An extension of the $\tau$-value to games with coalition structures

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Abstract: We introduce the coalitional $\tau$-value, which is an extension of the $\tau$-value for TU-games to games with a coalition structure. We identify a class of TU-games that satisfy the property that for every game in this class and every coalition structure on its player set it holds that the coalitional $\tau$-value can be defined for the corresponding game with a coalition structure. We study properties of the coalitional $\tau$-value and provide an axiomatic characterization of this allocation rule. We use the coalitional $\tau$-value to study bankruptcy problems and the determination of aircraft landing fees.

Key Words: Game theory, TU-games, $\tau$-value, bankruptcy problems, airport games.

1This work was partially supported by the Spanish Ministry for Education and Culture (grant PB98-0613-C02-02). Anne van den Nouweland gratefully acknowledges the support of the University of Santiago de Compostela. The authors appreciate the useful comments made by Michael Maschler and by two anonymous referees.
1 Introduction

The traditional model of transferable utility games (TU-games for short) is not suitable to study situations in which some players have organized themselves into subgroups to try and get a better bargaining position vis a vis the other players. To allow for this possibility, Owen (1977) introduced games with a priori unions, which later became also known as games with coalition structures. A game with a priori unions not only specifies the values of all possible coalitions of players, but it also explicitly describes a system of unions according to which the players have organized themselves. In his paper, Owen modified the Shapley value for TU-games to fit the more general framework of games with a priori unions. The allocation rule that he introduced has become known as the Owen value. It is computed in two steps. In the first step, the situation is considered at the level of the unions. The Owen value prescribes an allocation of the total utility to the unions according to the Shapley value of the induced game that is played among the unions. In the second step, the Owen value determines a division of the allocation of a union among its members. This step is carried out in the spirit of the Shapley value and it takes into account the members’ possibilities of joining other unions.

In this paper, we introduce and study an extension of the $\tau$-value, which was introduced in Tijs (1981), to the setting of games with a priori unions. We call this value the coalitional $\tau$-value. Like the Owen value, the coalitional $\tau$-value is computed in two steps, first at the level of the unions and then at the level of individual players. Both steps are carried out in the spirit of the $\tau$-value. The coalitional $\tau$-value, like the Owen value, determines an allocation of the total utility available to the players. The unions are negotiation units that are formed for the purpose of influencing the bargaining power of the players in the grand coalition. This distinguishes our coalitional $\tau$-value from the $\tau$-value for games with coalition structures that was introduced and studied by Driessen and Tijs (1992). In that paper, unions of players act as stand-alone entities that have to distribute their stand-alone values among their members. Hence, there is no bargaining among unions but only within unions.

Similar to the $\tau$-value for TU-games, the coalitional $\tau$-value cannot be defined for every game with a priori unions. However, we identify a class of TU-games that satisfy the property that for every game in this class and
every system of a priori unions of its player set it holds that the coalitional
\( \tau \)-value can be defined for the corresponding game with a priori unions. We
conclude our study of the coalitional \( \tau \)-value in general by considering several
of its properties and providing an axiomatic characterization of this allocation
rule.

We then proceed by applying the coalitional \( \tau \)-value to study bankruptcy
problems and the determination of aircraft landing fees. Bankruptcy games
were introduced by O’Neill (1982). Curiel et al (1987) used the \( \tau \)-value
in their study of bankruptcy games. They showed that the \( \tau \)-value of a
bankruptcy game has a natural equivalent in the bankruptcy problem itself
in the form of the so-called adjusted proportional rule (\( AP \)-rule for short).
None of the authors who have studied bankruptcy problems, however, have
considered the possibility that creditors may consolidate their claims to try
and strengthen their position when the remaining assets of the debtor are
divided. Using the coalitional \( \tau \)-value allows us to take this possibility into
account. We show that the coalitional \( \tau \)-value of a bankruptcy game with a
priori unions has a natural equivalent in the bankruptcy problem itself in the
form of an extension of the adjusted proportional rule. We also provide an
axiomatic characterization of the coalitional \( \tau \)-value for bankruptcy games
with a priori unions that exploits the specific aspects of such games.

