{\mathbf a}:=E_{i_1}^{a_1}T_{s_{i_1}}(E_{i_2}^{a_2})\cdots T_{s_{i_1}\cdots s_{i_{m-1}}}(E_{i_m}^{a_m}). \tag{5.3}$$

It follows from [19] and (5.1) that for all $w\in W$, $i\in I$ such that $\ell(ws_i)=\ell(w)+1$, we have

$$T_w(E_i),\,T_{w^{-1}}^{-1}(E_i)\in U_q^+,\quad T_w(F_i),\,T_{w^{-1}}^{-1}(F_i)\in U_q^-. \tag{5.4}$$

Thus, the $E_{\mathbf i}^{\mathbf a}\in U_q^+$. It follows from [19, Proposition 40.2.1] that the $E_{\mathbf i}^{\mathbf a}$ are linearly independent. Let $U_q^+(w,1)$ be the $\Bbbk$-subspace of U_q^+ spanned by the $E_{\mathbf i}^{\mathbf a}$, $\mathbf a\in\mathbb Z_{\ge0}^m$. Let $U_q^+(w)'=T_w(U_q^+)\cap U_q^+$.

Conjecture 5.3 ([25, Proposition 2.10] and [17, Theorem 1.1])**.** *For any $\mathfrak g$ we have a unique (tensor) factorization $U_q^+=U_q^+(w)\cdot U_q^+(w)'$. In particular, $U_q^+(w,1)=U_q^+(w)$.*

We retain an elementary proof for the special case of $\mathfrak g$ semisimple here for reader's convenience, since the arguments in [17, 25] are rather long and non-trivial.

Proposition 5.4. *If $\mathfrak g$ is semisimple then $U_q^+(w)=U_q^+(w,1)$.*

Proof. We need the following

Lemma 5.5. *For any Kac-Moody Lie algebra $\mathfrak g$, $U_q^+(w,1)\subset U_q^+(w)$.*

Proof. Since $U_q^+(w,1)$ is contained in the subalgebra of U_q^+ generated by the $T_{u_r}(E_{i_r})$, $1\le r\le m$, where $u_r=s_{i_1}\cdots s_{i_{r-1}}$, it suffices to prove that $T_w^{-1}(T_{u_r}(E_{i_r}))\in\mathcal K U_q^-$, $1\le r\le m$. Indeed, write $w=u_rs_{i_r}v_r$ where $v_r=s_{i_{r+1}}\cdots s_m$. Since $\ell(w)=\ell(u_r)+\ell(v_r)+1$ we have by (5.4)

$$T_w^{-1}(T_{u_r}(E_{i_r}))=T_{v_r^{-1}}^{-1}(T_{i_r}^{-1}(E_{i_r}))=T_{v_r^{-1}}^{-1}(K_{-i_r}^{-1}\diamond F_{i_r})\in\mathcal K U_q^-. \qquad\square$$

To prove the inclusion $U_q^+(w)\subset U_q^+(w,1)$ for $\mathfrak g$ semisimple, let $w_\circ$ be the longest element in W and set $w'=w^{-1}w_\circ$. Since $\ell(w)+\ell(w')=\ell(w_\circ)$, we can choose a reduced word $\mathbf i_\circ$ for $w_\circ$ which is the concatenation of reduced words $\mathbf i$ and $\mathbf i'$ for w and w' respectively. Then by [19, Corollary 40.2.2], monomials $E_{\mathbf i_\circ}^{\mathbf a\mathbf a'}$, $\mathbf a\in\mathbb Z_{\ge0}^{\ell(w)}$, $\mathbf a'\in\mathbb Z_{\ge0}^{\ell(w')}$ form a basis of U_q^+. Observe that $E_{\mathbf i_\circ}^{\mathbf a}=E_{\mathbf i}^{\mathbf a}T_w(E_{\mathbf i'}^{\mathbf a'})\in U_q^+(w)T_w(E_{\mathbf i'}^{\mathbf a'})$. Let $u\in U_q^+(w)$. Then we can write $u=\sum_{\mathbf a'\in\mathbb Z_{\ge0}^{\ell(w')}}c_{\mathbf a'}T_w(E_{\mathbf i'}^{\mathbf a'})$, where $c_{\mathbf a'}\in U_q^+(w)$. Then

$$T_w^{-1}(u)=\sum_{\mathbf a'\in\mathbb Z_{\ge0}^{\ell(w')}}T_w^{-1}(c_{\mathbf a'})E_{\mathbf i'}^{\mathbf a'}.$$

By definition of $U_q^+(w)$, $T_w^{-1}(c_{\mathbf a'})\in\mathcal K U_q^-$. Note that the triangular decomposition $U_q(\tilde{\mathfrak g})\cong\mathcal K\otimes U_q^-\otimes U_q^+$ implies that the $E_{\mathbf i'}^{\mathbf a'}$ are linearly independent over $\mathcal K U_q^-$. Therefore, $T_w^{-1}(u)\in\mathcal K U_q^-$ if and only if $c_{\mathbf a'}=0$ unless $\mathbf a'=0$. $\square$

5.3. **Proof of Theorem 3.11.** We will often need the following identity, which is an immediate consequence of (A.35) (cf. [19, Lemma 1.4.4])

$$\langle F_i^r, E_i^r\rangle = q_i^{\binom{r}{2}}\langle r\rangle_{q_i}!, \qquad r\in\mathbb{Z}_{\ge 0},\ i\in I. \tag{5.5}$$

Let $w\in W$ and let $w=s_{i_1}\cdots s_{i_m}$ be its reduced decomposition. Denote $\mathbf{i}=(i_1,\dots,i_m)\in I^m$ and set $w_r=s_{i_1}\cdots s_{i_r}$, $0\le r\le m$. Given $\mathbf{a}\in\mathbb{Z}^I_{\ge 0}$, let $\mu_{\mathbf{i}}(\mathbf{a}):=q^{-\frac12\sum_{r=1}^m a_r\langle w_{r-1}^{-1}\rangle\cdot\alpha_{i_r}}$ (cf. (5.2)). Let $U^\pm_{\mathbb{Z}}(w,1)=U^\pm_q(w,1)\cap U^\pm_{\mathbb{Z}}$. We need the following Lemma.

Lemma 5.6. *The elements $\{\mu_{\mathbf{i}}(\mathbf{a})E_{\mathbf{i}}^{\mathbf{a}}\,:\,\mathbf{a}\in\mathbb{Z}^{\ell(w)}_{\ge 0}\}$ (respectively, $\{\mu_{\mathbf{i}}(\mathbf{a})F_{\mathbf{i}}^{\mathbf{a}}\,:\,\mathbf{a}\in\mathbb{Z}^{\ell(w)}_{\ge 0}\}$ where $F_{\mathbf{i}}^{\mathbf{a}}=(E_{\mathbf{i}}^{\mathbf{a}})^{*t}$) form a $\mathbb{Z}[q,q^{-1}]$-basis of $U^+_{\mathbb{Z}}(w,1)$ (respectively, $U^-_{\mathbb{Z}}(w,1)$). Moreover,*

$$\langle \mu_{\mathbf{i}}(\mathbf{a})F_{\mathbf{i}}^{\mathbf{a}}, \mu_{\mathbf{i}}(\mathbf{a}')E_{\mathbf{i}}^{\mathbf{a}'}\rangle\in\mathbb{Z}[q,q^{-1}]$$

and equals zero unless $\mathbf{a}=\mathbf{a}'$.

Proof. Set

$$\check E_{\mathbf{i}}^{\mathbf{a}} = E_{i_1}^{\langle a_1\rangle}T''_{w_1,-1}(E_{i_2}^{\langle a_2\rangle})\cdots T''_{w_{m-1},-1}(E_{i_m}^{\langle a_m\rangle}).$$

Then by (5.2), $\check E_{\mathbf{i}}^{\mathbf{a}} = \mu_{\mathbf{i}}(\mathbf{a})^{-1}\big(\prod_{i=1}^m\langle a_i\rangle_{q_i}!\big)^{-1}E_{\mathbf{i}}^{\mathbf{a}}$. We also set $\check E_{\mathbf{i}}^{\mathbf{a}}=\check F_{\mathbf{i}}^{\mathbf{a}}$. It follows from [19, Proposition 41.1.4] that the monomials $\{\check E_{\mathbf{i}}^{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{Z}^m_{\ge 0}}$ (respectively $\{\check F_{\mathbf{i}}^{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{Z}^m_{\ge 0}}$) form a $\mathbb{Z}[q,q^{-1}]$-basis of ${}_{\mathbb{Z}}U^+(w,1)$ (respectively, ${}_{\mathbb{Z}}U^-(w,1)$), where ${}_{\mathbb{Z}}U^\pm(w,1)={}_{\mathbb{Z}}U^\pm\cap U_q^+(w,1)$. Moreover, it follows from [19, Proposition 38.2.3] and (5.5) that

$$\langle \check F_{\mathbf{i}}^{\mathbf{a}},\check E_{\mathbf{i}}^{\mathbf{a}'}\rangle=\delta_{\mathbf{a},\mathbf{a}'}q^{\sum_{r=1}^m a_r\eta(w_{r-1}(\alpha_{i_r}))}\prod_{r=1}^N\langle F_{i_r}^{\langle a_r\rangle},E_{i_r}^{\langle a_r\rangle}\rangle=\delta_{\mathbf{a},\mathbf{a}'}q^{\sum_{r=1}^N a_r\eta(w_{r-1}(\alpha_{i_r}))}\prod_{r=1}^N q_{i_r}^{\binom{a_r}{2}}\frac{1}{\langle a_r\rangle_{q_{i_r}}!}.$$

This implies that

$$\langle \check F_{\mathbf{i}}^{\mathbf{a}},\mu(\mathbf{a}')E_{\mathbf{i}}^{\mathbf{a}'}\rangle\in\delta_{\mathbf{a},\mathbf{a}'}q^{\mathbb{Z}}$$

and so $\{\mu(\mathbf{a})E_{\mathbf{i}}^{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{Z}^m_{\ge 0}}$ (respectively, $\{\mu(\mathbf{a})F_{\mathbf{i}}^{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{Z}^m_{\ge 0}}$) is a $\mathbb{Z}[q,q^{-1}]$-basis of $U^+_{\mathbb{Z}}$ (respectively, of $U^-_{\mathbb{Z}}$). Finally,

$$\langle\mu(\mathbf{a})F_{\mathbf{i}}^{\mathbf{a}},\mu(\mathbf{a}')E_{\mathbf{i}}^{\mathbf{a}'}\rangle\in\mu(\mathbf{a})^2\delta_{\mathbf{a},\mathbf{a}'}\mathbb{Z}[q,q^{-1}].$$

Since $\mu(\mathbf{a})^2\in q^{\mathbb{Z}}$, the last assertion follows. □

Proof of Theorem 3.11. Suppose that $\mathfrak{g}$ is semisimple and that $w=w_\circ$ is the longest element in W. Then $U^\pm_{\mathbb{Z}}(w_\circ)=U^\pm_{\mathbb{Z}}$ and by Lemma 5.6, $U^\pm_{\mathbb{Z}}$ admit a pair of bases $\mathbf{B}_\pm$ such that $\langle\mathbf{B}_-,\mathbf{B}_+\rangle\subset\mathbb{Z}[q,q^{-1}]$. Thus, $\langle U^-_{\mathbb{Z}},U^+_{\mathbb{Z}}\rangle=\mathbb{Z}[q,q^{-1}]$. The same argument as in the proof of Proposition 3.9 shows that $\langle\mathbf{B}_{\mathfrak{n}_-},\mathbf{B}_{\mathfrak{n}_+}\rangle\subset\mathbb{Z}[q,q^{-1}]$. □

5.4. **Braid group action for $U_q(\mathfrak{sl}_2)$.** Retain the notation from §4.1.

Lemma 5.7. *We have, for all $a_\pm\in\mathbb{Z}$, $m_\pm\in\mathbb{Z}_{\ge 0}$*

$$T(K_-^{a_-}K_+^{a_+}\diamond F^{m_-}\bullet E^{m_+})=K_-^{-a_--m_-}K_+^{-a_+-m_+}\diamond F^{m_+}\bullet E^{m_-}.$$

Proof. We claim that $T(C^{(r)})=(K_+K_-)^{-r}C^{(r)}$. Indeed, this is obvious for $r=0$ and easily seen to hold for $r=1$. Then by induction hypothesis we have

$$\begin{aligned}T(C^{(r+1)})=T(C)T(C^{(r)})-(K_+K_-)^{-1}T(C^{(r-1)})&=(K_-K_+)^{-r-1}(CC^{(r)}-K_-K_+C^{(r-1)})\\&=(K_-K_+)^{-r-1}C^{(r+1)}.\end{aligned}$$

Since $T(E^{m_+})=K_+^{-m_+}\diamond F^{m_+}$, $T(F^{m_-})=K_-^{-m_-}\diamond E^{m_-}$ and the $C^{(r)}$, $r\ge 0$ are central we obtain, setting $m=\min(m_-,m_+)$

$$T(K_-^{a_-}K_+^{a_+}\diamond F^{m_-}\bullet E^{m_+})=K_-^{-a_-+m}K_+^{-a_++m}\diamond(K_-^{-m_-+m}\diamond E^{m_--m})C^{(m)}(K_+^{-m_++m}\diamond F^{m_+-m})$$

$$= K_-^{-a_- - m_-} K_+^{-a_+ - m_+} \diamond F^{m_+} \bullet E^{m_-}. \qquad \square$$

5.5. **Braid group action on elements of $\mathbf{B}_{\mathfrak{n}_+}$.** Retain the notation of §3.5. It follows from Proposition 3.22(b) that for any element $b_+ \in \mathbf{B}_{\mathfrak{n}_+}$ and $r \in \mathbb{Z}_{\ge 0}$ there exists a unique $b'_+ \in \mathbf{B}_{\mathfrak{n}_+}$ such that $\partial_i^{(top)}(b_+) = \partial_i^{(top)}(b'_+)$ and $\ell_i(b'_+) = \ell_i(b_+) + r$. We denote this element by $\tilde{\partial}_i^{-r}(b_+)$. Observe that

$$\tilde{\partial}_i^{-r}(b_+) = \tilde{\partial}_i^{-r-\ell_i(b_+)} \partial_i^{(top)}(b_+). \tag{5.6}$$

Proposition 5.8. *For all $b_+ \in \mathbf{B}_{\mathfrak{n}_+} \cap \ker \partial_i$, $r \in \mathbb{Z}_{\ge 0}$, $i \in I$ we have*

$$E_i^r \,\square\, b_+ - \tilde{\partial}_i^{-r}(b_+) \in \sum_{b'_+ \in \mathbf{B}_{\mathfrak{n}_+} \,:\, \ell_i(b'_+) < r} q\mathbb{Z}[q] b'_+, \tag{5.7}$$

where for any $x \in U_q^+$ and $r \in \mathbb{Z}_{\ge 0}$ homogeneous we denote

$$E_i^r \,\square\, x := q_i^{-\frac{1}{2} r \alpha_i^\vee(x)} E_i^r x, \qquad \alpha_i^\vee(x) := \alpha_i^\vee(\deg x).$$

Proof. First, note that for $b_+ \in \ker \partial_i$, $\ell_i(E_i^r \,\square\, b_+) = r = \ell_i(\tilde{\partial}_i^{-r}(b_+))$ and by Corollary 3.19

$$\partial_i^{(r)}(E_i^r \,\square\, b_+ - \tilde{\partial}_i^{-r}(b_+)) = \partial_i^{(top)}(b_+) - \partial_i^{(top)}(b_+) = 0,$$

hence by Proposition 3.22(a) and Corollary 3.7

$$E_i^r \,\square\, b_+ - \tilde{\partial}_i^{-r}(b_+) \in \sum_{b'_+ \in \mathbf{B}_{\mathfrak{n}_+} \,:\, \ell_i(b'_+) < r} \mathbb{Z}[q, q^{-1}] b'_+. \tag{5.8}$$

Given $\lambda = (\lambda_i)_{i \in I} \in \mathbb{Z}^I$ and $i \in I$, define $\Bbbk$-linear operators on U_q^+ by

$$F_{i;\lambda}(x) = \frac{q_i^{-\lambda_i + \frac{1}{2}\alpha_i^\vee(x)} E_i x - q_i^{-\frac{1}{2}\alpha_i^\vee(x) + \lambda_i} x E_i}{q_i^{-1} - q_i}, \qquad K_{i;\lambda}(x) = q_i^{\lambda_i - \alpha_i^\vee(x)} x,$$

The following result is well-known (see e.g. [3, Section 3] and also Proposition A.35).

Lemma 5.9. *For any $\lambda \in \mathbb{Z}^I$, the assignments $E_i \mapsto \partial_i$, $F_i \mapsto F_{i;\lambda}$, $K_i \mapsto K_{i;\lambda}$ define a structure of a $U_q(\mathfrak{g})$-module on U_q^+. Moreover, the submodule $\mathscr{V}_\lambda$ of U_q^+ generated by 1 is simple and if $\lambda \in \mathbb{Z}_{\ge 0}^I$ then $\mathscr{V}_\lambda = \{x \in U_q^+ \,:\, \ell_i(x^*) \le \lambda_i\}$ and is integrable.*

Remark 5.10. Here we use the "standard" generators of $U_q(\mathfrak{g})$.

We need the following technical fact which is easy to check by induction.