Airport games describe the problem of the determination of aircraft land-
ing fees at an airport. These games have been studied by several authors, in-
cluding Littlechild and Owen (1973, 1976), Littlechild and Thompson (1977),
and Dubey (1982). More recently, Tijs and Driessen (1986) and Driessen
(1988) used the \( \tau \)-value in the context of airport games to study the determi-
nation of aircraft landing fees. In these papers, it is not taken into account
that airplanes are part of airlines and therefore organized into a priori unions
in a natural way. To be able to take this into account, Vázquez-Brage et al
(1997) used the model of TU-games with a priori unions and the Owen value
to analyze the determination of aircraft landing fees. In the current paper,
we use the coalitional \( \tau \)-value to study the determination of aircraft landing
fees.

The paper is organized as follows. In section 2 we provide the definition of
the \( \tau \)-value and we introduce the coalitional \( \tau \)-value. In this section, we also
show that the class of games with a priori unions for which the coalitional
\( \tau \)-value is defined is reasonably large. In section 3 we study properties of
the coalitional \( \tau \)-value and we provide an axiomatic characterization for it. In
section 4 we use the coalitional $\tau$-value for bankruptcy problems with unions and relate it to the AP-rule. In this section, we also provide an axiomatic characterization of the coalitional $\tau$-value for bankruptcy games with unions that exploits the specific aspects of such games. In section 5 we apply the coalitional $\tau$-value to airport games with unions (airlines). We illustrate our results in an example that uses data of Labacolla airport in Santiago de Compostela, Spain, for the first three months of 1993.

2 The coalitional $\tau$-value

In this section we introduce and discuss an extension of the $\tau$-value that allows us to take into account the possibility that some players -because of personal or political affinities- may be more likely to act together than others. For inspiration we look at the seminal paper by Owen (1977), who extended the Shapley value to such a setting. Owen modelled the affinities between players by introducing a partition of the player set into a priori unions. His extension of the Shapley value, the Owen value, is based on the idea that if there are a priori unions, then negotiations over payoffs are conducted in two stages. In the first stage, negotiations take place between the unions to determine the share of each of the unions. In the second stage, negotiations are conducted between the players in each union to establish a division of the union’s share among its players. While doing so, the possibilities of the different members of the union have to be taken into account. These possibilities include their defection from the union to join one or more of the other unions. We propose a similar method to define an extension of the $\tau$-value to games with a priori unions, which we name the coalitional $\tau$-value.

We start with some preliminary notations, definitions and results. A cooperative game with transferable utility (a TU-game) is a pair $(N, v)$, where $N = \{1, 2, ..., n\}$ is the set of players, and $v : 2^N \rightarrow \mathbb{R}$ is the characteristic function that assigns to each coalition $S \in 2^N$ its value $v(S)$, with $v(\emptyset) = 0$. It is customary to identify a TU-game $(N, v)$ with its characteristic function $v$. We denote by $G(N)$ the set of all TU-games with player set $N$.

A solution concept is a method to determine for each game $(N, v)$ one or more allocations of the value that is obtainable if all players cooperate, $v(N)$, among the players. Many solution concepts have been proposed to allocate $v(N)$ among the players. One of the earliest and perhaps most compelling
ones is the core. For a TU-game \( v \in G(N) \), its core \( C(v) \) is defined by

\[
C(v) := \{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N) \quad \text{and} \quad \sum_{i \in S} x_i \geq v(S), \quad \text{for all} \ S \subset N \}.
\]

Hence, the core of a game consists of all allocations of \( v(N) \) among the players that are stable against deviations by coalitions \( S \subset N \), because for each coalition it holds that its players together already get at least as much as they can guarantee themselves by splitting off. While the core is certainly very compelling as a solution concept, it also has drawbacks. One of these is that it might be empty, in which case it generates no acceptable allocations, and another one is that it might contain many allocations, so that it does not have much of a bite.

In the current paper we concentrate on the \( \tau \)-value, which was introduced in Tijs (1981). While the \( \tau \)-value does also not generate an allocation for all TU-games, it does so for all games that have a non-empty core. Moreover, when the \( \tau \)-value generates an allocation, then it is unique. The \( \tau \)-value is based on the idea of a compromise between an upper and a lower value for each player in the game. Let \( v \in G(N) \) be a TU-game. The vector \( M(v) \in \mathbb{R}^N \) with coordinates

\[
M_i(v) := v(N) - v(N \setminus \{i\})
\]

is called the upper vector of \( v \). \( M_i(v) \) can be regarded as the maximal payoff that player \( i \) can expect to get in the game in the sense that if he claims more, then it is advantageous for the other players to exclude him from the grand coalition \( N \) and divide the value \( v(N \setminus \{i\}) \) among themselves. \( M_i(v) \) is also called the utopia payoff for player \( i \). The vector \( m(v) \in \mathbb{R}^N \) with coordinates

\[
m_i(v) := \max_{S \ni i} \left( v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \right)
\]

is called the lower vector of \( v \). For each player \( i \in N \), the value \( m_i(v) \) can be regarded as his minimal right in the sense that he can guarantee himself this payoff by offering the members of a suitable coalition their utopia payoff, which is a good deal to them, and taking the remainder for himself. It makes sense to consider a compromise between the lower and the upper vectors if the following two conditions are satisfied.
• \( m(v) \leq M(v) \)

• \( \sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v) \)

A game \( v \in G(N) \) that satisfies these two conditions is said to be \textit{quasi balanced}. We denote the class of quasi balanced games with player set \( N \) by \( QBG(N) \). For a quasi balanced game \( v \in QBG(N) \), the \( \tau \)-value of \( v \), denoted by \( \tau(v) \), is the unique compromise between the upper and lower vectors of the game that establishes an allocation of the value \( v(N) \). Thus,

\[
\tau(v) := m(v) + \alpha(M(v) - m(v)),
\]

where \( \alpha \in [0,1] \) is such that \( \sum_{i \in N} \tau_i(v) = v(N) \).

Note that the \( \tau \)-value is defined only for quasi balanced games. The class of quasi balanced games contains all games that have a non-empty core. In fact, Tijs and Lipperts (1982) prove that if \( v \in G(N) \) has a non-empty core, then \( m(v) \leq x \leq M(v) \) for every \( x \in C(v) \). Another property of the \( \tau \)-value is that it is \textit{individually rational}, i.e., \( \tau_i(v) \geq v(i) \) for all \( v \in QBG(N) \) and all \( i \in N \) (see, for instance, Driessen (1988)). For more information on the \( \tau \)-value we refer the reader to Tijs and Otten (1993).

We now define games with a priori unions and generalize the definition of the \( \tau \)-value to such games.

A game with a priori unions is a triple \((N,v,P)\), where \((N,v)\) is a TU-game and \( P = \{P_1, P_2, \ldots, P_m\} \) is a partition of the player set \( N \). The partition \( P \) is referred to as a system of a priori unions or a coalition structure. We denote the set of all games with a priori unions and player set \( N \) by \( U(N) \) and we usually suppress the player set \( N \), so that we identify a game with a priori unions with its characteristic function and its system of unions. Following Owen (1977), we interpreted a game with a priori unions as a negotiation situation where there is a structure of a priori unions (the partition) which conditions the negotiations. In a first round, negotiations take place between the unions, and in a second round they take place within the unions.

The negotiations between the unions are described by the so-called quotient game, whose players are the unions and in which the worth of a collection of unions is the worth obtainable by their joint members. We identify a union \( P_k \) with its number \( k \) and denote the set of unions by \( M := \{1,2,\ldots,m\} \). Then, formally, the \textit{quotient game} \( v^P \in G(M) \) is defined by
\[ v^P(L) = v(\bigcup_{k \in L} P_k) \] for all \( L \subset M \). If the quotient game is quasi balanced, then we can use the \( \tau \)-value to find an allocation of the joint worth \( v(N) \) among the unions. Hence, union \( k \) will get \( \tau_k(v^P) \) to divide among its members.