Lemma 5.11. *For all $\lambda \in \mathbb{Z}^I$, $r \in \mathbb{Z}_{\ge 0}$ and $x \in U_q^+$ homogeneous*

$$q_i^{r(\lambda_i - \alpha_i^\vee(x) - 1) - \binom{r}{2}} F_{i;\lambda}^r(x) = (1 - q_i^2)^{-r} q_i^{-\frac{1}{2} r \alpha_i^\vee(x)} \sum_{k=0}^{r} (-1)^k q_i^{k(2\lambda_i - \alpha_i^\vee(x) - 2r + 2) + k(k-1)} \begin{bmatrix} r \\ k \end{bmatrix}_{q_i^2} E_i^{r-k} x E_i^k.$$

This immediately implies that

$$\begin{aligned} E_i^r \,\square\, x = {} & q_i^{r(\lambda_i - \alpha_i^\vee(x)) - \binom{r+1}{2}} (1 - q_i^2)^r F_{i;\lambda}^r(x) \\ & + q_i^{-\frac{r}{2}\alpha_i^\vee(x)} \sum_{k=1}^{r} (-1)^{k+1} q_i^{k(2\lambda_i - \alpha_i^\vee(x) + k - 2r + 1)} \begin{bmatrix} r \\ k \end{bmatrix}_{q_i^2} E_i^{r-k} x E_i^k. \end{aligned} \tag{5.9}$$

Let $b_+ \in \mathbf{B}_{\mathfrak{n}_+} \cap \ker \partial_i$. It follows by an obvious induction from [18, Proposition 5.3.1] that

$$q_i^{r\varphi_i(b_+) - \binom{r+1}{2}} (1 - q_i^2)^r F_{i;\lambda}^r(b_+) = \prod_{t=0}^{r-1} (1 - q_i^{2(\varphi_i(b_+) - t)}) \tilde{\partial}_i^{-r}(b_+) + \sum_{b'_+ \in \mathbf{B}_{\mathfrak{n}_+} \,:\, \ell_i(b'_+) < r} q\mathbb{Q}[q] b'_+,$$

where $\varphi_i(b_+)=\lambda_i-\alpha_i^\vee(b_+)$. Combining this identity with (5.9) we obtain

$$\begin{aligned}E_i^r\diamond b_+&=\prod_{t=0}^{r-1}(1-q_i^{2(\lambda_i-\alpha_i^\vee(b_+)-t)})\tilde\partial_i^{-r}(b_+)\\&+q_i^{-\frac{r}{2}\alpha_i^\vee(b_+)}\sum_{k=1}^r(-1)^{k+1}q_i^{k(2\lambda_i-\alpha_i^\vee(b_+)+k-2r+1)}\begin{bmatrix}r\\k\end{bmatrix}_{q_i^2}E_i^{r-k}b_+E_i^k+\sum_{b'_+\in\mathbf{B}_{\mathfrak{n}_+}}q\mathbb{Q}[q]b'_+.\end{aligned}\tag{5.10}$$

By Corollary 3.7 we have for all $1\le k\le r$

$$q_i^{-\frac12 r\alpha_i^\vee(b_+)+k(2\lambda_i-\alpha_i^\vee(b_+)+k-2r+1)}E_i^{r-k}b_+E_i^k=q_i^{2\lambda_i}\sum_{b'_+\in\mathbf{B}_{\mathfrak{n}_+}}C_{b_+;r,k}^{b'_+}b'_+,\qquad C_{b_+;r,k}^{b'_+}\in\mathbb{Z}[q,q^{-1}].$$

Since only finitely many terms in this sum are non-zero, there exists $\lambda_i\in\mathbb{Z}_{\ge0}$, $\lambda_i\ge\alpha_i^\vee(b_+)+r$ such that $q_i^{2\lambda_i}C_{b_+,;r,k}^{b'_+}\in q\mathbb{Z}[q]$ for all $b'_+\in\mathbf{B}_{\mathfrak{n}_+}$, $1\le k\le r$. Therefore, it follows from (5.10) that

$$E_i^r\diamond b_+-\tilde\partial_i^{-r}(b_+)\in\sum_{b'_+\in\mathbf{B}_{\mathfrak{n}_+}}q\mathbb{Q}[q]b'_+.$$

It remains to apply (5.8). □

Corollary 5.12. *For any $b_+\in\mathbf{B}_{\mathfrak{n}_+}$ we have*

$$b_+-E_i^{\ell_i(b_+)}\diamond\partial_i^{(top)}(b_+)\in\sum_{b'_+\in\mathbf{B}_{\mathfrak{n}_+}\cap\ker\partial_i,\,0\le r<\ell_i(b_+)}q\mathbb{Z}[q]E_i^r\diamond b'_+.\tag{5.11}$$

Proof. It follows from the Theorem that the elements $\{E_i^r\diamond b_+\,:\,b_+\in\mathbf{B}_{\mathfrak{n}_+}\cap\ker\partial_i,\,r\ge0\}$ form a $\mathbb{Z}[q]$-basis of the lattice $\mathbb{Z}[q]\mathbf{B}_{\mathfrak{n}_+}$ and the transfer matrix is unitriangular with off-diagonal elements in $q\mathbb{Z}[q]$. Then the inverse matrix has the same property. □

We can now prove the following

Theorem 5.13. *For all $b_+\in\mathbf{B}_{\mathfrak{n}_+}$, $i\in I$*

$$T_i(b_+)=K_{+i}^{-\ell_i(b_+)}\diamond F_i^{\ell_i(b_+)}\bullet T_i((\partial_i^{(top)}(b_+)),\quad T_i^{-1}(b_+)=K_{-i}^{-\ell_i(b_+^*)}\diamond F_i^{\ell_i(b_+^*)}\bullet T_i^{-1}((\partial_i^{op})^{(top)}(b_+)).$$

In particular, all elements of $\mathbf{B}_{\mathfrak{n}_+}$ are tame.

Proof. We only prove the first identity, the proof of the second one being similar. We need the following crucial corollary of [20, Theorem 1.2].

Proposition 5.14. *T_i induces a bijection $\mathbf{B}_{\mathfrak{n}_+}\cap\ker\partial_i\to\mathbf{B}_{\mathfrak{n}_+}\cap\ker\partial_i^{op}$.*

Proof. It follows from [19, Lemma 38.1.3 and Proposition 38.1.6] that $T''_{i,-1}$ induces an isomorphism of algebras $\ker\partial_i=\ker\partial_{F_i}^{op}\to\ker\partial_i^{op}=\ker\partial_{F_i}$. Moreover [20, Theorem 1.2] implies that if $b\in\mathbf{B}^{\mathrm{can}}\cap\ker\partial_{E_i}^{op}$ then $T''_{i,-1}(b^{*t})\in(\mathbf{B}^{\mathrm{can}})^{*t}\cap\ker\partial_i^{op}$. Now, let $b_+=\delta_b^{*t}\in\mathbf{B}_{\mathfrak{n}_+}\cap\ker\partial_i$ and $b'\in\mathbf{B}^{\mathrm{can}}\cap\ker\partial_{E_i}^{op}$. Then it follows from (5.2) and [19, Proposition 38.2.1] that $\delta_{b,b'}=(\!|\delta_b,b'|\!)=q^{\frac12\nu}(\!|T_i(\delta_b^{*t})^{*t},T''_{i,-1}(b'^{*t})^{*t}|\!)=q^{\frac12\nu}(\!|T_i(\delta_b^{*t})^{*t},b''|\!)$, where $b''\in\mathbf{B}^{\mathrm{can}}\cap\ker\partial_{E_i}$ and $\nu\in\mathbb{Z}$ depends only on the degree of b. This implies that $T_i(\delta_b^{*t})=q^{-\frac12\nu}\delta_{b''}^{*t}$. But since T_i commutes with $\bar{\cdot}$, it follows that $\nu=0$. □

We have, for any $r>0$, $b'_+\in\mathbf{B}_{\mathfrak{n}_+}\cap\ker\partial_i$

$$\begin{aligned}T_i(E_i^r\diamond b'_+)&=q_i^{-\frac12 r\alpha_i^\vee(\deg b'_+)}T_i(E_i^r)T_i(b'_+)=q_i^{-\frac12 r\alpha_i^\vee(\deg b_+)}(K_{+i}^{-r}\diamond F_i^r)T_i(b_+)\\&=q_i^{-\frac12 r\alpha_i^\vee(\deg b_+)-\frac12 r\alpha_i^\vee(s_i(\deg b_+))}K_{+i}^{-r}\diamond(F_i^rT_i(b_+))=K_{+i}^{-r}\diamond(F_i^rT_i(b_+)).\end{aligned}$$

Then applying T_i to (5.11) yields

$$\begin{aligned}T_i(b_+) &= K_{+i}^{-\ell_i(b_+)} \diamond F_i^{\ell_i(b_+)} T_i(\partial_i^{(top)}(b_+)) + \sum_{\substack{0\le r<\ell_i(b_+)\\ b'_+\in\ker\partial_i\cap\mathbf{B}_{\mathfrak{n}_+}}} D^{b_+}_{b'_+;r} K_{+i}^{-r} \diamond (F_i^r T_i(b'_+))\\ &= K_{+i}^{-\ell_i(b_+)} \diamond \Big(F_i^{\ell_i(b_+)} T_i(\partial_i^{(top)}(b_+)) + \sum_{\substack{0<r\le\ell_i(b_+)\\ b'_+\in\ker\partial_i\cap\mathbf{B}_{\mathfrak{n}_+}}} D^{b_+}_{b'_+;\ell_i(b_+)-r} K_{+i}^{r} \diamond (F_i^{\ell_i(b_+)-r} T_i(b'_+)\Big).\end{aligned}$$

Since $\bar{\cdot}$ commutes with the T_i, this element is $\bar{\cdot}$-invariant. Since all $D^{b_+}_{b'_+;s} \in q\mathbb{Z}[q]$ and $T_i(b'_+) \in \mathbf{B}_{\mathfrak{n}_+}\cap\ker\partial_i^{op}$ for all $b'_+ \in \mathbf{B}_{\mathfrak{n}_+}\cap\ker\partial_i$, $T_i(b_+) = K_{+i}^{-\ell_i(b_+)}\diamond F_i^{\ell_i(b_+)}\circ T_i(\partial_i^{(top)}(b_+))$ by Theorem 1.3. But since for $b_+ \in \ker\partial_i^{op}$, $\overline{F_i b_+}$ in $U_q(\tilde{\mathfrak{g}})$ and in $\mathcal{H}_q^+(\mathfrak{g})$ coincide, it follows that $F_i^{\ell_i(b_+)} \bullet T_i(\partial_i^{(top)}(b_+)) = F_i^{\ell_i(b_+)} \circ T_i(\partial_i^{(top)}(b_+))$ by Theorem 1.5. □

Example 5.15. We now use the above Theorem to compute $F_i^r \bullet b_+$, $r \ge 0$, $b_+ \in \mathbf{B}_{\mathfrak{n}_+} \cap \ker\partial_i^{op}$ for $\mathfrak{g} = \mathfrak{sl}_3$. In this case $\mathbf{B}_{\mathfrak{n}_+}$ consists of elements

$$b_+(\mathbf{a}) := q^{\frac{1}{2}(a_1-a_2)(a_{12}-a_{21})} E_1^{a_1} E_2^{a_2} E_{12}^{a_{12}} E_{21}^{a_{21}}, \qquad \mathbf{a} = (a_1,a_2,a_{12},a_{21}) \in \mathbb{Z}^4_{\ge 0},\ \min(a_1,a_2) = 0.$$

Then

$$\mathbf{B}_{\mathfrak{n}_+} \cap \ker\partial_1^{op} = \{b_+(0,a_2,a_{12},0) \,:\, a_2,a_{12} \in \mathbb{Z}_{\ge 0}\}.$$

Since $T_1^{-1}(E_2) = E_{21}$, $T_1^{-1}(E_{12}) = E_2$ we have $T_1^{-1}(b_+(0,a_2,a_{12},0)) = b_+(0,a_{12},0,a_2)$. Then $F_1^r \bullet b_+(0,a_2,a_{12},0) = K_{+i}^r \diamond T_1(\tilde{\partial}_1^{-r}(b_+(0,a_{12},0,a_2))$. Since

$$\begin{aligned}&\ell_1(b_+(0,a'_2,a'_{12},a'_{21})) = a'_{12}, &&\partial_1^{(top)}(b_+(0,a'_2,a'_{12},a'_{21})) = b_+(0,a'_2+a'_{12},0,a'_{21})\\ &\ell_1(b_+(a'_1,0,a'_{12},a'_{21})) = a'_1+a'_{12}, &&\partial_1^{(top)}(b_+(a'_1,0,a'_{12},a'_{21})) = b_+(0,a'_1+a'_{12},0,a'_{21})\end{aligned}$$

we conclude that

$$\tilde{\partial}_1^{-r}(b_+(0,a_{12},0,a_2)) = \begin{cases} b_+(0,a_{12}-r,r,a_2), & 0\le r\le a_{12}\\ b_+(r-a_{12},0,a_{12},a_2), & r>a_{12}.\end{cases}$$

Since

$$b_+(a'_1,a'_2,a'_{12},a'_{21}) = \sum_{t=0}^{a'_{12}} (-1)^t q^{t(a'_1+a'_2+1)} \begin{bmatrix} a'_{12}\\ t\end{bmatrix}_{q^2} E_1^{a'_1+a'_{12}-t} \diamond b_+(0,a'_2+a'_{12}-t,0,a'_{21}+t), \tag{5.12}$$

we obtain

$$F_1^r \bullet b_+(0,a_2,a_{12},0) = \sum_{t=0}^{\min(r,a_{12})} (-1)^t q^{t(|a_{12}-r|+1)} \begin{bmatrix}\min(r,a_{12})\\ t\end{bmatrix}_{q^2} K_{+1}^t \diamond F_1^{r-t} b_+(0,a_2+t,a_{12}-t,0).$$

Then it is easy to see that $T_2(F_1^r \bullet b_+(0,a_2,a_{12},0)) = K_{+2}^{-a_2} \diamond b_-(0,a_2,0,r) \bullet E_1^{a_{12}} = (K_{+2}^{-a_2} \diamond F_1^{a_{12}} \bullet b_+(0,a_2,r,0))^t$. In a similar fashion, using T_1^{-1} we obtain

$$F_1^r \bullet b_+(0,a_2,0,a_{21}) = \sum_{t=0}^{\min(r,a_{21})} (-1)^t q^{-t(|a_{21}-r|+1)} \begin{bmatrix}\min(r,a_{21})\\ t\end{bmatrix}_{q^{-2}} K_{-1}^t \diamond F_1^{r-t} b_+(0,a_2+t,0,a_{21}-t).$$

5.6. **Wild elements of a double canonical basis.** Assume that $a_{ij} = a_{ji} = -a$, $d_i = d_j = 1$, $a \geq 2$ and consider elements $F_{ij} \bullet E_{ji}$ computed in §4.3. Then for $a = 2$ we have

$$T_i(F_{ij} \bullet E_{ij}) = K_{-i}^{-1} F_{iji} \bullet E_{ji^2} + K_{-i}^{-1} F_{ji^2} \bullet E_{ji^2},$$

while for $a = 3$

$$T_i(F_{ij} \bullet E_{ij}) = (3)_q K_{-i}^{-1} F_{iji^2} \bullet E_{ji^3} + (2)_q K_{-i}^{-1} F_{ji^3} \bullet E_{ji^3} + (2)_q F_{ji^2} \bullet E_{ji^2} + (2)_q K_{+i} K_{+j} F_i \bullet E_i \\ + K_{-j} {K_{-i}}^2 + {K_{+i}}^2 K_{+j} + {K_{-i}}^{-1} K_{+i}^3 K_{+j}.$$

Appendix A. Drinfeld and Heisenberg doubles

A.1. **Nichols algebras.** Let $\Bbbk$ be a field, let V be a $\Bbbk$-vector space and let $\Psi = \Psi_V : V \otimes V \to V \otimes V$ be a braiding, that is, Ψ is invertible and $\Psi_{1,2}\Psi_{2,3}\Psi_{1,2} = \Psi_{2,3}\Psi_{1,2}\Psi_{2,3}$ as endomorphisms of $V^{\otimes 3}$, where

$$\Psi_{i,i+1} = \mathrm{id}_V^{\otimes(i-1)} \otimes \Psi \otimes \mathrm{id}_V^{\otimes(n-i-1)} \in \mathrm{End}_\Bbbk(V^{\otimes n}), \qquad 1 \leq i < n.$$

Define $[n]_\Psi, [n]_\Psi! \in \mathrm{End}_\Bbbk(V^{\otimes n})$, $n \in \mathbb{Z}_{\geq 0}$, by

$$[n]_\Psi = \mathrm{id}_{V^{\otimes n}} + \Psi_{n-1,n} + \Psi_{n-1,n}\Psi_{n-2,n-1} + \cdots + \Psi_{n-1,n}\Psi_{n-2,n-1}\cdots\Psi_{1,2},$$
$$[n]_\Psi! = ([1]_\Psi \otimes \mathrm{id}_V^{\otimes n-1}) \circ ([2]_\Psi \otimes \mathrm{id}_V^{\otimes n-2}) \circ \cdots \circ [n]_\Psi,$$

In particular, $[0]_\Psi! = 1$ and $[1]_\Psi! = \mathrm{id}_V$. Furthermore, given σ in the symmetric group $\mathbb{S}_n$, let $\Psi_\sigma = \Psi_{i_1,i_1+1}\cdots\Psi_{i_r,i_r+1}$ where $\sigma = (i_1, i_1+1)\cdots(i_r, i_r+1)$ is a reduced expression. A standard argument shows that Ψ_σ depends only on σ and not on the reduced expression. In particular, if $\ell(\sigma) + \ell(\tau) = \ell(\sigma\tau)$, where $\ell(\sigma)$ denotes the length of any reduced expression of σ as a product of elementary transpositions, then $\Psi_\sigma\Psi_\tau = \Psi_{\sigma\tau}$. Then

$$[n]_\Psi! = \sum_{\sigma \in \mathbb{S}_n} \Psi_\sigma.$$

Let $\sigma_\circ : (1, \ldots, n) \mapsto (n, \ldots, 1)$ be the longest element of $\mathbb{S}_n$. Since $\ell(\sigma) + \ell(\sigma^{-1}\sigma_\circ) = \ell(\sigma) + \ell(\sigma_\circ\sigma^{-1}) = \ell(\sigma_\circ)$, it follows that $\Psi_\sigma\Psi_{\sigma^{-1}\sigma_\circ} = \Psi_{\sigma_\circ} = \Psi_{\sigma_\circ\sigma^{-1}}\Psi_\sigma$. This implies that

$$[n]_\Psi! = \Psi_{\sigma_\circ}[n]_{\Psi^{-1}}! = [n]_{\Psi^{-1}}!\Psi_{\sigma_\circ}. \tag{A.1}$$

Also, by [7, Proposition 5.5] or [8, Proposition 4.17], $\Psi_{\sigma_\circ}\Psi_\tau = \Psi_{\sigma_\circ\tau\sigma_\circ}\Psi_{\sigma_\circ}$ for all $\tau \in \mathbb{S}_n$, hence

$$[n]_\Psi!\Psi_{\sigma_\circ} = \Psi_{\sigma_\circ}[n]_\Psi! \tag{A.2}$$

Let $r, s > 0$. Then the element $\Psi_\sigma \in \mathrm{End}_\Bbbk(V^{\otimes(r+s)})$ where $\sigma : (1, \ldots, r+s) \mapsto (s+1, \ldots, r+s, 1 \ldots, s)$ defines a brading $\Psi_{V^{\otimes r}, V^{\otimes s}}$.

The tensor algebra $T(V) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V^{\otimes n}$ of V, where $V^{\otimes 0} = \Bbbk$, is the free associative algebra generated by V. The braiding Ψ extends to a braiding $\Psi_{T(V)} : T(V) \otimes T(V) \to T(V) \otimes T(V)$ via $\Psi_{T(V)}|_{V^{\otimes r}, V^{\otimes s}} = \Psi_{V^{\otimes r}, V^{\otimes s}}$. Then $T(V) \otimes T(V)$ can be endowed with a braided algebra structure via $m_{T(V)\otimes T(V)} := (m_{T(V)} \otimes m_{T(V)}) \circ (\mathrm{id}_{T(V)} \otimes \Psi_{T(V)} \otimes \mathrm{id}_{T(V)})$ where $m_{T(V)} : T(V) \otimes T(V) \to T(V)$ is the multiplication map. Furthermore, $T(V)$ becomes a braided bialgebra with the coproduct defined by $\underline{\Delta}(v) = v \otimes 1 + 1 \otimes v$, $v \in V$, and the counit defined by $\varepsilon(1) = 1$, $\varepsilon(v) = 0$, $v \in V$.

The Woronowicz symmetrizer $\mathrm{Wor}(\Psi) : T(V) \to T(V)$ is the linear map defined by

$$\mathrm{Wor}(\Psi)|_{V^{\otimes n}} = [n]_\Psi!.$$

It turns out (cf. [2, 15]) that $\ker \mathrm{Wor}(\Psi)$ is a bi-ideal of $T(V)$. Note that (A.1) implies that $\ker \mathrm{Wor}(\Psi) = \ker \mathrm{Wor}(\Psi^{-1})$.

Definition A.1. The quotient $T(V)/\ker \mathrm{Wor}(\Psi)$ is called the Nichols-Woronowicz algebra $\mathcal{B}(V, \Psi)$ of (V, Ψ).

The algebra $\mathcal{B}(V,\Psi)$ is thus a braided bialgebra, where the braiding $\Psi_{\mathcal{B}(V,\Psi)}$ on $\mathcal{B}(V,\Psi)\otimes\mathcal{B}(V,\Psi)$ is induced by $\Psi_{T(V)}$. By construction, $\mathcal{B}(V,\Psi)$ is $\mathbb{Z}_{\ge 0}$-graded, $\mathcal{B}^r(V,\Psi)$ being the canonical image of $V^{\otimes r}$. Since $V\cap\ker\mathrm{Wor}(\Psi)=0$, V identifies with its canonical image in $\mathcal{B}(V,\Psi)$ and can be shown to coincide with the space of primitive elements in $\mathcal{B}(V,\Psi)$.

The braided antipode S_Ψ on $T(V)$ is defined by $S_\Psi|_{V^{\otimes n}}=(-1)^n\Psi_{\sigma_\circ}$ where $\sigma_\circ:(1,\dots,n)\mapsto(n,\dots,1)$ is the longest permutation in $\mathbb{S}_n$. It satisfies the usual properties, namely

$$m\circ(S_\Psi\otimes 1)\circ\underline{\Delta}=\varepsilon,\quad \underline{\Delta}\circ S_\Psi=(S_\Psi\otimes S_\Psi)\circ\Psi_{T(V)}\circ\underline{\Delta},\quad S_\Psi\circ m=m\circ\Psi_{T(V)}\circ(S_\Psi\otimes S_\Psi) \tag{A.3}$$

where $m=m_{T(V)}$ (see for example [15, §9.4.6]). By (A.1), S_Ψ preserves $\ker\mathrm{Wor}(\Psi)$ hence factors through to a map $S_\Psi:\mathcal{B}(V,\Psi)\to\mathcal{B}(V,\Psi)$ satisfying (A.3).

A.2. **Bar and star involutions.** Let $\bar{\cdot}:\Bbbk\to\Bbbk$ be a field involution and fix an additive involutive map $\bar{\cdot}:V\to V$ satisfying $\overline{xv}=\overline{x}\cdot\overline{v}$, $v\in V$, $x\in\Bbbk$ (we will call such a map anti-linear). There is a unique anti-linear algebra homomorphism $\tilde{\cdot}:T(V)\to T(V)$ whose restriction to $\Bbbk$ and V coincides with the corresponding $\bar{\cdot}$. We say that Ψ is *unitary* if $\tilde{\cdot}\circ\Psi\circ\tilde{\cdot}=\Psi^{-1}$. If Ψ is unitary then, by (A.1), $\ker\widetilde{\mathrm{Wor}}(\Psi)=\ker\mathrm{Wor}(\Psi)$, hence $\tilde{\cdot}$ factors through to an anti-linear algebra involution of $\mathcal{B}(V,\Psi)$.

Proposition A.2. $\Psi_{T(V)}(\tilde{\cdot}\otimes\tilde{\cdot})\circ\underline{\Delta}=\underline{\Delta}\circ\tilde{\cdot}$. *Moreover, the same identity holds for* $\mathcal{B}(V,\Psi)$.

Proof. Let $u\in V^{\otimes n}$, $n\ge 0$. We prove that $\Psi_{T(V)}(\tilde{\underline{u}}_{(1)}\otimes\tilde{\underline{u}}_{(2)})=\underline{\Delta}(\tilde{u})$ by induction on n. The identity is clear for $u\in V$. Furthermore, take $u\in V^{\otimes r}$, $v\in V^{\otimes s}$. Then

$$\begin{aligned}\underline{\Delta}(\widetilde{uv})&=\underline{\Delta}(\tilde{u})\underline{\Delta}(\tilde{v})=\Psi_{T(V)}(\tilde{\underline{u}}_{(1)}\otimes\tilde{\underline{u}}_{(2)})\Psi_{T(V)}(\tilde{v}_{(1)}\otimes\tilde{v}_{(2)})\\&=(m_{T(V)}\otimes m_{T(V)})\circ(1\otimes\Psi_{T(V)}\otimes 1)(\Psi_{T(V)}\otimes\Psi_{T(V)})(\tilde{\underline{u}}_{(1)}\otimes\tilde{\underline{u}}_{(2)}\otimes\tilde{v}_{(1)}\otimes\tilde{v}_{(2)})\end{aligned}$$

On the other hand,

$$\begin{aligned}\Psi_{T(V)}(\tilde{\cdot}\otimes\tilde{\cdot})\underline{\Delta}(uv)&=\Psi_{T(V)}(\tilde{\cdot}\otimes\tilde{\cdot})(m_{T(V)}\otimes m_{T(V)})(1\otimes\Psi_{T(V)}\otimes 1)(\underline{u}_{(1)}\otimes\underline{u}_{(2)}\otimes v_{(1)}\otimes v_{(2)})\\&=\Psi_{T(V)}(m_{T(V)}\otimes m_{T(V)})(1\otimes\Psi^{-1}_{T(V)}\otimes 1)(\tilde{\underline{u}}_{(1)}\otimes\tilde{\underline{u}}_{(2)}\otimes\tilde{v}_{(1)}\otimes\tilde{v}_{(2)}).\end{aligned}$$

So, the first assertion follows from the commutativity of the diagram

$$\begin{tikzcd}
U_1\otimes U_2\otimes U_3\otimes U_4 \arrow[r, "\mathrm{id}_{U_1}\otimes\Psi_{U_2,U_3}\otimes\mathrm{id}_{U_4}"] \arrow[d, "\Psi_{U_1\otimes U_2,U_3\otimes U_4}"'] & U_1\otimes U_3\otimes U_2\otimes U_4 \arrow[d, "\Psi_{U_1,U_3}\otimes\Psi_{U_2,U_4}"] \\
U_3\otimes U_4\otimes U_1\otimes U_2 & U_3\otimes U_1\otimes U_4\otimes U_2 \arrow[l, "\mathrm{id}_{U_3}\otimes\Psi_{U_1,U_4}\otimes\mathrm{id}_{U_2}"]
\end{tikzcd}$$

where $U_i=V^{\otimes r_i}$, $r_1+r_2=r$, $r_3+r_4=s$. The second assertion is immediate. □

Let $\tau_n\in\mathrm{End}(V^{\otimes n})$ be the map satisfying $v_1\otimes\cdots\otimes v_n\mapsto v_n\otimes\cdots\otimes v_1$, $v_i\in V$. We say that Ψ is *self-transposed* if $\Psi=\tau_2\Psi\tau_2$. Define ${}^*\in\mathrm{End}\,T(V)$ by ${}^*|_{V^{\otimes n}}=\tau_n$. Then * is the unique anti-automorphism of $T(V)$ whose restriction to $\Bbbk$ and V is the identity. Since for a self-transposed Ψ we have $\tau_n\Psi_{i,i+1}\tau_n=\Psi_{n-i,n-i+1}$, $1\le i\le n-1$, it follows that

$$\tau_n\Psi_\sigma\tau_n=\Psi_{\sigma_\circ\sigma\sigma_\circ},\qquad\sigma\in\mathbb{S}_n. \tag{A.4}$$

This implies that $[n]_\Psi!\circ\tau_n=\tau_n\circ[n]_\Psi!$ hence * preserves $\ker\mathrm{Wor}(\Psi)$ and so factors through to an anti-automorphism of $\mathcal{B}(V,\Psi)$.

Lemma A.3. *Suppose that Ψ is self-transposed. Then $\underline{\Delta}\circ{}^*={}^*\otimes{}^*\circ\underline{\Delta}^{op}$ on $T(V)$, where $\underline{\Delta}^{op}(u)=\underline{u}_{(2)}\otimes\underline{u}_{(1)}$ in Sweedler's notation, $u\in T(V)$. Moreover, the same identity holds on $\mathcal{B}(V,\Psi)$.*

Proof. The assertion clearly holds for $v\in V$. Let $u\in V^{\otimes r}$, $v\in V^{\otimes s}$. By the induction hypothesis,

$$\begin{aligned}\underline{\Delta}((uv)^*)&=\underline{\Delta}(v^*)\underline{\Delta}(u^*)=(\underline{v}^*_{(2)}\otimes\underline{v}^*_{(1)})(\underline{u}^*_{(2)}\otimes\underline{u}^*_{(1)})\\&=(m_{T(V)}\otimes m_{T(V)})(\underline{v}^*_{(2)}\otimes\Psi_{T(V)}(v^*_{(1)}\otimes\underline{u}^*_{(2)})\otimes\underline{u}^*_{(1)})\\&=(m_{T(V)}\otimes m_{T(V)})(\underline{v}^*_{(2)}\otimes\Psi_{T(V)}((\underline{u}_{(2)}\otimes\underline{v}_{(1)})^*)\otimes\underline{u}^*_{(1)}).\end{aligned}$$

By (A.4) we have $\Psi_{T(V)}\circ{}^*={}^*\circ\Psi_{T(V)}$, hence

$$\begin{aligned}\underline{\Delta}((uv)^*)&=(m_{T(V)}\otimes m_{T(V)})(\underline{v}^*_{(2)}\otimes\Psi_{T(V)}(\underline{u}_{(2)}\otimes\underline{v}_{(1)})^*\otimes\underline{u}^*_{(1)})\\&=({}^*\otimes{}^*)\circ((m_{T(V)}\otimes m_{T(V)})(\underline{u}_{(1)}\otimes\Psi_{T(V)}(\underline{u}_{(2)}\otimes\underline{v}_{(1)})\otimes\underline{v}_{(2)}))={}^*\otimes{}^*\underline{\Delta}^{op}(uv).\qquad\square\end{aligned}$$

If Ψ is both self-transposed and unitary we can define $\bar{\cdot}=\tilde{\cdot}\circ{}^*$, which is the unique anti-linear anti-involution of $T(V)$ and $\mathcal{B}(V,\Psi)$ whose restriction to V coincides with $\bar{\cdot}$. Clearly, $\bar{\cdot}\circ\Psi\circ\bar{\cdot}=\Psi^{-1}$. We also have

$$\underline{\Delta}\circ\bar{\cdot}=\Psi_{T(V)}\circ(\bar{\cdot}\otimes\bar{\cdot})\circ\underline{\Delta}^{op}=(\bar{\cdot}\otimes\bar{\cdot})\circ\Psi^{-1}_{T(V)}\circ\underline{\Delta}^{op}\tag{A.5}$$

A.3. **Pairing and quasi-derivations.** Let $V^\star$ be another $\Bbbk$-vector space with a braiding $\Psi^\star: V^\star\otimes V^\star\to V^\star\otimes V^\star$. Suppose that there exists a pairing $\langle\cdot,\cdot\rangle: V^\star\otimes V\to\Bbbk$ and let $\langle\cdot,\cdot\rangle'$ be the natural pairing $T(V^\star)\otimes T(V)\to\Bbbk$ defined by

$$\langle f_1\otimes\cdots\otimes f_r, v_1\otimes\cdots\otimes v_r\rangle'=\prod_{k=1}^r\langle f_k,v_k\rangle,\qquad f_k\in V^\star,\ v_k\in V,\ 1\le k\le r,$$

while $\langle (V^\star)^{\otimes r},V^{\otimes s}\rangle=0$ if $r\ne s$. If $\Psi^\star$ is the adjoint of Ψ with respect to $\langle\cdot,\cdot\rangle'|_{V^{\star\otimes 2}\otimes V^{\otimes 2}}$ define $\langle\cdot,\cdot\rangle: T(V^\star)\otimes T(V)\to\Bbbk$ by

$$\langle f,u\rangle=\langle f,\mathrm{Wor}(\Psi)(u)\rangle'=\langle \mathrm{Wor}(\Psi^\star)(f),u\rangle',\qquad f\in T(V^\star),\quad u\in T(V).$$

The following Lemma is standard.

Lemma A.4. *Suppose that $\Psi^\star$ is the adjoint of Ψ. Then*

(a) *for all $f,f'\in T(V^\star)$, $v,v'\in T(V)$ we have*

$$\langle ff',v\rangle=\langle f,\underline{v}_{(1)}\rangle\langle f',\underline{v}_{(2)}\rangle,\qquad \langle f,vv'\rangle=\langle \underline{f}_{(1)},v\rangle\langle\underline{f}_{(2)},v'\rangle,$$

where $\underline{\Delta}(v)=\underline{v}_{(1)}\otimes\underline{v}_{(2)}$ and $\underline{\Delta}(f)=\underline{f}_{(1)}\otimes\underline{f}_{(2)}$ in Sweedler's notation.

(b) *Let $\langle\cdot,\cdot\rangle: V^\star\otimes V\to\Bbbk$ be non-degenerate. Then* $\ker\mathrm{Wor}(\Psi)=\{v\in T(V)\,:\,\langle T(V^\star),v\rangle=0\}$, *and* $\ker\mathrm{Wor}(\Psi^\star)=\{f\in T(V^\star)\,:\,\langle f,T(V)\rangle=0\}$. *In particular, $\langle\cdot,\cdot\rangle$ induces a non-degenerate pairing $\langle\cdot,\cdot\rangle:\mathcal{B}(V^\star,\Psi^\star)\otimes\mathcal{B}(V,\Psi)\to\Bbbk$ satisfying* (a).

Remark A.5. The same construction works if $\Psi^\star$ is the adjoint of Ψ^{-1}.

Lemma A.6. $\langle f,S_\Psi(u)\rangle=\langle S_{\Psi^\star}(f),u\rangle$ *for all $f\in T(V^\star)$, $u\in T(V)$ (respectively, $f\in\mathcal{B}(V^\star,\Psi^\star)$, $u\in\mathcal{B}(V,\Psi)$).*

Proof. We may assume, without loss of generality, that $f\in V^{\star\otimes n}$, $u\in V^{\otimes n}$. Let $\sigma_\circ$ be the longest permutation in $\mathbb{S}_n$. Then

$$\langle f,S_\Psi(u)\rangle=(-1)^n\langle f,[n]_\Psi!\Psi_{\sigma_\circ}(u)\rangle'=(-1)^n\langle f,\Psi_{\sigma_\circ}[n]_\Psi!(u)\rangle'=(-1)^n\langle\Psi^\star_{\sigma_\circ^{-1}}(f),u\rangle=\langle S_{\Psi^\star}(f),u\rangle,$$

where we used (A.2). The assertion for Nichols algebras is now immediate. $\square$

Proposition A.7. (a) *Suppose that Ψ is unitary and $\langle\cdot,\cdot\rangle$ satisfies $\overline{\langle\overline{f},\overline{v}\rangle}=-\langle f,v\rangle$, $f\in V^\star$, $v\in V$. Then for all $f\in T(V^\star)$, $v\in T(V)$ we have $\overline{\langle\tilde{f},\tilde{v}\rangle}=\langle f,S_\Psi^{-1}(v)\rangle$ and $S_\Psi^{-1}(\tilde{v})=\widetilde{S_\Psi(v)}$.*

(b) *Suppose that Ψ is self-transposed. Then $\langle f^*,v\rangle=\langle f,v^*\rangle$ for all $f\in T(V^\star)$, $v\in T(V)$.*

(c) *Suppose that the assumptions of* (a) *and* (b) *hold. Then $\overline{\langle\overline{f},\overline{v}\rangle}=\langle f,S_\Psi^{-1}(v)\rangle$ and $S_\Psi^{-1}(\overline{v})=\overline{S_\Psi(v)}$ for all $f\in T(V^\star)$, $v\in T(V)$.*

(d) *Identities* (a)–(c) *hold in corresponding Nichols algebras.*

Proof. To prove (a) we use induction on the degree in $T(V)$. The induction base is given by the assumption. Suppose that the identity is established for all $f \in V^{\star \otimes r}$, $v \in V^{\otimes r}$, $r < n$. Note that the induction hypothesis implies

$$\overline{\langle \tilde{f} \otimes \tilde{g}, \tilde{u} \otimes \tilde{v} \rangle} = \langle f \otimes g, (S_\Psi^{-1} \otimes S_\Psi^{-1})(u \otimes v) \rangle, \quad f \in V^{\star \otimes r},\ g \in V^{\star \otimes s},\ u \in V^{\otimes r},\ v \in V^{\otimes s},\ 0 < r+s \le n.$$

Furthermore, $S_\Psi^{-1} = S_{\Psi^{-1}}$. Hence for all $u \in V^{\otimes n}$, $f \in V^{\star \otimes r}$, $g \in V^{\star \otimes s}$ with $0 < r+s = n$ we have

$$\begin{aligned}\overline{\langle \tilde{f} g, \tilde{u} \rangle} = \overline{\langle \tilde{f} \otimes \tilde{g}, \underline{\Delta}(\tilde{u}) \rangle} &= \overline{\langle \tilde{f} \otimes \tilde{g}, \Psi_{T(V)}(\tilde{\cdot} \otimes \tilde{\cdot}) \underline{\Delta}(u) \rangle} = \overline{\langle \tilde{f} \otimes \tilde{g}, (\tilde{\cdot} \otimes \tilde{\cdot}) \Psi_{T(V)}^{-1} \underline{\Delta}(u) \rangle} \\ &= \langle f \otimes g, (S_\Psi^{-1} \otimes S_\Psi^{-1}) \Psi_{T(V)}^{-1} \underline{\Delta}(u) \rangle = \langle f \otimes g, \underline{\Delta}(S_{\Psi^{-1}}(u)) \rangle = \langle fg, S_\Psi^{-1}(u) \rangle,\end{aligned}$$

where we used (A.3) and Proposition A.2. The identity $S_\Psi^{-1}(\tilde{v}) = \widetilde{S_\Psi(v)}$ is a direct consequence of the unitarity of Ψ and the definition of S_Ψ.

To prove (b), we also use induction on the degree in $T(V)$. The induction base is obvious. Suppose that the identity is established for all $f \in V^{\star \otimes r}$, $v \in V^{\otimes r}$, $r < n$. Then for all $u \in V^{\otimes n}$, $f \in V^{\star \otimes r}$, $g \in V^{\star \otimes s}$ with $0 < r+s = n$ we have

$$\begin{aligned}\langle (fg)^*, u \rangle = \langle g^* f^*, u \rangle &= \langle g^* \otimes f^*, \underline{\Delta}(u) \rangle = \langle g^*, \underline{u}_{(1)} \rangle \langle f^*, \underline{u}_{(2)} \rangle = \langle g, \underline{u}_{(1)}^* \rangle \langle f, \underline{u}_{(2)}^* \rangle \\ &= \langle f \otimes g, \underline{u}_{(2)}^* \otimes \underline{u}_{(1)}^* \rangle = \langle f \otimes g, ({}^* \otimes {}^*) \underline{\Delta}^{op}(u) \rangle = \langle f \otimes g, \underline{\Delta}(u^*) \rangle = \langle fg, u^* \rangle,\end{aligned}$$

where we used Lemma A.3.

To prove (c) note that by (a) and (b) we have

$$\overline{\langle \overline{f}, \overline{g} \rangle} = \langle f^*, S_\Psi^{-1}(g^*) \rangle = \langle f, (S_\Psi^{-1}(g^*))^* \rangle.$$

Let $f \in V^{\star \otimes n}$, $g \in V^{\otimes n}$. Then $S_\Psi^{-1}(g^*)^* = (-1)^n \tau_n \Psi_{\sigma_\circ}^{-1} \tau_n(v) = (-1)^n \Psi_{\sigma_\circ}^{-1}(v) = S_\Psi^{-1}(v)$, where we used (A.4). Part (d) is immediate. □

Suppose that for every $n > 0$, there exists an invertible $L_n \in \operatorname{End}(V^{\otimes n})$ such that $L_n^2 = (-1)^n S_\Psi \circ {}^*$, $L_n \circ \bar{\cdot} = \bar{\cdot} \circ L_n^{-1}$ and $L_n \circ {}^* = {}^* \circ L_n$. Let $L \in \operatorname{End}(T(V))$ be the linear operator defined by $L|_{V^{\otimes n}} = L_n$ and define $(\cdot,\cdot) : \mathcal{B}(V^\star, \Psi^\star) \otimes \mathcal{B}(V, \Psi) \to \Bbbk$ by

$$(f, v) = \langle f, L^{-1}(v) \rangle.$$

Lemma A.8. *Suppose that Ψ is self-transposed and unitary. Then for all $f \in \mathcal{B}^r(V^\star, \Psi^\star)$, $v \in \mathcal{B}^s(V, \Psi)$ we have*

$$\overline{(\overline{f}, \tilde{v})} = (-1)^r \delta_{r,s} (f, v).$$

Proof. Let $f \in \mathcal{B}^r(V^\star, \Psi^\star)$, $v \in \mathcal{B}^r(V, \Psi)$, the case $r \ne s$ being trivial. Then

$$\overline{(\overline{f}, \tilde{v})} = \overline{\langle \overline{f}, L_r^{-1}(\overline{v}^*) \rangle} = (-1)^r \langle f, L_r^{-2}((L_r(v^*))^*) \rangle = (-1)^r \langle f, L_r^{-1}(v) \rangle = (-1)^r (f, v). \qquad \square$$

Given $f \in \mathcal{B}(V^\star, \Psi^\star)$, $v \in \mathcal{B}(V, \Psi)$ define $\Bbbk$-linear operators $\partial_f, \partial_f^{op} : \mathcal{B}(V, \Psi) \to \mathcal{B}(V, \Psi)$, $\partial_v, \partial_v^{op} : \mathcal{B}(V^\star, \Psi^\star) \to \mathcal{B}(V^\star, \Psi^\star)$ by

$$\begin{aligned} &\partial_v(g) = \underline{g}_{(1)} \langle \underline{g}_{(2)}, v \rangle, && \partial_v^{op}(g) = \langle \underline{g}_{(1)}, v \rangle \underline{g}_{(2)}, && f, g \in \mathcal{B}(V^\star, \Psi^\star) \\ &\partial_f(u) = \underline{u}_{(1)} \langle f, \underline{u}_{(2)} \rangle, && \partial_f^{op}(u) = \langle f, \underline{u}_{(1)} \rangle \underline{u}_{(2)}, && u, v \in \mathcal{B}(V, \Psi). \end{aligned} \tag{A.6}$$

Then for all $f, g \in \mathcal{B}(V^\star, \Psi^\star)$, $u, v \in \mathcal{B}(V, \Psi)$

$$\begin{aligned} \langle f, uv \rangle &= \langle \partial_v(f), u \rangle = \langle \partial_u^{op}(f), v \rangle \\ \langle fg, u \rangle &= \langle f, \partial_g(u) \rangle = \langle g, \partial_f^{op}(u) \rangle. \end{aligned} \tag{A.7}$$

The definitions immediately imply that if $f \in \mathcal{B}(V^\star, \Psi^\star)$, $v \in \mathcal{B}(V, \Psi)$ are homogeneous then ∂_f, ∂_f^{op}, ∂_v, ∂_v^{op} are homogeneous. Moreover, if say $f \in \mathcal{B}^r(V^\star, \Psi^\star)$, $v \in \mathcal{B}^k(V, \Psi)$ then $\partial_f(v), \partial_f^{op}(v) \in \sum_{k'=0}^{k-r} \mathcal{B}^{k'}(V, \Psi)$ and $\partial_v(f), \partial_v^{op}(f) \in \sum_{r'=0}^{r-k} \mathcal{B}^{r'}(V^\star, \Psi^\star)$. Thus, ∂_f, ∂_f^{op}, ∂_v, ∂_v^{op} are locally nilpotent.

Lemma A.9. (a) *The assignment $v \mapsto \partial_v$, $v \in \mathcal{B}(V, \Psi)$ (respectively, $f \mapsto \partial_f$, $f \in \mathcal{B}(V^\star, \Psi^\star)$) defines a homomorphism of algebras $\mathcal{B}(V, \Psi) \to \operatorname{End}_{\Bbbk} \mathcal{B}(V^\star, \Psi^\star)$ (respectively, $\mathcal{B}(V^\star, \Psi^\star) \to \operatorname{End}_{\Bbbk} \mathcal{B}(V, \Psi)$).*

(b) *The assignment $v \mapsto \partial_v^{op}$, $v \in \mathcal{B}(V, \Psi)$ (respectively, $f \mapsto \partial_f^{op}$, $f \in \mathcal{B}(V^\star, \Psi^\star)$) defines an anti-homomorphism of algebras $\mathcal{B}(V, \Psi) \to \operatorname{End}_{\Bbbk} \mathcal{B}(V^\star, \Psi^\star)$ (respectively, $\mathcal{B}(V^\star, \Psi^\star) \to \operatorname{End}_{\Bbbk} \mathcal{B}(V, \Psi)$).*

(c) *For all $u, v \in \mathcal{B}(V, \Psi)$, $f, g \in \mathcal{B}(V^\star, \Psi^\star)$ we have $\partial_u \partial_v^{op} = \partial_v^{op} \partial_u$ and $\partial_f \partial_g^{op} = \partial_g^{op} \partial_f$.*

Proof. We have for all $u \in \mathcal{B}(V, \Psi)$, $f, g \in \mathcal{B}(V^\star, \Psi^\star)$

$$\partial_f \partial_g(u) = \langle g, \underline{u}_{(2)} \rangle \partial_f(\underline{u}_{(1)}) = \langle g, \underline{u}_{(3)} \rangle \langle f, \underline{u}_{(2)} \rangle \underline{u}_{(1)} = \langle fg, \underline{u}_{(2)} \rangle \underline{u}_{(1)} = \partial_{fg}(u),$$
$$\partial_f^{op} \partial_g^{op}(u) = \langle g, \underline{u}_{(1)} \rangle \partial_f^{op}(\underline{u}_{(2)}) = \langle g, \underline{u}_{(1)} \rangle \langle f, \underline{u}_{(2)} \rangle \underline{u}_{(3)} = \langle gf, \underline{u}_{(1)} \rangle \underline{u}_{(2)} = \partial_{gf}^{op}(u),$$

and

$$\partial_f \partial_g^{op}(u) = \partial_f(\underline{u}_{(2)}) \langle g, \underline{u}_{(1)} \rangle = \langle f, \underline{u}_{(3)} \rangle \underline{u}_{(2)} \langle g, \underline{u}_{(1)} \rangle = \langle f, \underline{u}_{(2)} \rangle \partial_g^{op}(\underline{u}_{(1)}) = \partial_g^{op} \partial_f(u).$$

The corresponding statements for operators ∂_v, $v \in \mathcal{B}(V, \Psi)$ are checked similarly. □

A.4. **Double smash products.** Let H be a bialgebra with the comultiplication Δ_H and the counit ε_H. Denote by H^{cop} the bialgebra H with the opposite comultiplication. Suppose that C is an $H^{cop} \otimes H$-module algebra. In other words, we have two commuting left actions $\triangleright$ and $\tilde{\triangleright}$ of H on C satisfying

$$h \triangleright (cc') = (h_{(1)} \triangleright c)(h_{(2)} \triangleright c'), \quad h \tilde{\triangleright} (cc') = (h_{(2)} \tilde{\triangleright} c)(h_{(1)} \tilde{\triangleright} c'), \quad h \in H,\, c, c' \in C,$$

where $\Delta_H(h) = h_{(1)} \otimes h_{(2)}$. Define $\mathcal{D}_{H,H^{cop}}(C)$ as $C \otimes H$ with the product

$$(c \otimes 1) \cdot (1 \otimes h) = c \otimes h, \qquad (h \otimes 1) \cdot (1 \otimes c) = (h_{(1)} \triangleright h_{(3)} \tilde{\triangleright} c) h_{(2)}.$$

Proposition A.10. *$\mathcal{D}_{H,H^{cop}}(C)$ is an associative algebra. Moreover, H and C identify with subalgebras of $\mathcal{D}_{H,H^{cop}}(C)$.*

Proof. The only non-trivial identity to check is $h \cdot (cc') = (h \cdot c) \cdot c'$ for all $h \in H$, $c, c' \in C$. We have

$$\begin{aligned}(h \cdot c) \cdot c' &= (h_{(1)} \triangleright h_{(3)} \tilde{\triangleright} c) \cdot (h_{(2)} \cdot c') = (h_{(1)} \triangleright h_{(5)} \tilde{\triangleright} c) \cdot (h_{(2)} \triangleright h_{(4)} \tilde{\triangleright} c') \cdot h_{(3)} \\ &= (h_{(1)} \triangleright ((h_{(4)} \tilde{\triangleright} c) \cdot (h_{(3)} \tilde{\triangleright} c'))) \cdot h_{(2)} = (h_{(1)} \triangleright h_{(3)} \tilde{\triangleright} (cc')) \cdot h_{(2)} = h \cdot (c \cdot c').\end{aligned}$$ □

Remark A.11. Note that if $\triangleright$ (respectively, $\tilde{\triangleright}$) is trivial, that is $h \triangleright c = \varepsilon_H(h) c$ (respectively, $h \tilde{\triangleright} c = \varepsilon_H(h) c$), then $\mathcal{D}_{H,H^{cop}}(C) = C \rtimes H$ (respectively, $C \rtimes H^{cop}$).

Suppose now that C is a bialgebra and that Δ_C, ε_C are homomorphisms of $H^{cop} \otimes H$-modules, where $H^{cop} \otimes H$ acts naturally on $C \otimes C$ and $\Bbbk$. Thus, $\Delta_C(h \triangleright h' \tilde{\triangleright} c) = (h' \tilde{\triangleright} c_{(1)}) \otimes (h \triangleright c_{(2)})$ and $\varepsilon_C(h \triangleright h' \tilde{\triangleright} c) = \varepsilon_C(c) \varepsilon_H(h) \varepsilon_H(h')$ for all $c \in C$, $h, h' \in H$.

Proposition A.12. *Suppose that the actions $\triangleright$, $\tilde{\triangleright}$ satisfy*

$$h_{(2)} \triangleright c_{(1)} \otimes h_{(1)} \tilde{\triangleright} c_{(2)} = \varepsilon_H(h) \Delta(c), \qquad c \in C,\, h \in H. \tag{A.8}$$

Then $\mathcal{D}_{H,H^{cop}}(C)$ is a bialgebra with the comultiplication and the counit defined by $\Delta(c \cdot h) = \Delta_C(c) \cdot \Delta_H^{op}(h)$ and $\varepsilon(c \cdot h) = \varepsilon_C(c) \varepsilon_H(h)$, $c \in C$, $h \in H$, and C, H^{cop} identify with its subbialgebras. If both C and H are Hopf algebras and

$$S_C(h \triangleright h' \tilde{\triangleright} c) = S_H^{-2}(h') \triangleright h \tilde{\triangleright} S_C(c), \qquad c \in C,\, h, h' \in H \tag{A.9}$$

then $\mathcal{D}_{H,H^{cop}}(C)$ is a Hopf algebra with the antipode defined by $S(c\cdot h)=S_H^{-1}(h)\cdot S_C(c)$ and C, H^{cop} identify with its Hopf subalgebras.

Proof. We need to check that $\Delta(h\cdot c)=\Delta_H^{op}(h)\cdot\Delta_C(c)$ for all $c\in C$, $h\in H$. Indeed

$$\begin{aligned}\Delta(h\cdot c)&=\Delta((h_{(1)}\triangleright h_{(3)}\tilde{\triangleright} c)\cdot h_{(2)})=\Delta_C(h_{(1)}\triangleright h_{(4)}\tilde{\triangleright} c)\cdot(h_{(3)}\otimes h_{(2)})\\&=(h_{(4)}\tilde{\triangleright} c_{(1)}\otimes h_{(1)}\triangleright c_{(2)})\cdot(h_{(3)}\otimes h_{(2)})=\varepsilon(h_{(3)})(h_{(5)}\tilde{\triangleright} c_{(1)}\otimes h_{(1)}\triangleright c_{(2)})\cdot(h_{(4)}\otimes h_{(2)})\\&=(h_{(4)}\triangleright h_{(6)}\tilde{\triangleright} c_{(1)})\cdot h_{(5)}\otimes(h_{(1)}\triangleright h_{(3)}\tilde{\triangleright} c_{(2)})\cdot h_{(2)}\\&=h_{(2)}\cdot c_{(1)}\otimes h_{(1)}\cdot c_{(2)}=\Delta_H^{op}(h)\cdot\Delta_C(c).\end{aligned}$$

The property of ε is obvious. For the antipode, we have

$$\begin{aligned}S(h\cdot c)&=S_H^{-1}(h_{(2)})\cdot S_C(h_{(1)}\triangleright h_{(3)}\tilde{\triangleright} c)=S_H^{-1}(h_{(2)})\cdot S_H^{-2}(h_{(3)})\triangleright h_{(1)}\tilde{\triangleright} S_C(c)\\&=S_H^{-1}(h_{(4)})S_H^{-2}(h_{(5)})\triangleright S_H^{-1}(h_{(2)})h_{(1)}\tilde{\triangleright} S_C(c)\cdot S_H^{-1}(h_{(3)})\\&=S_C(c)\cdot S_H^{-1}(h).\end{aligned}$$

□

Denote $H^{\boldsymbol{op}}$ the opposite algebra and coalgebra of H. Note that we can endow $H^{op}\otimes C^{op}$ with an associative algebra structure via

$$c\cdot h=h_{(2)}\cdot(h_{(1)}\triangleright h_{(3)}\tilde{\triangleright} c).$$

Denote the resulting algebra $\mathcal{D}_{H^{\boldsymbol{op}},H^{op}}(C^{op})$. The following proposition is immediate.

Proposition A.13. *The map $\tau: C\otimes H\to H\otimes C$, $c\otimes h\mapsto h\otimes c$ is an isomorphism of algebras $\mathcal{D}_{H,H^{cop}}(C)^{op}\to\mathcal{D}_{H^{\boldsymbol{op}},H^{op}}(C^{op})$. Moreover, if (A.8) and (A.9) hold then τ is an isomorphism of Hopf algebras $\mathcal{D}_{H,H^{cop}}(C)^{\boldsymbol{op}}\to\mathcal{D}_{H^{\boldsymbol{op}},H^{op}}(C^{\boldsymbol{op}})$*

Let $\bar{\cdot}$ be a field involution on $\Bbbk$ and suppose that it extends to an anti-linear anti-involutions of algebras C and H. Assume that $\bar{\cdot}$ is an anti-linear involution of coalgebras for H. Note that then we have $\overline{S_H(h)}=S_H^{-1}(\overline{h})$, $h\in H$. Extend $\bar{\cdot}$ to an anti-linear map $\mathcal{D}_{H,H^{op}}(C)\to\mathcal{D}_{H,H^{op}}(C)$ by

$$\overline{c\cdot h}=\overline{h}\cdot\overline{c}.$$

Lemma A.14. *Suppose that*

$$\overline{h_{(2)}}\triangleright\overline{h_{(1)}\triangleright c}=\varepsilon_H(\overline{h})\overline{c}=\overline{h_{(1)}}\tilde{\triangleright}\overline{h_{(2)}\tilde{\triangleright} c},\qquad h\in H,\,c\in C.\tag{A.10}$$

Then $\bar{\cdot}$ is an anti-linear anti-involution of the algebra $\mathcal{D}_{H,H^{op}}(C)$.

Proof. We have

$$\begin{aligned}\overline{h\cdot c}=\overline{(h_{(1)}\triangleright h_{(3)}\tilde{\triangleright} c)\cdot h_{(2)}}=\overline{h_{(2)}}\cdot\overline{(h_{(1)}\triangleright h_{(3)}\tilde{\triangleright} c)}&=\overline{h_{(2)}}\triangleright\overline{h_{(4)}}\tilde{\triangleright}\overline{(h_{(1)}\triangleright h_{(5)}\tilde{\triangleright} c)}\cdot\overline{h_{(3)}}\\&=\overline{\varepsilon_H(h_{(4)})}\,\overline{h_{(2)}}\triangleright\overline{(h_{(1)}\triangleright c)}\cdot\overline{h_{(3)}}=\varepsilon_H(\overline{h_{(1)}})\triangleright\overline{c}\cdot\overline{h_{(2)}}=\overline{c}\cdot\overline{h}.\end{aligned}$$

This shows that $\bar{\cdot}$ is a well-defined anti-linear anti-involution of $\mathcal{D}_{H,H^{cop}}(C)$. □

Remark A.15. It is easy to check that (A.10) holds if

$$\overline{h\triangleright c}=S_H^{-1}(\overline{h})\triangleright\overline{c},\qquad\overline{h\tilde{\triangleright} c}=S_H(\overline{h})\tilde{\triangleright}\overline{c},\qquad h\in H,\,c\in C.\tag{A.11}$$

A.5. **Bialgebra pairings and doubles of bialgebras.** We will now consider a special case of the double smash product construction. Given bialgebras H and C, $\phi\in\operatorname{Hom}_{\Bbbk}(C\otimes H,\Bbbk)$ is said to be a *bialgebra pairing* if for all $c,c'\in C$ and $h,h'\in H$

$$\phi(cc',h)=\phi(c,h_{(1)})\phi(c',h_{(2)}),\ \phi(c,hh')=\phi(c_{(1)},h)\phi(c_{(2)},h'),\ \phi(c,1)=\varepsilon_C(c),\ \phi(1,h)=\varepsilon_H(h).$$

If both C and H are Hopf algebras, a bialgebra pairing ϕ is called a Hopf pairing if

$$\phi(S_C(c),h)=\phi(c,S_H(h)),\qquad c\in C,\,h\in H.$$

Given a bialgebra pairing $\phi : C \otimes H \to \Bbbk$, define

$$h \underset{\phi}{\triangleright} c = c_{(1)}\phi(c_{(2)}, h), \quad c \underset{\phi}{\triangleleft} h = c_{(2)}\phi(c_{(1)}, h), \quad c \underset{\phi}{\triangleright} h = h_{(1)}\phi(c, h_{(2)}), \quad h \underset{\phi}{\triangleleft} c = h_{(2)}\phi(c, h_{(1)}) \tag{A.12}$$

The following is easily checked.

Lemma A.16. *Let ϕ, ϕ' be two bialgebra pairings $C \otimes H \to \Bbbk$. Then $\underset{\phi}{\triangleright}$, $\underset{\phi'}{\triangleleft}$ define a structure of an H- (respectively, a C-) bimodule algebra on C (respectively, on H). Moreover,*

$$\Delta_C(h \underset{\phi}{\triangleright} c \underset{\phi'}{\triangleleft} h') = (c_{(1)} \underset{\phi'}{\triangleleft} h') \otimes (h \underset{\phi}{\triangleright} c_{(2)}). \tag{A.13}$$

Given two bialgebra pairings $\phi_+, \phi_- : C \otimes H \to \Bbbk$ define $\mathcal{D}_{\phi_+,\phi_-}(C, H)$ as $\mathcal{D}_{H,H^{cop}}(C)$ where $h \triangleright c = h \underset{\phi_+}{\triangleright} c$ and $h \tilde{\triangleright} c = c \underset{\phi_-}{\triangleleft} S_H^{-1}(h)$. Thus, in $\mathcal{D}_{\phi_+,\phi_-}(C, H)$ we have

$$\begin{aligned} h \cdot c &= c_{(2)} \cdot h_{(2)}\phi_-(c_{(1)}, S_H^{-1}(h_{(3)}))\phi_+(c_{(3)}, h_{(1)}) \\ &= (h_{(1)} \underset{\phi_+}{\triangleright} c \underset{\phi_-}{\triangleleft} S_H^{-1}(h_{(3)})) \cdot h_{(2)} = c_{(2)} \cdot (S_C^{-1}(c_{(1)}) \underset{\phi_-}{\triangleright} h \underset{\phi_+}{\triangleleft} c_{(3)}) \end{aligned} \tag{A.14}$$

We abbreviate $\mathcal{D}_\phi(C, H) = \mathcal{D}_{\phi,\phi}(C, H)$

Proposition A.17. *Let H be a Hopf algebra, C be a bialgebra and $\phi, \phi_\pm : C \otimes H \to \Bbbk$ be bialgebra pairings.*

(a) *$\mathcal{D}_{\phi_+,\phi_-}(C, H)$ is an associative algebra and C, H identify with its subalgebras.*
(b) *$\mathcal{D}_\phi(C, H)$ is a bialgebra and C, H^{cop} identify with its sub-bialgebras. Moreover, if C is a Hopf algebra and ϕ is a Hopf pairing then $\mathcal{D}_\phi(C, H)$ is a Hopf algebra.*

Proof. Part (a) is immediate from Proposition A.10. To prove (b) note that by (A.13) we only need to check that (A.8) and (A.9) hold. Indeed

$$\begin{aligned} h_{(2)} \underset{\phi}{\triangleright} c_{(1)} \otimes c_{(2)} \underset{\phi}{\triangleleft} S_H^{-1}(h_{(1)}) &= \phi(c_{(2)}, h_{(2)})\phi(c_{(3)}, S_H^{-1}(h_{(1)}))c_{(1)} \otimes c_{(4)} \\ &= \phi(c_{(2)}, h_{(2)}S_H^{-1}(h_{(1)}))c_{(1)} \otimes c_{(3)} = \varepsilon_H(h)c_{(1)} \otimes c_{(2)} = \varepsilon_H(h)\Delta(c). \end{aligned}$$

Finally, to prove (A.9) note that

$$\begin{aligned} S_C(h \triangleright c \triangleleft S_H^{-1}(h')) &= \phi(c_{(3)}, h)\phi(c_{(1)}, S_H^{-1}(h'))S_C(c_{(2)}) \\ &= \phi(S_C(c_{(3)}), S_H^{-1}(h))\phi(S_C(c_{(1)}), S_H^{-2}(h'))S_C(c_{(2)}) = S_H^{-2}(h') \triangleright S_C(c) \triangleleft S_H^{-1}(h). \end{aligned}$$ ☐

Note the following useful identity in $\mathcal{D}_{\phi^+,\phi^-}(C, H)$

$$c \cdot h = h_{(2)} \cdot (S_H^{-1}(h_{(1)}) \underset{\phi_+}{\triangleright} c \underset{\phi_-}{\triangleleft} h_{(3)}), \qquad c \in C,\ h \in H. \tag{A.15}$$

The following is a straightforward consequence of (A.12) and (A.14).

Proposition A.18. *H is a left (respectively, right) $\mathcal{D}_\phi(C, H)$-module algebra via $c \triangleright h' = c \underset{\phi}{\triangleright} h'$ and $h \triangleright h' = h_{(2)}h'S_H^{-1}(h_{(1)})$ (respectively, via $h' \triangleleft c = h' \underset{\phi}{\triangleleft} c$ and $h' \triangleleft h = S_H^{-1}(h_{(2)})h'h_{(1)}$), $c \in C$, $h, h' \in H$. Moreover, if C is a Hopf algebra then C is a left (respectively, right) $\mathcal{D}_\phi(C, H)$-module algebra via $h \triangleright c' = h \underset{\phi}{\triangleright} c'$, $c \triangleright c' = c_{(2)}c'S_C^{-1}(c_{(1)})$ (respectively, via $c' \triangleleft h = c' \underset{\phi}{\triangleleft} h$, $c' \triangleleft c = S_C^{-1}(c_{(2)})c'c_{(1)}$), $c, c' \in C$, $h \in H$.*

The compatibility conditions from Lemma A.14 read

$$\overline{c_{(1)}}\phi_+(\overline{c_{(2)}}, \overline{h_{(2)}})\overline{\phi_+(c_{(3)}, h_{(1)})} = \varepsilon_H(\overline{h})\overline{c} = \overline{c_{(3)}}\phi_-(\overline{c_{(2)}}, S_H^{-1}(\overline{h_{(1)}}))\overline{\phi_-(c_{(1)}, S_H^{-1}(h_{(2)}))} \tag{A.16}$$

and are satisfied if

$$\overline{\phi_\pm(\overline{c}, \overline{h})} = \phi_\pm(c, S_H^{-1}(h)), \qquad c \in C,\ h \in H.$$

A.6. **Bosonization of Nichols algebras.** Suppose that V is a left Yetter-Drinfeld module over a Hopf algebra H with the comultiplication Δ_H and the antipode S_H. That is, V is a left H-module with the action denoted by $\triangleright$ and a left H-comodule with the co-action $\delta : V \to H \otimes V$. We use the Sweedler-type notation $\delta(v) = v^{(-1)} \otimes v^{(0)}$. The action and co-action are compatible, that is

$$\delta(h \triangleright v) = h_{(1)} v^{(-1)} S_H(h_{(3)}) \otimes h_{(2)} \triangleright v^{(0)}, \qquad h \in H,\, v \in V, \tag{A.17}$$

where $(\Delta_H \otimes 1)\Delta_H = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$.

The category ${}^H_H\mathscr{Y}\mathscr{D}$ of left Yetter-Drinfeld modules over H is a braided tensor category with the braiding $\Psi : V \otimes W \to W \otimes V$ being given by

$$\Psi_{V,W}(v \otimes w) = v^{(-1)} \triangleright w \otimes v^{(0)}, \qquad v \in V,\, w \in W. \tag{A.18}$$

Note that

$$\Psi^{-1}_{V,W}(w \otimes v) = v^{(0)} \otimes S_H^{-1}(v^{(-1)}) \triangleright w. \tag{A.19}$$

In particular, $T(V)$ is a braided Hopf algebra in the category ${}^H_H\mathscr{Y}\mathscr{D}$. We will denote the corresponding Nichols algebra by $\mathcal{B}(V)$.

Consider now the algebra $T(V) \rtimes H = T(V) \otimes H$ with the cross-relation

$$h \cdot u = (h_{(1)} \triangleright u) \cdot h_{(2)}. \tag{A.20}$$

It has a co-algebra structure defined by

$$\Delta(v) = v \otimes 1 + \delta(v), \quad \Delta(h) = \Delta_H(h), \qquad \varepsilon(v) = 0, \quad \varepsilon(h) = \varepsilon_H(h), \quad v \in V,\, h \in H. \tag{A.21}$$

It is easy to check, using (A.17), that this comultiplication and counit extend to homomorphisms of respective algebras.

Lemma A.19. *Let $u \in T(V)$. Then $\Delta(u) = \underline{u}_{(1)}\underline{u}_{(2)}^{(-1)} \otimes \underline{u}_{(2)}^{(0)}$, where $\underline{\Delta}(u) = \underline{u}_{(1)} \otimes \underline{u}_{(2)}$.*

Proof. For $v \in V$ there is nothing to prove. Suppose that the identity holds for all $u \in V^{\otimes r}$, $r < n$. Let $u \in V^{\otimes r}$, $v \in V^{\otimes s}$, $r, s > 0$, $r + s = n$. Then

$$\underline{\Delta}(uv) = \underline{\Delta}(u)\underline{\Delta}(v) = (\underline{u}_{(1)} \otimes 1)\Psi(\underline{u}_{(2)} \otimes \underline{v}_{(1)})(1 \otimes \underline{v}_{(2)}) = \underline{u}_{(1)}(\underline{u}_{(2)}^{(-1)} \triangleright \underline{v}_{(1)}) \otimes \underline{u}_{(2)}^{(0)}\underline{v}_{(2)},$$

whence

$$\begin{aligned}\Delta(uv) &= (\underline{u}_{(1)}\underline{u}_{(2)}^{(-1)} \otimes \underline{u}_{(2)}^{(0)})(\underline{v}_{(1)}\underline{v}_{(2)}^{(-1)} \otimes \underline{v}_{(2)}^{(0)}) = \underline{u}_{(1)}\underline{u}_{(2)}^{(-1)}\underline{v}_{(1)}\underline{v}_{(2)}^{(-1)} \otimes \underline{u}_{(2)}^{(0)}\underline{v}_{(2)}^{(0)}\\ &= \underline{u}_{(1)}(\underline{u}_{(2)}^{(-2)} \triangleright \underline{v}_{(1)})\underline{u}_{(2)}^{(-1)}\underline{v}_{(2)}^{(-1)} \otimes \underline{u}_{(2)}^{(0)}\underline{v}_{(2)}^{(0)} = \underline{uv}_{(1)}\underline{uv}_{(2)}^{(-1)} \otimes \underline{uv}_{(2)}^{(0)}. \qquad \square\end{aligned}$$

Denote by $\underline{S}$ the braided antipode on $T(V)$ corresponding to the braiding $\Psi_{V,V}$. Note that $\underline{S}$ is a morphism in the category ${}^H_H\mathscr{Y}\mathscr{D}$ hence commutes with the action and the co-action of H. Define $S : T(V) \rtimes H \to T(V) \rtimes H$ by

$$S(uh) = S_H(u^{(-1)}h)\underline{S}(u^{(0)}). \tag{A.22}$$

Lemma A.20. *S is an antipode for $T(V) \rtimes H$. Moreover, S is invertible and*

$$S^{-1}(uh) = S_H^{-1}(h)\underline{S}^{-1}(u^{(0)})S_H^{-1}(u^{(-1)}), \qquad u \in T(V),\, h \in H. \tag{A.23}$$

Proof. By definition, we have $S(uh) = S(h)S(u)$, $u \in T(V)$, $h \in H$. Furthermore, using (A.17), we obtain

$$\begin{aligned}S(hu) &= S((h_{(1)} \triangleright u)h_{(2)}) = S_H(h_{(1)} \triangleright u)^{(-1)}h_{(2)})\underline{S}((h_{(1)} \triangleright u)^{(0)})\\ &= S_H(h_{(1)}u^{(-1)}S_H(h_{(3)})h_{(4)})\underline{S}(h_{(2)} \triangleright u^{(0)}) = S_H(h_{(1)}u^{(-1)})\underline{S}(h_{(2)} \triangleright u^{(0)})\\ &= S_H(u^{(-1)})S_H(h_{(1)})(h_{(2)} \triangleright \underline{S}(u^{(0)}) = S_H(u^{(-1)})(S_H(h_{(2)})h_{(3)}) \triangleright \underline{S}(u^{(0)})S_H(h_{(1)})\\ &= S_H(u^{(-1)})\underline{S}(u^{(0)})S_H(h) = S(u)S(h).\end{aligned}$$

To prove that S is an anti-endomorphism of $T(V)\rtimes H$, it remains to show that $S(uv)=S(v)S(u)$ for all $u,v\in T(V)$. Indeed,

$$\begin{aligned}S(uv)&=S_H((uv)^{(-1)})\underline{S}((uv)^{(0)})=S_H(v^{(-1)})S_H(u^{(-1)})((\underline{S}(u^{(0)}))^{(-1)}\triangleright\underline{S}(v^{(0)}))(\underline{S}(u^{(0)}))^{(0)}\\&=S_H(v^{(-1)})S_H(u^{(-2)})(u^{(-1)}\triangleright\underline{S}(v^{(0)}))\underline{S}(u^{(0)})\\&=S_H(v^{(-1)})(S_H(u^{(-2)})u^{(-1)}\triangleright\underline{S}(v^{(0)}))S_H(u^{(-3)})\underline{S}(u^{(0)})\\&=S_H(v^{(-1)})\underline{S}(v^{(0)}))S_H(u^{(-1)})\underline{S}(u^{(0)})=S(v)S(u).\end{aligned}$$

We have

$$m(S\otimes 1)\Delta(u)=S(\underline{u}_{(1)}\underline{u}_{(2)}^{(-1)})\underline{u}_{(2)}^{(0)}=S_H(\underline{u}_{(1)}^{(-1)}\underline{u}_{(2)}^{(-1)})\underline{S}(\underline{u}_{(1)}^{(0)})\underline{u}_{(2)}^{(0)}=S_H((\Delta(u))^{(-1)})\varepsilon(u^{(0)})=\varepsilon(u).$$

On the other hand,

$$\begin{aligned}m(1\otimes S)\Delta(u)=\underline{u}_{(1)}\underline{u}_{(2)}^{(-1)}S(\underline{u}_{(2)}^{(0)})=\underline{u}_{(1)}\underline{u}_{(2)}^{(-2)}S_H(\underline{u}_{(2)}^{(-1)})\underline{S}(\underline{u}_{(2)}^{(0)})&=\underline{u}_{(1)}\varepsilon_H(\underline{u}_{(2)}^{(-1)})\underline{S}(\underline{u}_{(2)}^{(0)})\\&=\underline{u}_{(1)}\underline{S}(\underline{u}_{(2)})=\varepsilon(u).\end{aligned}$$

Define $\tilde{S}:T(V)\rtimes H\to T(V)\rtimes H$ by $\tilde{S}(uh)=S_H^{-1}(h)\underline{S}^{-1}(u^{(0)})S_H^{-1}(u^{(-1)})$. Then we have

$$\begin{aligned}\tilde{S}(hu)&=\tilde{S}((h_{(1)}\triangleright u)h_{(2)})=S_H^{-1}(h_{(2)})\underline{S}^{-1}((h_{(1)}\triangleright u)^{(0)})S_H^{-1}((h_{(1)}\triangleright u)^{(-1)})\\&=S_H^{-1}(h_{(4)})\underline{S}^{-1}(h_{(2)}\triangleright u^{(0)})S_H^{-1}(h_{(1)}u^{(-1)}S_H(h_{(3)}))\\&=S_H^{-1}(h_{(4)})(h_{(2)}\triangleright\underline{S}^{-1}(u^{(0)}))h_{(3)}S_H^{-1}(u^{(-1)})S_H^{-1}(h_{(1)})\\&=S_H^{-1}(h_{(3)})h_{(2)}\underline{S}^{-1}(u^{(0)})S_H^{-1}(u^{(-1)})S_H^{-1}(h_{(1)})\\&=\underline{S}^{-1}(u^{(0)})S_H^{-1}(u^{(-1)})S_H^{-1}(h)=\tilde{S}(u)S_H^{-1}(h).\end{aligned}$$

Now

$$\begin{aligned}S\tilde{S}(uh)=S(S_H^{-1}(h)\underline{S}^{-1}(u^{(0)})S_H^{-1}(u^{(-1)}))&=u^{(-1)}S(\underline{S}(u^{(0)}))h\\&=u^{(-2)}S_H(u^{(-1)})u^{(0)}h=\varepsilon_H(u^{(-1)})u^{(0)}h=uh,\end{aligned}$$

while

$$\begin{aligned}\tilde{S}S(uh)=\tilde{S}(S_H(u^{(-1)}h)\underline{S}(u^{(0)}))&=\tilde{S}(\underline{S}(u^{(0)}))u^{(-1)}h\\&=u^{(0)}S_H^{-1}(u^{(-1)})u^{(-2)}h=u^{(0)}\varepsilon_H(u^{(-1)})h=uh.\end{aligned}$$

Thus, $\tilde{S}$ is the inverse of S. □

Observe that $\ker\operatorname{Wor}(\Psi)$ is a bi-ideal in $T(V)\rtimes H$. In particular, we can consider the quotient of $T(V)\rtimes H$ by that ideal which is isomorphic to $\mathcal{B}(V)\rtimes H$. Clearly, Lemmata A.19 and A.20 hold in $\mathcal{B}(V)\rtimes H$.

Let $\bar{\cdot}$ be a field involution on $\Bbbk$ and fix its extension to V as in §A.2. Suppose that $\overline{h\triangleright v}=S_H^{-1}(\overline{h})\triangleright\overline{v}$ and that $(\bar{\cdot}\otimes\bar{\cdot})\circ\delta\circ\bar{\cdot}=\delta$.

Lemma A.21. *Suppose that Ψ is self-transposed. Then Ψ is also unitary, that is $\bar{\cdot}\otimes\bar{\cdot}\circ\Psi=\Psi^{-1}\circ\bar{\cdot}$*

Proof. Since Ψ is self-transposed, it follows that

$$u^{(-1)}\triangleright v\otimes u^{(0)}=v^{(0)}\otimes v^{(-1)}\triangleright u \tag{A.24}$$

Applying $\bar{\cdot}\otimes\bar{\cdot}$ to both sides yields

$$(\bar{\cdot}\otimes\bar{\cdot})\circ\Psi(u\otimes v)=\overline{u^{(-1)}\triangleright v}\otimes\overline{u^{(0)}}=\overline{v^{(0)}}\otimes\overline{v^{(-1)}\triangleright u}=\overline{v^{(0)}}\otimes S_H^{-1}(\overline{v^{(-1)}})\triangleright\overline{u}=\Psi^{-1}(\overline{u}\otimes\overline{v}),$$

where we used (A.19). □

Thus, if (A.24) holds, $\mathcal{B}(V)$ admits the anti-linear anti-involution $\bar{\cdot}$. Then by Lemma A.14, (A.11) and Remark A.11, $\bar{\cdot}$ extends uniquely to an anti-linear anti-involution on $\mathcal{B}(V)\rtimes H$ such that $\overline{v\cdot h}=\overline{h}\cdot\overline{v}$, $v\in V$, $h\in H$. Thus, we obtain the following

Lemma A.22. *Suppose that $\Psi: V\otimes V\to V\otimes V$ is self-transposed, $\bar{\cdot}$ commutes with the co-action on V and $\overline{h\triangleright v}=S_H^{-1}(\overline{h})\triangleright\overline{v}$. Then $\bar{\cdot}$ extends to an anti-linear algebra anti-involution of $\mathcal{B}(V)\rtimes H$.*

A.7. **Drinfeld double.** Let C, H be Hopf algebras and fix a Hopf pairing $\xi: C\otimes H\to\Bbbk$. Let V (respectively, $V^\star$) be an object in ${}^H_H\mathscr{YD}$ (respectively, in ${}^C_C\mathscr{YD}$). Then we have a right C-module (respectively, H-module) structure on V (respectively, $V^\star$) defined by

$$f\triangleleft h=\xi(f^{(-1)},h)f^{(0)},\qquad v\triangleleft c=\xi(c,v^{(-1)})v^{(0)},\qquad f\in V^\star,\,v\in V,\,c\in C,\,h\in H.\tag{A.25}$$

Assume that a pairing $\langle\cdot,\cdot\rangle: V^\star\otimes V\to\Bbbk$ satisfies

$$\langle f,h\triangleright v\rangle=\langle f\triangleleft h,v\rangle,\qquad\langle c\triangleright f,v\rangle=\langle f,v\triangleleft c\rangle,\qquad f\in V^\star,\,v\in V,\,c\in C,\,h\in H.\tag{A.26}$$

Lemma A.23. *Suppose that (A.26) holds. Then the braiding $\Psi^\star$ is the adjoint of Ψ with respect to $\langle\cdot,\cdot\rangle': V^{\star\otimes 2}\otimes V^{\otimes 2}\to\Bbbk$ in the notation of §A.3.*

Proof. We need to show that for all $f,g\in V^\star$, $u,v\in V$

$$\langle\Psi^\star(f\otimes g),u\otimes v\rangle'=\langle f\otimes g,\Psi(u\otimes v)\rangle'$$

which, by the definition of Ψ and $\Psi^\star$ is equivalent to

$$\langle f^{(-1)}\triangleright g,u\rangle\langle f^{(0)},v\rangle=\langle f,u^{(-1)}\triangleright v\rangle\langle g,u^{(0)}\rangle$$

But, using (A.25) and (A.26) we obtain

$$\begin{aligned}\langle f^{(-1)}\triangleright g,u\rangle\langle f^{(0)},v\rangle&=\langle g,u\triangleleft f^{(-1)}\rangle\langle f^{(0)},v\rangle=\langle g,u^{(0)}\rangle\xi(f^{(-1)},u^{(-1)})\langle f^{(0)},v\rangle\\&=\langle g,u^{(0)}\rangle\langle f\triangleleft u^{(-1)},v\rangle=\langle g,u^{(0)}\rangle\langle f,u^{(-1)}\triangleright v\rangle.\end{aligned}$$

□

Thus, we can define the pairing $\langle\cdot,\cdot\rangle: T(V^\star)\otimes T(V)\to\Bbbk$ as in §A.3. Note that (A.26) holds for all $f\in T(V^\star)$, $v\in T(V)$. Clearly, we can replace $T(V)$, $T(V^\star)$ by the corresponding Nichols algebras.

It should be noted that V is not an H-C bimodule with respect to the actions $\triangleright$ and $\triangleleft$. Given $c\in C$, $h\in H$ define, for all $v\in V$, $f\in V^\star$

$$v\triangleleft(c\cdot h)=S_H^{-1}(h)\triangleright(v\triangleleft c),\qquad(c\cdot h)\triangleright f=c\triangleright(f\triangleleft S_H^{-1}(h)).\tag{A.27}$$

Lemma A.24. *V (respectively, $V^\star$) is a right (respectively, left) Yetter-Drinfeld module over $\mathcal{D}_\xi(C,H)$, with the right coaction on V defined by $\delta_R(v)=v^{(0)}\otimes v^{(-1)}$, the left coaction on $V^\star$ defined by $\delta(f)=f^{(-1)}\otimes f^{(0)}$ and the left (right) action defined by (A.27).*

Proof. Let $c\in C$, $h\in H$ and $v\in V$. By definition, we have $v\triangleleft(c\cdot h)=(v\triangleleft c)\triangleleft h$. On the other hand,

$$\begin{aligned}(v\triangleleft h)\triangleleft c&=\xi(c,(S_H^{-1}(h)\triangleright v)^{(-1)})(S_H^{-1}(h)\triangleright v)^{(0)}=\xi(c,S_H^{-1}(h_{(3)})v^{(-1)}h_{(1)})S_H^{-1}(h_{(2)})\triangleright v^{(0)}\\&=\xi(c_{(1)},S_H^{-1}(h_{(3)}))\xi(c_{(3)},h_{(1)})\xi(c_{(2)},v^{(-1)})S_H^{-1}(h_{(2)})\triangleright v^{(0)}\\&=\xi(c_{(1)},S_H^{-1}(h_{(3)}))\xi(c_{(3)},h_{(1)})S_H^{-1}(h_{(2)})\triangleright(v\triangleleft c_{(2)})\\&=\xi(c_{(1)},S_H^{-1}(h_{(3)}))\xi(c_{(3)},h_{(1)})S_H^{-1}(h_{(2)})(v\triangleleft(c_{(2)}\cdot h_{(2)})=v\triangleleft(h\cdot c).\end{aligned}$$

Thus, (A.27) defines a right $\mathcal{D}_\xi(C,H)$-module structure on V. It remains to verify that this action is compatible with the right co-action. Recall that H^{cop} identifies with a sub-bialgebra of $\mathcal{D}_\xi(C,H)$, hence we only need to check the compatibility condition for $c\in C$. We have

$$(v^{(0)}\triangleleft c_{(2)})\otimes S_C(c_{(1)})v^{(-1)}c_{(3)}=(v^{(0)}\triangleleft c_{(2)})\otimes S_C(c_{(1)})c_{(4)}v^{(-2)}\xi(c_{(3)},S_H^{-1}(v^{(-1)}))\xi(c_{(5)},v^{(-3)})$$

$$\begin{aligned}
&= v^{(0)} \otimes S_C(c_{(1)})c_{(4)}v^{(-3)}\xi(S_C^{-1}(c_{(3)}), v^{(-2)})\xi(c_{(2)}, v^{(-1)})\xi(c_{(5)}, v^{(-4)})\\
&= v^{(0)} \otimes S_C(c_{(1)})c_{(4)}v^{(-2)}\xi(S_C^{-1}(c_{(3)})c_{(2)}, v^{(-1)})\xi(c_{(5)}, v^{(-3)})\\
&= v^{(0)} \otimes S_C(c_{(1)})c_{(2)}v^{(-1)}\xi(c_{(3)}, v^{(-2)}) = v^{(0)} \otimes v^{(-1)}\xi(c, v^{(-2)}) = \xi(c, v^{(-1)})\delta_R(v^{(0)})\\
&= \delta_R(v \triangleleft c).
\end{aligned}$$

The assertion for $V^\star$ is proved similarly. □

Definition A.25. Fix pairings $\langle\cdot,\cdot\rangle_\pm : V^\star \otimes V \to \Bbbk$ such that (A.26) holds for both of them. The algebra $\mathcal{U}_\xi(V^\star, C, V, H)$ is generated by V, $V^\star$, $\mathcal{D}_\xi(C,H)$ subjects to the following relations

- (i) The subalgebra generated by V (respectively, $V^\star$) and $\mathcal{D}_\xi(C,H)$ is isomorphic to $\mathcal{D}_\xi(C,H) \ltimes \mathcal{B}(V)$ (respectively, $\mathcal{B}(V^\star) \rtimes D_\xi(C,H)$)
- (ii) $[v, f] = f^{(-1)}\langle f^{(0)}, v\rangle_+ - v^{(-1)}\langle f, v^{(0)}\rangle_-$, $f \in V^\star$, $v \in V$.

Proposition A.26. *The algebra $\mathcal{U}_\xi(V^\star, C, V, H)$ is isomorphic to the braided double $\mathcal{B}(V^\star) \rtimes \mathcal{D}_\xi(C,H) \ltimes \mathcal{B}(V)$ in the sense of [2] and admits a triangular decomposition. In particular, if $\langle\cdot,\cdot\rangle_-$ equals zero, $\mathcal{U}_\xi(V^\star, C, V, H)$ is the Heisenberg double.*

Proof. Define $\beta : V^\star \otimes V \to \mathcal{D}_\xi(C,H)$ by $\beta = \beta_+ - \beta_-$ where $\beta_+(f,v) = f^{(-1)}\langle f^{(0)}, v\rangle_+$ and $\beta_-(f,v) = v^{(-1)}\langle f, v^{(0)}\rangle_-$, $f \in V^\star$, $v \in V$. Then in $\mathcal{U}_\xi(V^\star, C, V, H)$ we have $[v, f] = \beta(f, v)$. Thus, $\mathcal{U}_\xi(V^\star, C, V, H)$ is a braided double. By [2, Theorem A], it remains to prove that

$$x_{(1)}\beta_\pm(f, v \triangleleft x_{(2)}) = \beta_\pm(x_{(1)} \triangleright f, v)x_{(2)}, \qquad x \in \mathcal{D}_\xi(C,H),\ v \in V,\ f \in V^\star. \tag{A.28}$$

Using Lemma A.24, we obtain

$$\begin{aligned}
\beta_+(x_{(1)} \triangleright f, v)x_{(2)} &= (x_{(1)} \triangleright f)^{(-1)}\langle (x_{(1)} \triangleright f)^{(0)}, v\rangle_+ x_{(2)} = x_{(1)}f^{(-1)}S(x_{(3)})x_{(4)}\langle x_{(2)} \triangleright f^{(0)}, v\rangle_+\\
&= x_{(1)}f^{(-1)}\langle f^{(0)}, v \triangleleft x_{(2)}\rangle_+ = x_{(1)}\beta_+(f, v \triangleright x_{(2)}),
\end{aligned}$$

while

$$\begin{aligned}
x_{(1)}\beta_-(f, v \triangleleft x_{(2)}) &= \langle f, v^{(0)} \triangleleft x_{(3)}\rangle_- x_{(1)}S(x_{(2)})v^{(-1)}x_{(4)} = \langle f, v^{(0)} \triangleleft x_{(1)}\rangle_- v^{(-1)}x_{(2)}\\
&= v^{(-1)}\langle x_{(1)} \triangleright f, v^{(0)}\rangle_- x_{(2)} = \beta_-(x_{(1)} \triangleright f, v)x_{(2)}.
\end{aligned}$$
□

We now obtain another presentation of $\mathcal{U}_\xi(V^\star, C, V, H)$. Given a pairing $\langle\cdot,\cdot\rangle : V^\star \otimes V \to \Bbbk$ satisfying (A.26), define $\phi : \mathcal{B}(V^\star) \rtimes C \otimes \mathcal{B}(V) \rtimes H \to \Bbbk$ by $\phi(fc, vh) = \langle f, v\rangle\xi(c, h)$.

Lemma A.27. *ϕ is a Hopf pairing.*

Proof. We have

$$\begin{aligned}
&\phi((fc)_{(1)}, vh)\phi((fc)_{(2)}, v'h') = \langle \underline{f}_{(1)}, v\rangle\langle \underline{f}_{(2)}^{(0)}, v'\rangle\xi(\underline{f}_{(2)}^{(-1)}c_{(1)}, h)\xi(c_{(2)}, h')\\
&\quad= \langle \underline{f}_{(1)}, v\rangle\langle \underline{f}_{(2)}^{(0)}, v'\rangle\xi(\underline{f}_{(2)}^{(-1)}, h_{(1)})\xi(c_{(1)}, h_{(2)})\xi(c_{(2)}, h') = \langle \underline{f}_{(1)}, v\rangle\langle \underline{f}_{(2)} \triangleleft h_{(1)}, v'\rangle\xi(c, h_{(2)}h')\\
&\quad= \langle \underline{f}_{(1)}, v\rangle\langle \underline{f}_{(2)}, h_{(1)} \triangleright v'\rangle\xi(c, h_{(2)}h') = \langle f, v(h_{(1)} \triangleright v')\rangle\xi(c, h_{(2)}h')\\
&\quad= \phi(fc, v(h_{(1)} \triangleright v')h_{(2)}h') = \phi(fc, (vh)\cdot(v'h')),
\end{aligned}$$

where we used (A.26). Similarly,

$$\begin{aligned}
&\phi(fc, (vh)_{(1)})\phi(f'c', (vh)_{(2)}) = \langle f, \underline{v}_{(1)}\rangle\langle f', \underline{v}_{(2)}^{(0)}\rangle\xi(c, \underline{v}_{(2)}^{(-1)}h_{(1)})\xi(c', h_{(2)})\\
&\quad= \langle f, \underline{v}_{(1)}\rangle\langle f', \underline{v}_{(2)}^{(0)}\rangle\xi(c_{(1)}, \underline{v}_{(2)}^{(-1)})\xi(c_{(2)}, h_{(1)})\xi(c', h_{(2)}) = \langle f, \underline{v}_{(1)}\rangle\langle f', \underline{v}_{(2)} \triangleleft c_{(1)}\rangle\xi(c_{(2)}c', h)\\
&\quad= \langle f, \underline{v}_{(1)}\rangle\langle c_{(1)} \triangleright f', \underline{v}_{(2)}\rangle\xi(c_{(2)}c', h) = \langle f(c_{(1)} \triangleright f'), v\rangle\xi(c_{(2)}c', h)\\
&\quad= \phi(f(c_{(1)} \triangleright f')c_{(2)}c', vh) = \phi((fc)\cdot(f'c'), vh).
\end{aligned}$$

Clearly, $\phi(fc,1)=\varepsilon(f)\varepsilon_C(c)$ while $\phi(1,vh)=\varepsilon(v)\varepsilon_H(h)$. Finally, we have

$$\begin{aligned}\phi(S(fc),vh)&=\phi(S_C(f^{(-1)}c)\underline{S}(f^{(0)}),vh)=\phi(S_C(f^{(-1)}c_{(2)})\triangleright\underline{S}(f^{(0)})S_C(f^{(-2)}c_{(1)}),vh)\\&=\langle S_C(f^{(-1)}c_{(2)})\triangleright\underline{S}(f^{(0)}),v\rangle\xi(S_C(f^{(-2)}c_{(1)}),h)\\&=\langle\underline{S}(f^{(0)}),v^{(0)}\rangle\xi(S_C(f^{(-1)}c_{(2)}),v^{(-1)})\xi(S_C(f^{(-2)}c_{(1)}),h)\\&=\langle f^{(0)},\underline{S}(v^{(0)})\rangle\xi(f^{(-1)}c_{(2)},S_H(v^{(-1)}))\xi(f^{(-2)}c_{(1)},S_H(h))\\&=\langle f^{(0)},\underline{S}(v^{(0)})\rangle\xi(f^{(-1)}c,S_H(h)S_H(v^{(-1)}))=\langle f^{(0)},\underline{S}(v^{(0)})\rangle\xi(f^{(-1)}c,S_H(v^{(-1)}h))\\&=\langle f^{(0)},\underline{S}(v^{(0)})\rangle\xi(f^{(-1)},S_H(v^{(-1)}h_{(2)}))\xi(c,S_H(v^{(-2)}h_{(1)}))\\&=\langle f,S_H(v^{(-1)}h_{(2)})\triangleright\underline{S}(v^{(0)})\rangle\xi(c,S_H(v^{(-2)}h_{(1)}))\\&=\phi(fc,S_H(v^{(-1)}h_{(2)})\triangleright\underline{S}(v^{(0)})S_H(v^{(-2)}h_{(1)}))\\&=\phi(fc,S_H(v^{(-1)}h)\underline{S}(v^{(0)}))=\phi(fc,S(vh)).\end{aligned}$$

□

Theorem A.28. *The algebra $\mathcal{U}_\xi(V^\star,C,V,H)$ is isomorphic to $\mathcal{D}_{\phi_+,\phi_-}(\mathcal{B}(V^\star)\rtimes C,\mathcal{B}(V)\rtimes H)$ where $\phi_\pm(fc,vh)=\langle f,v\rangle_\pm\xi(c,h)$. In particular, for all $v\in\mathcal{B}(V)$, $f\in\mathcal{B}(V^\star)$ we have in $\mathcal{U}_\xi(V^\star,C,V,H)$*

$$v\cdot f=\underline{f}^{(0)}_{(2)}\underline{f}^{(-1)}_{(3)}\cdot\underline{v}_{(2)}\underline{v}^{(-2)}_{(3)}\xi(\underline{f}^{(-1)}_{(2)}\underline{f}^{(-2)}_{(3)},S_H^{-1}(\underline{v}^{(-1)}_{(3)}))\langle\underline{f}_{(1)},\underline{S}^{-1}(\underline{v}^{(0)}_{(3)})\rangle_-\langle\underline{f}^{(0)}_{(3)},\underline{v}_{(1)}\rangle_+ \tag{A.29}$$

Moreover, if $\langle\cdot,\cdot\rangle_+=\langle\cdot,\cdot\rangle_-$ then $\mathcal{U}_\xi(V^\star,C,V,H)$ is a Hopf algebra with the comultiplication defined by $\Delta(f)=f\otimes1+f^{(-1)}\otimes f^{(0)}$, $\Delta(v)=1\otimes v+v^{(0)}\otimes v^{(-1)}$, $\Delta(c)=\Delta_C(c)$, $\Delta(h)=\Delta_H^{op}(h)$, $v\in V$, $f\in V^\star$, $c\in C$, $h\in H$.

Proof. Let $\mathcal{D}=\mathcal{D}_{\phi_+,\phi_-}(\mathcal{B}(V^\star)\rtimes C,\mathcal{B}(V)\rtimes H)$. Clearly, the subalgebra of $\mathcal{U}:=\mathcal{U}_\xi(V^\star,C,V,H)$ generated by $V^\star$ and C identifies with $\mathcal{B}(V^\star)\rtimes C$. Likewise, the subalgebra generated by V and H identifies with $\mathcal{B}(V)\rtimes H$ since

$$(h_{(1)}\triangleright v)\cdot h_{(2)}=h_{(3)}\cdot((h_{(1)}\triangleright v)\triangleleft h_{(2)})=h_{(3)}\cdot(S_H^{-1}(h_{(2)})h_{(1)}\triangleright v)=h\cdot v,\qquad h\in H,\,v\in V.$$

Furthermore, in $\mathcal{D}$ we have, for all $v\in V$, $f\in V^\star$, $c\in C$ and $h\in H$

$$\begin{aligned}v\cdot c&=c_{(2)}v_{(2)}\phi_-(c_{(1)},S^{-1}(v_{(3)}))\phi_+(c_{(3)},v_{(1)})=c_{(1)}v^{(0)}\phi_+(c_{(2)},v^{(-1)})\\&=c_{(1)}v^{(0)}\xi(c_{(2)},v^{(-1)})=c_{(1)}(v\triangleleft c_{(2)})\end{aligned}$$

while

$$h\cdot f=f_{(2)}h_{(2)}\phi_-(f_{(1)},S_H^{-1}(h_{(3)}))\phi_+(f_{(3)},h_{(1)})=f^{(0)}h_{(1)}\phi_-(f^{(-1)},S_H^{-1}(h_{(2)}))=(h_{(2)}\triangleright f)\cdot h_{(1)}$$

and

$$\begin{aligned}v\cdot f&=f_{(2)}v_{(2)}\phi_-(f_{(1)},S^{-1}(v_{(3)}))\phi_+(f_{(3)},v_{(1)})=v^{(-1)}\phi_-(f,S^{-1}(v^{(0)}))+f\cdot v+f^{(-1)}\phi_+(f^{(0)},v)\\&=f\cdot v+f^{(-1)}\langle f^{(0)},v\rangle_+-v^{(-1)}\langle f,v^{(0)}\rangle_-.\end{aligned}$$

Thus, all relations between generators of $\mathcal{D}$ hold in $\mathcal{U}$, hence we have a homomorphism of algebras $\mathcal{D}\to\mathcal{U}$, which is clearly an isomorphism of vector spaces.

It remains to prove (A.29). Observe that Lemma A.19 implies that

$$(\Delta\otimes1)\Delta(v)=\Delta(\underline{v}_{(1)}\underline{v}^{(-1)}_{(2)})\otimes\underline{v}^{(0)}_{(2)}=\underline{v}_{(1)}\underline{v}^{(-1)}_{(2)}\underline{v}^{(-2)}_{(3)}\otimes\underline{v}^{(0)}_{(2)}\underline{v}^{(-1)}_{(3)}\otimes\underline{v}^{(0)}_{(3)}=v_{(1)}\otimes v_{(2)}\otimes v_{(3)}$$

and similarly for $(\Delta\otimes1)\Delta(f)$. Then by (A.14) and Lemma A.20 we have

$$\begin{aligned}v\cdot f&=f_{(2)}v_{(2)}\phi_-(f_{(1)},S^{-1}(v_{(3)}))\phi_+(f_{(3)},v_{(1)})\\&=\underline{f}^{(0)}_{(2)}\underline{f}^{(-1)}_{(3)}\underline{v}^{(0)}_{(2)}\underline{v}^{(-1)}_{(3)}\phi_-(\underline{f}_{(1)}\underline{f}^{(-1)}_{(2)}\underline{f}^{(-2)}_{(3)},S^{-1}(\underline{v}^{(0)}_{(3)}))\phi_+(\underline{f}^{(0)}_{(3)},\underline{v}_{(1)}\underline{v}^{(-1)}_{(2)}\underline{v}^{(-2)}_{(3)})\\&=\underline{f}^{(0)}_{(2)}\underline{f}^{(-1)}_{(3)}\underline{v}_{(2)}\underline{v}^{(-1)}_{(3)}\phi_-(\underline{f}_{(1)}\underline{f}^{(-1)}_{(2)}\underline{f}^{(-2)}_{(3)},S^{-1}(\underline{v}^{(0)}_{(3)}))\langle\underline{f}^{(0)}_{(3)},\underline{v}_{(1)}\rangle_+\end{aligned}$$

$$= \underline{f}_{(2)}^{(0)}\underline{f}_{(3)}^{(-1)}\underline{v}_{(2)}\underline{v}_{(3)}^{(-2)}\phi_-(\underline{f}_{(1)}\underline{f}_{(2)}^{(-1)}\underline{f}_{(3)}^{(-2)},\underline{S}^{-1}(\underline{v}_{(3)}^{(0)})S_H^{-1}(\underline{v}_{(3)}^{(-1)}))\langle\underline{f}_{(3)}^{(0)},\underline{v}_{(1)}\rangle_+$$
$$= \underline{f}_{(2)}^{(0)}\underline{f}_{(3)}^{(-1)}\cdot\underline{v}_{(2)}\underline{v}_{(3)}^{(-2)}\xi(\underline{f}_{(2)}^{(-1)}\underline{f}_{(3)}^{(-2)},S_H^{-1}(\underline{v}_{(3)}^{(-1)}))\langle\underline{f}_{(1)},\underline{S}^{-1}(\underline{v}_{(3)}^{(0)})\rangle_-\langle\underline{f}_{(3)}^{(0)},\underline{v}_{(1)}\rangle_+. \qquad \square$$

The identity (A.29) can be also written in the following form

$$v\cdot f = \underline{f}_{(2)}^{(0)}\underline{f}_{(3)}^{(-1)}\underline{v}_{(3)}^{(-2)}(\underline{v}_{(2)}\triangleleft v_{(3)}^{(-3)})\xi(\underline{f}_{(2)}^{(-1)}\underline{f}_{(3)}^{(-2)},S_H^{-1}(\underline{v}_{(3)}^{(-1)}))\langle\underline{f}_{(1)},\underline{S}^{-1}(\underline{v}_{(3)}^{(0)})\rangle_-\langle\underline{f}_{(3)}^{(0)},\underline{v}_{(1)}\rangle_+.$$

Note that if $\langle\cdot,\cdot\rangle_-=0$ on $V^\star\otimes V$, we obtain

$$v\circ_+ f = \underline{f}_{(1)}\underline{f}_{(2)}^{(-1)}\langle\underline{f}_{(2)}^{(0)},\underline{v}_{(1)}\rangle_+\underline{v}_{(2)} = \underline{f}_{(1)}\beta_+(\underline{f}_{(2)},\underline{v}_{(1)})\underline{v}_{(2)}, \qquad v\in\mathcal{B}(V),\ f\in\mathcal{B}(V^\star),$$

where $\beta_+:\mathcal{B}(V^\star)\otimes\mathcal{B}(V)\to C$ is defined by

$$\beta_+(f,v) = f^{(-1)}\langle f^{(0)},v\rangle_+, \qquad v\in\mathcal{B}(V),\ f\in\mathcal{B}(V^\star).$$

We denote the corresponding braided double $\mathcal{B}(V^\star)\rtimes C\ltimes\mathcal{B}(V)$ by $\mathcal{H}_+(V^\star,C,V)$. Similarly, if $\langle\cdot,\cdot\rangle_+=0$ on $V^\star\otimes V$ we have

$$\begin{aligned}
v\circ_- f &= \underline{f}_{(2)}^{(0)}\cdot\underline{v}_{(1)}\underline{v}_{(2)}^{(-2)}\xi(\underline{f}_{(2)}^{(-1)},S_H^{-1}(\underline{v}_{(2)}^{(-1)}))\langle\underline{f}_{(1)},\underline{S}^{-1}(\underline{v}_{(2)}^{(0)})\rangle_-\\
&= (\underline{v}_{(2)}^{(-1)}\triangleright\underline{f}_{(2)})\underline{v}_{(1)}\underline{v}_{(2)}^{(-2)}\langle\underline{S}^{-1}(\underline{f}_{(1)}),\underline{v}_{(2)}^{(0)}\rangle_- = (\beta_-(\underline{S}^{-1}(\underline{f}_{(1)}),\underline{v}_{(2)}^{(0)})\triangleright\underline{f}_{(2)})\underline{v}_{(1)}\underline{v}_{(2)}^{(-1)}\\
&= (\underline{v}_{(2)}^{(-1)}\triangleright\underline{f}_{(2)})\underline{v}_{(2)}^{(-2)}(\underline{v}_{(1)}\triangleleft\underline{v}_{(2)}^{(-3)})\langle\underline{S}^{-1}(\underline{f}_{(1)}),\underline{v}_{(2)}^{(0)}\rangle_-\\
&= \beta_-(\underline{S}^{-1}(\underline{f}_{(1)}),\underline{v}_{(2)}^{(0)})\underline{f}_{(2)}(\underline{v}_{(1)}\triangleleft\underline{v}_{(2)}^{(-1)})
\end{aligned}$$

where $\beta_-:\mathcal{B}(V^\star)\otimes\mathcal{B}(V)\to H$ is defined by

$$\beta_-(f,v) = \langle f,v^{(0)}\rangle_-v^{(-1)}, \qquad v\in\mathcal{B}(V),\ f\in\mathcal{B}(V^\star).$$

The corresponding braided double is denoted $\mathcal{H}_-(V^\star,H,V)$. Clearly, $\mathcal{H}_\pm(V^\star,C,V)$ naturally identify with subspaces of $\mathcal{U}_\xi(V^\star,C,V,H)$.

Consider also some special cases of (A.29). If $f\in V^\star$ we have

$$[v,f] = f^{(-1)}\langle f^{(0)},\underline{v}_{(1)}\rangle_+\underline{v}_{(2)} - \underline{v}_{(1)}\langle f,\underline{v}_{(2)}^{(0)}\rangle_-\underline{v}_{(2)}^{(-1)}, \qquad v\in\mathcal{B}(V). \tag{A.30}$$

Similarly, if $v\in V$ we have

$$\begin{aligned}
[v,f] &= \underline{f}_{(1)}\underline{f}_{(2)}^{(-1)}\langle\underline{f}_{(2)}^{(0)},v\rangle_+ - \underline{f}_{(2)}^{(0)}v^{(-2)}\xi(\underline{f}_{(2)}^{(-1)},S_H^{-1}(v^{(-1)}))\langle\underline{f}_{(1)},v^{(0)}\rangle_-\\
&= \underline{f}_{(1)}\underline{f}_{(2)}^{(-1)}\langle\underline{f}_{(2)}^{(0)},v\rangle_+ - (v^{(-1)}\triangleright\underline{f}_{(2)})v^{(-2)}\langle\underline{f}_{(1)},v^{(0)}\rangle_-\\
&= \underline{f}_{(1)}\langle\underline{f}_{(2)}^{(0)},v\rangle_+\underline{f}_{(2)}^{(-1)} - v^{(-1)}\langle\underline{f}_{(1)},v^{(0)}\rangle_-\underline{f}_{(2)}
\end{aligned}\tag{A.31}$$

Let $\bar{\cdot}$ be a field involution of $\Bbbk$. Suppose that it extends to V, $V^\star$, C and H and that ξ satisfies

$$\overline{\xi(\overline{c},\overline{h})} = \xi(c,S_H^{-1}(h))$$

and $\overline{c\triangleright f} = S_C^{-1}(\overline{c})\triangleright\overline{f}$, $\overline{h\triangleright v} = S_H^{-1}(\overline{h})\triangleright\overline{v}$. Then $\bar{\cdot}$ extends to an anti-linear algebra anti-involution and coalgebra involution of $\mathcal{D}_\xi(C,H)$. Moreover, we have

$$\overline{h\triangleright f} = \overline{\xi(f^{(-1)},S_H^{-1}(h))f^{(0)}} = \xi(\overline{f^{(-1)}},\overline{h})\overline{f}^{(0)} = S_H(\overline{h})\triangleright\overline{f}.$$

Since H^{cop} identifies with a sub-bialgebra of $\mathcal{D}_\xi(C,H)$, it follows that for all $x\in\mathcal{D}_\xi(C,H)$ we have $\overline{x\triangleright f} = S^{-1}(\overline{x})\triangleright\overline{f}$. Assuming that $\Psi^\star$ is self-transposed, it follows from Lemma A.22 that $\bar{\cdot}$ extends to an anti-linear algebra anti-involution of $\mathcal{B}(V^\star)\rtimes\mathcal{D}_\xi(C,H)$. Similarly, $\overline{v\triangleleft x} = \overline{v}\triangleleft S^{-1}(\overline{x})$ for all $x\in\mathcal{D}_\xi(C,H)$ and $v\in V$, whence $\bar{\cdot}$ extends to an anti-linear algebra anti-involution of $\mathcal{D}_\xi(C,H)\ltimes\mathcal{B}(V)$.

Proposition A.29. *Suppose that $\overline{\langle f,v\rangle_\pm} = -\langle\overline{f},\overline{v}\rangle_\pm$, $f\in V^\star$, $v\in V$. Then $\bar{\cdot}$ extends to an anti-linear algebra anti-involution of $\mathcal{U}_\xi(V^\star,C,V,H)$.*

Proof. Define $\bar{\cdot}$ on $\mathcal{U} = \mathcal{U}_\xi(V^\star, C, V, H)$ by $\overline{f \cdot x \cdot v} = \overline{v} \cdot \overline{x} \cdot \overline{f}$, $x \in \mathcal{D}_\xi(C, H)$, $v \in \mathcal{B}(V)$, $f \in \mathcal{B}(V^\star)$. Since the restrictions of $\bar{\cdot}$ to $\mathcal{D}_\xi(C,H) \ltimes \mathcal{B}(V)$ and $\mathcal{B}(V^\star) \rtimes \mathcal{D}_\xi(C,H)$ are well-defined anti-linear algebra anti-involutions, it remains to prove that $\overline{[v,f]} = [\overline{f}, \overline{v}]$ for all $v \in V$, $f \in V^\star$. Indeed,

$$[\bar{f}, \bar{v}] = -\overline{f^{(-1)}}\langle \overline{f^{(0)}}, \overline{v}\rangle_+ + \overline{v^{(-1)}}\langle \overline{f}, \overline{v^{(0)}}\rangle_- = \overline{f^{(-1)}\langle f^{(0)}, v\rangle_+ - v^{(-1)}\langle f, v^{(0)}\rangle_-} = \overline{[v,f]}. \qquad \square$$

A.8. **Diagonal braidings.** We now consider an important special case of the constructions discussed above. Let Γ be an abelian monoid and fix its bicharacter $\chi : \Gamma \times \Gamma \to \Bbbk^\times$. Let $V = \bigoplus_{\alpha\in\Gamma} V_\alpha$ be a Γ-graded vector space over $\Bbbk$. Define a braiding $\Psi : V \otimes V \to V \otimes V$ by $\Psi(v \otimes v') = \chi(\alpha, \alpha')v' \otimes v$, where $v \in V_\alpha$, $v' \in V_{\alpha'}$. Furthermore, let $V^\star = \bigoplus_{\alpha\in\Gamma} V^\star$ be another Γ-graded vector space over $\Bbbk$ and let $\langle\cdot,\cdot\rangle : V^\star \otimes V \to \Bbbk$ be any pairing satisfying $\langle V^\star_\alpha, V_{\alpha'}\rangle = 0$ if $\alpha \neq \alpha' \in \Gamma$. Then $\langle\cdot,\cdot\rangle$ is non-degenerate provided that its restrictions to $V^\star_\alpha \otimes V_\alpha$ are non-degenerate for all $\alpha \in \Gamma$. If $\Psi^\star$ is the adjoint of Ψ with respect to the form $\langle\cdot,\cdot\rangle'$ in the notation of §A.3 then it is easy to see that $\Psi^\star(f \otimes f') = \chi(\alpha', \alpha)f' \otimes f$, $f \in V^\star_\alpha$, $f' \in V^\star_{\alpha'}$. Henceforth we will assume that $\langle\cdot,\cdot\rangle$ is non-degenerate and denote $\Gamma_0 = \{\alpha \in \Gamma : V_\alpha \neq 0\} = \{\alpha \in \Gamma : V^\star_\alpha \neq 0\}$. We will always assume that Γ is generated by Γ_0.

The algebras $T(V)$, $\mathcal{B}(V,\Psi)$, $T(V^\star)$ and $\mathcal{B}(V^\star,\Psi^\star)$ are naturally Γ-graded. By abuse of notation we write $\chi(x,y) = \chi(\deg x, \deg y)$ where x, y are homogeneous elements of $T(V)$, $\mathcal{B}(V,\Psi)$ or $T(V^\star)$, $\mathcal{B}(V^\star,\Psi^\star)$ and $\deg x$ denotes the degree of x with respect to Γ. Note that if $u \in \mathcal{B}(V,\Psi)$ is homogeneous and $\underline{\Delta}(u) = \underline{u}_{(1)} \otimes \underline{u}_{(2)}$ in Sweedler's notation then $\deg u = \deg \underline{u}_{(1)} + \deg \underline{u}_{(2)}$. Furthermore, if $u, v \in T(V)$ are homogeneous then $\Psi(u \otimes v) = \chi(u,v)v \otimes u$ hence $\underline{\Delta}(uv) = \chi(\underline{u}_{(2)}, \underline{v}_{(1)})\underline{u}_{(1)}\underline{v}_{(1)} \otimes \underline{u}_{(2)}\underline{v}_{(2)}$.

Lemma A.30. *For $f \in \mathcal{B}(V^\star,\Psi^\star)$, $v \in \mathcal{B}(V,\Psi)$ homogeneous, $\langle f, v\rangle = 0$ unless $\deg f = \deg v$.*

Proof. This statement clearly holds for $f \in V^\star$, $v \in V$. Let $f \in \mathcal{B}^{r-1}(V^\star,\Psi^\star)$ and $v \in \mathcal{B}^r(V,\Psi)$ be homogeneous. Then for all $\alpha \in \Gamma_0$, $F_\alpha \in V^\star_\alpha$, $\langle F_\alpha f, v\rangle = \langle F_\alpha, \underline{v}_{(1)}\rangle\langle f, \underline{v}_{(2)}\rangle$ which is zero unless $\deg \underline{v}_{(1)} = \alpha$ and $\deg \underline{v}_{(2)} = \deg f$, whence $\deg v = \alpha + \deg f$. Since $\mathcal{B}^r(V^\star,\Psi^\star) \subset \sum_{\alpha\in\Gamma_0} V^\star_\alpha \mathcal{B}^{r-1}(V^\star,\Psi^\star)$, the assertion follows. $\square$

Lemma A.31. *For all $f, g \in \mathcal{B}(V^\star,\Psi^\star)$, $u, v \in \mathcal{B}(V,\Psi)$ homogeneous*

$$\begin{aligned}
\partial_f(uv) &= \chi(\underline{f}_{(1)}, v)\chi(\underline{f}_{(1)}, \underline{f}_{(2)})^{-1}\partial_{\underline{f}_{(1)}}(u)\partial_{\underline{f}_{(2)}}(v)\\
\partial^{op}_f(uv) &= \chi(u, \underline{f}_{(2)})\chi(\underline{f}_{(1)}, \underline{f}_{(2)})^{-1}\partial^{op}_{\underline{f}_{(1)}}(u)\partial^{op}_{\underline{f}_{(2)}}(v),\\
\partial_u(fg) &= \chi(g, \underline{u}_{(1)})\chi(\underline{u}_{(2)}, \underline{u}_{(1)})^{-1}\partial_{\underline{u}_{(1)}}(f)\partial_{\underline{u}_{(2)}}(g)\\
\partial^{op}_u(fg) &= \chi(\underline{u}_{(2)}, f)\chi(\underline{u}_{(2)}, \underline{u}_{(1)})^{-1}\partial^{op}_{\underline{u}_{(1)}}(f)\partial^{op}_{\underline{u}_{(2)}}(g).
\end{aligned} \tag{A.32}$$

In particular, for all $E_\alpha \in V_\alpha$, $F_\alpha \in V^\star_\alpha$

$$\partial_{F_\alpha}(uv) = \chi(\alpha, \deg v)\partial_{F_\alpha}(u)v + u\partial_{F_\alpha}(v), \quad \partial^{op}_{F_\alpha}(uv) = \partial^{op}_{F_\alpha}(u)v + \chi(\deg u, \alpha)u\partial^{op}_{F_\alpha}(v) \tag{A.33}$$

and

$$\partial_{E_\alpha}(fg) = \chi(\deg g, \alpha)\partial_{E_\alpha}(f)g + f\partial_{E_\alpha}(g), \quad \partial^{op}_{E_\alpha}(fg) = \partial^{op}_{E_\alpha}(f)g + \chi(\alpha, \deg f)f\partial^{op}_{E_\alpha}(g). \tag{A.34}$$

Proof. We prove only the first identity; others are proved similarly. We have

$$\begin{aligned}
\partial_f(uv) &= \langle f, \underline{uv}_{(2)}\rangle \underline{uv}_{(1)} = \chi(\underline{u}_{(2)}, \underline{v}_{(1)})\langle f, \underline{u}_{(2)}\underline{v}_{(2)}\rangle \underline{u}_{(1)}\underline{v}_{(1)}\\
&= \chi(\underline{u}_{(2)}, \underline{v}_{(1)})\langle \underline{f}_{(1)}, \underline{u}_{(2)}\rangle\langle \underline{f}_{(2)}, \underline{v}_{(2)}\rangle \underline{u}_{(1)}\underline{v}_{(1)} = \chi(\underline{f}_{(1)}, v)\chi(\underline{f}_{(1)}, \underline{f}_{(2)})^{-1}\partial_{\underline{f}_{(1)}}(u)\partial_{\underline{f}_{(2)}}(v),
\end{aligned}$$

where we used that $\chi(x, \underline{v}_{(1)})\chi(x, \underline{v}_{(2)}) = \chi(x, v)$ for all $x, v \in \mathcal{B}(V,\Psi)$ and Lemma A.30. $\square$

An obvious induction together with (A.7) implies then that

$$\partial_{E_\alpha}(F_\alpha^r) = \partial^{op}_{E_\alpha}(F_\alpha^r) = \langle F_\alpha, E_\alpha\rangle [r]_{\chi_\alpha} F_\alpha^{r-1}, \qquad \langle F_\alpha^r, E_\alpha^r\rangle = (\langle F_\alpha, E_\alpha\rangle)^r [r]_{\chi_\alpha}!, \qquad r \in \mathbb{Z}_{\geq 0}, \quad \text{(A.35)}$$

where we abbreviate $\chi_\alpha := \chi(\alpha, \alpha)$. Note also the following identity (cf. [19, Lemma 1.4.2])

$$\underline{\Delta}(F_\alpha^r) = \sum_{r'+r''=r} \begin{bmatrix} r \\ r' \end{bmatrix}_{\chi_\alpha} F_\alpha^{r'} \otimes F_\alpha^{r''}. \tag{A.36}$$

Clearly, Ψ is self-transposed provided that χ is symmetric, that is $\chi(\gamma, \gamma') = \chi(\gamma', \gamma)$ for all $\gamma, \gamma' \in \Gamma$. In that case, if the V_α are finite dimensional for all $\alpha \in \Gamma_0$, $\mathcal{B}(V^\star, \Psi^\star)$ is isomorphic to $\mathcal{B}(V, \Psi)$ as a braided bialgebra.

If χ is symmetric, let $v = v_1 \cdots v_r \in \mathcal{B}^r(V, \Psi)$ where $v_i \in V_{\alpha_i}$ and so $\alpha_i \in \Gamma_0$. The definition of the braided antipode (cf. §A.1) immediately implies that

$$\underline{S}(v) = \underline{S}_\Psi(v) = (-1)^r \Big(\prod_{1 \leq i < j \leq r} \chi(\alpha_i, \alpha_j) \Big) v^*.$$

If $\alpha_1 + \cdots + \alpha_r = \alpha'_1 + \cdots + \alpha'_s$ with $\alpha_i, \alpha'_j \in \Gamma_0$, $1 \leq i \leq r$, $1 \leq j \leq s$ implies that $r = s \pmod 2$ and $\prod_{1 \leq i<j \leq r} \chi(\alpha_i, \alpha_j) = \prod_{1 \leq i < j \leq s} \chi(\alpha'_i, \alpha'_j)$ (which is manifestly the case if Γ is freely generated by Γ_0) we can define a unique character $\operatorname{sgn} : \Gamma \to \{\pm 1\}$ with $\operatorname{sgn}(\alpha) = -1$, $\alpha \in \Gamma_0 \cup \{0\}$ and a function $\gamma : \Gamma \to \Bbbk^\times$ satisfying $\gamma(\alpha) = 1$, $\alpha \in \Gamma_0 \cup \{0\}$ and

$$\chi(\alpha, \alpha') = \chi_\gamma(\alpha, \alpha') := \frac{\gamma(\alpha + \alpha')}{\gamma(\alpha)\gamma(\alpha')}, \qquad \alpha, \alpha' \in \Gamma.$$

Then for any $v \in \mathcal{B}(V)$ homogeneous

$$\underline{S}(v) = \operatorname{sgn}(v)\gamma(v)v^*, \tag{A.37}$$

where we abbreviate $\operatorname{sgn}(v) := \operatorname{sgn}(\deg v)$ and $\gamma(v) := \gamma(\deg v)$. We will say that Γ affords a sign character if there exists a character $\operatorname{sgn} : \Gamma \to \{\pm 1\}$ satisfying $\operatorname{sgn}(\alpha) = -1$, $\alpha \in \Gamma_0$.

Suppose that $\bar{\cdot} : V \to V$ preserves the Γ-grading. Then the braiding Ψ is unitary if and only if $\overline{\chi(\alpha, \alpha')} = \chi(\alpha', \alpha)^{-1}$. The following is an immediate consequence of (A.37) and Proposition A.7(c,d).

Proposition A.32. *Suppose that Γ affords a sign character, $\chi = \chi_\gamma$ with $\gamma(\alpha) = 1$, $\alpha \in \Gamma_0 \cup \{0\}$ and $\overline{\chi(\alpha, \alpha')} = \chi(\alpha', \alpha)^{-1}$ for all $\alpha, \alpha' \in \Gamma_0$. Assume that the pairing $\langle \cdot, \cdot \rangle : V^\star \otimes V \to \Bbbk$ satisfies $\langle \overline{F_\alpha}, \overline{E_\alpha} \rangle = -\langle F_\alpha, E_\alpha \rangle$, $\alpha \in \Gamma_0$, $E_\alpha \in V_\alpha$, $F_\alpha \in V_\alpha^\star$. Then for all $f \in T(V^\star)$, $u \in T(V)$ or $f \in \mathcal{B}(V^\star, \Psi^\star)$, $u \in \mathcal{B}(V, \Psi)$ homogeneous we have*

$$\overline{\langle \overline{f}, \overline{u^*} \rangle} = \operatorname{sgn}(u)\gamma(u)^{-1}\langle f, u \rangle.$$

Suppose that $\gamma(\alpha)$ is a square in $\Bbbk$ for all $\alpha \in \Gamma$ and fix $\gamma^{\frac{1}{2}} : \Gamma \to \Bbbk^\times$. Set $\chi^{\frac{1}{2}} = \chi_{\gamma^{\frac{1}{2}}}$. The operator $L_n : V^{\otimes n} \to V^{\otimes n}$ defined on $u \in V^{\otimes n}$ homogeneous by $L_n(u) = \gamma(u)^{\frac{1}{2}} u$ clearly satisfies $L_n^2(u) = (-1)^n S_\Psi(u^*)$, commutes with * and is unitary with respect to $\bar{\cdot}$. The following is straightforward corollary of Lemma A.8.

Corollary A.33. *In the assumptions of Proposition A.32, the form $(\cdot, \cdot) : \mathcal{B}(V^\star, \Psi^\star) \otimes \mathcal{B}(V, \Psi) \to \Bbbk$ is defined for $u \in \mathcal{B}(V, \Psi)$ homogeneous and $f \in \mathcal{B}(V^\star, \Psi^\star)$ by $(f, u) = (\gamma^{\frac{1}{2}}(u))^{-1} \langle f, u \rangle$ and satisfies*

$$\overline{(\overline{f}, \overline{u}^*)} = \operatorname{sgn}(u)(f, u)$$

and for all $f, f' \in \mathcal{B}(V^\star, \Psi^\star)$, $u, u' \in \mathcal{B}(V, \Psi)$ homogeneous

$$(ff', u) = (\chi^{\frac{1}{2}}(f, f'))^{-1}(f, \underline{u}_{(1)})(f', \underline{u}_{(2)}), \quad (f, uu') = (\chi^{\frac{1}{2}}(u, u'))^{-1}(\underline{f}_{(1)}, u)(\underline{f}_{(2)}, u').$$

A.9. **Drinfeld double in the diagonal case.** Let $H = \Bbbk[\Gamma \oplus \Gamma] \cong \Bbbk[\Gamma] \otimes \Bbbk[\Gamma]$ be the monoidal bialgebra of $\Gamma \oplus \Gamma$ with a basis $K_{\alpha,\alpha'}$, $\alpha, \alpha' \in \Gamma$. Denote by H^+ (respectively, H^-) the subalgebra of H generated by the $K_{0,\alpha}$ (respectively, $K_{\alpha,0}$), $\alpha \in \Gamma$; clearly, $H^\pm \cong \Bbbk[\Gamma]$. Let $\widehat{H}$ (respectively, $\widehat{H}^\pm$) be localizations of the corresponding algebras at $K_{\alpha,\alpha'}$ (respectively, $K_{0,\alpha}$, $K_{\alpha,0}$), $\alpha, \alpha' \in \Gamma$. Then $\widehat{H}$ identifies with $\mathcal{D}_{\xi_\chi}(\widehat{H}^-, \widehat{H}^+)$ where the Hopf pairing $\xi_\chi : \widehat{H}^- \otimes \widehat{H}^+ \to \Bbbk$ is defined by $\xi_\chi(K_{\alpha,0}, K_{0,\alpha'}) = \chi(\alpha', \alpha)$.

Let V, $V^\star$ be Γ-graded $\Bbbk$-vector spaces as in §A.8. We regard V (respectively, $V^\star$) as left Yetter-Drinfeld $\widehat{H}^+$- (respectively, $\widehat{H}^-$)-module via

$$\begin{aligned} &K_{0,\alpha} \triangleright v = \chi(\alpha, \beta) v, \qquad \delta(v) = K_{0,\beta} \otimes v \\ &K_{\alpha,0} \triangleright f = \chi(\beta, \alpha) f, \qquad \delta(f) = K_{\beta,0} \otimes f, \qquad \alpha, \beta \in \Gamma, \qquad v \in V_\beta,\ f \in V^\star_\beta. \end{aligned}$$

Then by (A.25) we have

$$v \triangleleft K_{\alpha,0} = \chi(\beta, \alpha) v, \qquad f \triangleleft K_{0,\alpha} = \chi(\alpha, \beta) f,$$

and we can regard V (respectively, $V^\star$) as a right (respectively, left) Yetter-Drinfeld module over $\widehat{H}$ as in (A.27), with $v \triangleleft K_{\alpha,\alpha'} = \chi(\alpha', \beta)^{-1} \chi(\beta, \alpha) v$ and $K_{\alpha,\alpha'} \triangleright f = \chi(\beta, \alpha) \chi(\alpha', \beta)^{-1} f$.

Let $\langle \cdot, \cdot \rangle_\pm$ be pairings $V^\star \otimes V \to \Bbbk$ satisfying the assumptions of §A.8. Clearly, (A.26) holds. Denote by $\partial^\pm_f, \partial^{\pm op}_f : \mathcal{B}(V) \to \mathcal{B}(V)$ and $\partial^\pm_v, \partial^{\pm op}_v : \mathcal{B}(V^\star) \to \mathcal{B}(V^\star)$, $v \in \mathcal{B}(V)$, $f \in \mathcal{B}(V^\star)$, the linear operators corresponding to the respective pairings $\langle \cdot, \cdot \rangle_\pm$, as defined in §A.3. Consider now the algebra $\mathcal{U}_\chi(V^\star, V)$ which is the subalgebra of $\widehat{\mathcal{U}}_\chi(V^\star, V) := \mathcal{U}_{\xi_\chi}(V^\star, \widehat{H}^-, V, \widehat{H}^+)$ generated by $V^\star$, V and $H^\pm$. In particular, we have the following cross-relations

$$\begin{aligned} &K_{\alpha,\alpha'} E_\beta = \chi(\beta, \alpha)^{-1} \chi(\alpha', \beta) E_\beta K_{\alpha,\alpha'}, \qquad K_{\alpha,\alpha'} F_\beta = \chi(\beta, \alpha) \chi(\alpha', \beta)^{-1} F_\beta K_{\alpha,\alpha'} \\ &[E_\alpha, F_\beta] = K_{\beta,0} \langle F_\beta, E_\alpha \rangle_- - K_{0,\alpha} \langle F_\beta, E_\alpha \rangle_+, \qquad E_\alpha \in V_\alpha,\ F_\beta \in V^\star_\beta,\ \alpha, \alpha', \beta \in \Gamma. \end{aligned} \tag{A.38}$$

If $\langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle_-$ then, by Theorem A.28, $\widehat{\mathcal{U}}_\chi(V^\star, V)$ is a Hopf algebra with the comultiplication defined by

$$\Delta(F_\alpha) = F_\alpha \otimes 1 + K_{\alpha,0} \otimes F_\alpha, \qquad \Delta(E_\alpha) = 1 \otimes E_\alpha + E_\alpha \otimes K_{0,\alpha}. \tag{A.39}$$

and the antipode

$$S(F_\alpha) = -K_{\alpha,0}^{-1} F_\alpha, \qquad S(E_\alpha) = -E_\alpha K_{0,\alpha}^{-1} \tag{A.40}$$

for all $\alpha \in \Gamma$, $E_\alpha \in V_\alpha$ and $F_\alpha \in V^\star_\alpha$.

Lemma A.34. *For all $\alpha \in \Gamma$, $E_\alpha \in V_\alpha$, $F_\alpha \in V^\star_\alpha$, $v \in \mathcal{B}(V)$, $f \in \mathcal{B}(V^\star)$ we have in $\mathcal{U}_\chi(V^\star, V)$*

$$[v, F_\alpha] = K_{\alpha,0} \partial^{+\,op}_{F_\alpha}(v) - \partial^-_{F_\alpha}(v) K_{0,\alpha}, \qquad [E_\alpha, f] = \partial^+_{E_\alpha}(f) K_{\alpha,0} - K_{0,\alpha} \partial^{-\,op}_{E_\alpha}(f). \tag{A.41}$$

Proof. This is immediate from §A.3, (A.30), (A.31) and the fact that if $\langle f, E_\alpha \rangle_\pm \neq 0$ (respectively, $\langle F_\alpha, v \rangle_\pm \neq 0$) then $\delta(f) = K_{\alpha,0} \otimes f$ (respectively, $\delta(v) = K_{0,\alpha} \otimes v$). □

The following result is an easy consequence of Proposition A.18, Lemma A.19, (A.39) and (A.40).

Proposition A.35. *Let $\langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle_-$. Then $\mathcal{B}(V) \rtimes H^+$ is a left (respectively right) $\widehat{\mathcal{U}}_\chi(V^\star, V)$-module algebra via $F_\alpha \triangleright v = \partial_{F_\alpha}(v) K_{0,\alpha}$, $E_\alpha \triangleright v = E_\alpha v - K_{0,\alpha} v K_{0,\alpha}^{-1} E_\alpha$ (respectively, $v \triangleleft F_\alpha = \partial^{op}_{F_\alpha}(v)$, $v \triangleleft E_\alpha = K_{0,\alpha}^{-1}[v, E_\alpha]$), $v \in \mathcal{B}(V)$, $\alpha \in \Gamma$, $E_\alpha \in V_\alpha$, $F_\alpha \in V^\star_\alpha$. Similarly, $\mathcal{B}(V^\star) \rtimes H^-$ is a left (respectively, right) $\widehat{\mathcal{U}}_\chi(V^\star, V)$-module algebra via $E_\alpha \triangleright f = \partial_{E_\alpha}(f) K_{\alpha,0}$, $F_\alpha \triangleright f = [F_\alpha, f] K_{\alpha,0}^{-1}$ (respectively, $f \triangleleft E_\alpha = \partial^{op}_{E_\alpha}(f)$, $f \triangleleft F_\alpha = f F_\alpha - F_\alpha K_{\alpha,0}^{-1} f K_{\alpha,0}$), $f \in \mathcal{B}(V^\star)$, $\alpha \in \Gamma$, $E_\alpha \in V_\alpha$, $F_\alpha \in V^\star_\alpha$.*

Given $f \in \mathcal{B}(V^\star)$, $v \in \mathcal{B}(V)$ homogeneous, we obtain by (A.29)

$$v \cdot f = \underline{f}_{(2)} K_{\deg \underline{f}_{(3)},0} \cdot \underline{v}_{(2)} K_{0,\deg \underline{v}_{(3)}} \chi(\underline{f}_{(2)}, \underline{v}_{(3)})^{-1} \chi(\underline{f}_{(3)}, \underline{v}_{(3)})^{-1} \langle \underline{f}_{(1)}, \underline{S}^{-1}(\underline{v}_{(3)}) \rangle_- \langle \underline{f}_{(3)}, \underline{v}_{(1)} \rangle_+ \tag{A.42}$$

Since $\langle \underline{f}_{(1)}, \underline{S}^{-1}(\underline{v}_{(3)})\rangle_- = 0$ unless $\deg \underline{v}_{(3)} = \deg \underline{f}_{(1)}$ this can be written in the following form

$$v \cdot f = (\chi(\underline{f}_{(2)}, \underline{f}_{(1)})\chi(\underline{f}_{(2)}, \underline{f}_{(3)})\chi(\underline{f}_{(3)}, \underline{f}_{(1)}))^{-1}\langle \underline{f}_{(1)}, \underline{S}^{-1}(\underline{v}_{(3)})\rangle_- \langle \underline{f}_{(3)}, \underline{v}_{(1)}\rangle_+ \times \\ K_{\deg \underline{f}_{(3)},0}\underline{f}_{(2)}\underline{v}_{(2)}K_{0,\deg \underline{v}_{(3)}}, \tag{A.43}$$

where, as before, we abbreviate $\chi(x, y) := \chi(\deg x, \deg y)$. The following Proposition generalizes [19, Proposition 3.1.7] and is an immediate consequence of (A.43) and (A.37).

Proposition A.36. *Suppose that Γ affords the sign character, $\chi = \chi_\gamma$ with $\gamma : \Gamma \to \Bbbk^\times$ satisfying $\gamma(\alpha) = 1$, $\alpha \in \Gamma_0 \cup \{0\}$. Then for all $f \in \mathcal{B}(V^\star)$, $v \in \mathcal{B}(V)$ homogeneous we have in $\mathcal{U}_\chi(V^\star, V)$*

$$v \cdot f = \operatorname{sgn}(\underline{f}_{(1)})(\chi(\underline{f}_{(2)}, \underline{f}_{(1)})\chi(\underline{f}_{(2)}, \underline{f}_{(3)})\chi(\underline{f}_{(3)}, \underline{f}_{(1)})\gamma(\underline{f}_{(1)}))^{-1} \times \\ \langle \underline{f}_{(1)}, \underline{v}_{(3)}{}^*\rangle_- \langle \underline{f}_{(3)}, \underline{v}_{(1)}\rangle_+ K_{\deg \underline{f}_{(3)},0}\underline{f}_{(2)}\underline{v}_{(2)}K_{0,\deg \underline{v}_{(3)}}.$$

Suppose that $\langle \cdot, \cdot\rangle_- = 0$ (respectively, $\langle \cdot, \cdot\rangle_+ = 0$) on $V^\star \otimes V$. Then we obtain for $v \in \mathcal{B}(V)$, $f \in \mathcal{B}(V^\star)$ homogeneous

$$v \circ_+ f = \underline{f}_{(1)}K_{\deg \underline{f}_{(2)},0}\langle \underline{f}_{(2)}, \underline{v}_{(1)}\rangle_+ \underline{v}_{(2)}, \tag{A.44}$$

$$v \circ_- f = \operatorname{sgn}(\underline{f}_{(1)})(\chi(\underline{f}_{(2)}, \underline{f}_{(1)})\gamma(\underline{f}_{(1)}))^{-1}\langle \underline{f}_{(1)}, \underline{v}_{(2)}{}^*\rangle_- \underline{f}_{(2)}\underline{v}_{(1)}K_{0,\deg \underline{v}_{(2)}}. \tag{A.45}$$

We conclude this section with the following Lemma.

Lemma A.37. *Retain the assumptions of Proposition A.36.*

(a) *If $\overline{\chi(\alpha, \beta)} = \chi(\alpha, \beta)^{-1}$ for all $\alpha, \beta \in \Gamma$ and $\overline{\langle \overline{f}, \overline{v}\rangle_\pm} = -\langle f, v\rangle$, $f \in V^\star$, $v \in V$, then $\mathcal{U}_\chi(V^\star, V)$ admits a unique anti-linear anti-involution extending $\bar{\cdot} : V \to V$, $\bar{\cdot} : V^\star \to V^\star$ and satisfying $\bar{K}_{\alpha,\alpha'} = K_{\alpha,\alpha'}$, $\alpha, \alpha' \in \Gamma$.*

(b) *Suppose that $\langle \cdot, \cdot\rangle_+ = \langle \cdot, \cdot\rangle_-$. Then * extends to an anti-involution of $\mathcal{U}_\chi(V^\star, V)$ whose restrictions to V, $V^\star$ are the identity maps while $K_{\alpha,\alpha'}{}^* = K_{\alpha',\alpha}$, $\alpha, \alpha' \in \Gamma$.*

(c) *Any pair of graded isomorphisms $\xi_- : V^\star \to V$ and $\xi_+ : V \to V^\star$ satisfying $\xi_\pm \circ \xi_\mp = \mathrm{id}_{V^\mp}$ gives raise to an anti-involution ξ of $\mathcal{U}_\chi(V^\star, V)$ satisfying $\xi(K_{\alpha,\alpha'}) = K_{\alpha,\alpha'}$, $\alpha, \alpha' \in \Gamma$. Moreover, if the assumptions of part (a) hold and $\xi_\pm$ commute with $\bar{\cdot}$ then so does ξ.*

Proof. Part (a) is an immediate consequence of Proposition A.29. Part (b) follows from (A.38). To prove (c), note that $\xi_\pm$ define isomorphisms of braided bialgebras $\xi_+ : \mathcal{B}(V) \to \mathcal{B}(V^\star)$ (respectively, $\xi_- : \mathcal{B}(V^\star) \to \mathcal{B}(V)$) such that $\xi_+ \circ \xi_- = \mathrm{id}_{\mathcal{B}(V^\star)}$ and $\xi_- \circ \xi_+ = \mathrm{id}_{\mathcal{B}(V)}$. Define $\xi : \mathcal{U}_\chi(V^\star, V) \to \mathcal{U}_\chi(V^\star, V)$ by $\xi(f) = \xi_-(f^*)$, $f \in \mathcal{B}(V^\star)$, $\xi(v) = \xi_+(v^*)$, $v \in \mathcal{B}(V)$ and $\xi(h) = h$, $h \in H_\pm$. It remains to observe that (A.38) are preserved by ξ. □

For an anti-involution ξ commuting with $\bar{\cdot}$, define a pairing $(\!|\cdot, \cdot|\!) : \mathcal{B}(V^\star) \otimes \mathcal{B}(V^\star) \to \Bbbk$ by

$$(\!|f, g|\!) = (f, \xi(g^*)), \qquad f, g \in \mathcal{B}(V^\star)$$

in the above notation and that of Corollary A.33. In particular, we have for $f \in \mathcal{B}(V^\star)$ homogeneous

$$\overline{(\!|\overline{f}, \overline{g}^*|\!)} = \operatorname{sgn}(f)(\!|f, g|\!). \tag{A.46}$$

Since the braidings Ψ and $\Psi^\star$ are self-transposed in the sense of §A.2, $(\!|\cdot, \cdot|\!)$ is symmetric (note that this form is similar to the one defined in [19, §1.2.3]).

List of notation

$U_q(\tilde{\mathfrak{g}})$	p. 2	^{t}, *	p. 4	$C^{(m)}$	p. 24
$\mathcal{H}_q^{\pm}(\mathfrak{g})$	p. 3	$\widehat{U}_q(\tilde{\mathfrak{g}})$	p. 5	E_{ij}, F_{ij}, $E_{i^s j i^r}$, $F_{i^s j i^r}$	p. 28
$U_q^{\pm}$, $\mathcal{K}_{\pm}$, $\mathbf{K}_{\pm}$	p. 3	T_i	p. 5	$\underline{\Delta}$	p. 45
$\mathbf{B}_{\mathfrak{n}^{\pm}}$	p. 3, 16	$[a]_\nu$, $[a]_\nu!$, $\left[{a \atop b}\right]_\nu$	p. 14	$\mathcal{B}(V,\Psi)$	p. 45
$\bar{\cdot}$	p. 3	$(a)_\nu$, $(a)_\nu!$, $\binom{a}{b}_\nu$	p. 14	∂_v, ∂_v^{op}	p. 48
Γ, α_i, $\widehat{\Gamma}$, $\alpha_{\pm i}$, $\deg_{\widehat{\Gamma}}$	p. 3	$\langle a\rangle_\nu$, $\langle a\rangle_\nu!$	p. 14	$\mathcal{D}_{H,H^{cop}}(C)$	p. 49
$\diamond$	p. 3	χ, η, $\underline{\gamma}$	p. 15	$\mathcal{D}_{\phi_+,\phi_-}(C,H)$	p. 51
$\mathbf{d}_{b_-,b_+}$	p. 3	${}_{\mathbb{Z}}U^{\pm}$, $\overline{U}_{\mathbb{Z}}^{\pm}$	p. 15	$\mathcal{U}_\xi(V^\star,C,V,H)$	p. 55
$b_-\circ b_+$	p. 4	$\mathbf{B}^{\mathrm{can}}$	p. 16	$K_{\alpha,\alpha'}$	p. 60
$\mathbf{B}_{\mathfrak{g}}^+$	p. 4	$(\!\vert\cdot,\cdot\vert\!)$	p. 16, 61	sgn	p. 59
$b_-\bullet b_+$	p. 4	∂_i, ∂_i^{op}	p. 20	$\mathcal{U}_\chi(V^\star,V)$, $\widehat{\mathcal{U}}_\chi(V^\star,V)$	p. 60
$\mathbf{B}_{\tilde{\mathfrak{g}}}$	p. 4	ℓ_i, $\partial_i^{(top)}$	p. 20		

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Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
E-mail address: `arkadiy@math.uoregon.edu`

Department of Mathematics, University of California, Riverside, CA 92521.
E-mail address: `jacob.greenstein@ucr.edu`