An allocation of \( \tau_k(v^P) \) among the members of union \( P_k \) is in the spirit of the \( \tau \)-value. Hence, it is a compromise between an upper and a lower vector. In the determination of these vectors, we take into account the possibilities of the players to defect from their unions and join (groups of) other unions. The utopia payoff for a player \( i \in N \) is defined as

\[
M_i(v, P) := v(N) - v(N \setminus i).
\]

Player \( i \) cannot hope to get more, because then it would be beneficial for the other players to kick him out of the grand coalition altogether. Note that \( M_i(v, P) = M_i(v) \). The utopia payoff \( M_i(v, P) \) can also be interpreted as player \( i \)'s marginal contribution to the marginal contribution of his union to the worth of the grand coalition \( N \). To see this, suppose that \( i \in P_k \) and define the game \((M, v_{-i}^P)\) by \( v_{-i}^P(L) = v(\bigcup_{l \in L} P_l \setminus \{i\}) \) for all \( L \subseteq M \). It holds that

\[
M_i(v, P) = v^P(M) - v^P(M \setminus \{k\}) - (v_{-i}^P(M) - v_{-i}^P(M \setminus \{k\})).
\]

In defining the lower vector, our supposition is that a player can defect from his union and form a coalition with some of his fellow union members and with one or more of the remaining unions. He cannot, however, form a new coalition that includes some but not all players of another union. In this assumption we follow Owen (1977), which states “we will not, however, consider the possibility of a coalition with proper subsets of some of the other unions as that seems to represent, in some sense, a double order of difficulty”. Hence, to determine the minimal right of a player \( i \in P_k \), only coalitions in

\[
P(k) := \{S \subset N \mid S = \bigcup_{l \in L} P_l \cup T \text{ for some } L \subset M \setminus \{k\} \text{ and } T \subset P_k\}.
\]

are considered. Using this notation, the minimal right of a player \( i \in P_k \) is defined by

\[
m_i(v, P) := \max_{S \in P(k) : i \in S} \left( v(S) - \sum_{j \in S\setminus\{i\}} M_j(v, P) \right).
\]

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Player $i$ can guarantee himself the payoff $m_i(v, P)$ by offering the members of a suitable coalition their utopia payoffs and taking the remainder for himself. Note that the definitions of both minimal rights vectors readily imply that $m(v, P) \leq m(v)$.

A game with a priori unions $(v, P) \in U(N)$ is said to be quasi balanced if and only if the following three conditions are satisfied.

- $v^P \in QBG(M)$
- $m(v, P) \leq M(v, P)$
- $\sum_{i \in P_k} m_i(v, P) \leq \tau_k(v^P) \leq \sum_{i \in P_k} M_i(v, P)$ for all $P_k \in P$.

We denote the set of quasi balanced games with a priori unions and a fixed player set $N$ by $QBU(N)$. The coalitional $\tau$-value is defined for quasi balanced games as a compromise between the upper and lower vectors that for each union $P_k$ divides its $\tau$-value $\tau_k(v^P)$ among its member players.

**Definition 1** The coalitional $\tau$-value is a map $\tau : QBU(N) \to \mathbb{R}^N$ which assigns to every $(v, P) \in QBU(N)$ the vector $(\tau_i(v, P))_{i \in N}$ such that, for all $P_k \in P$ and all $i \in P_k$,

$$\tau_i(v, P) := m_i(v, P) + \alpha_k(M_i(v, P) - m_i(v, P)),$$

where, for each $k \in M$, $\alpha_k$ is such that $\sum_{i \in P_k} \tau_i(v, P) = \tau_k(v^P)$.

We illustrate Definition 1 with an example.

**Example 1** Let $N = \{1, 2, 3\}$, $P = \{\{1, 2\}, \{3\}\}$ and $v \in G(N)$ defined by $v(1) = v(2) = v(3) = 0$, $v(1, 2) = v(1, 3) = 1$, and $v(2, 3) = v(N) = 10$. We denote $P_1 = \{1, 2\}$ and $P_2 = \{3\}$ and we compute $v^P(P_1) = v(1, 2) = 1$, $v^P(P_2) = v(3) = 0$, and $v^P(P_1, P_2) = v(N) = 10$. Further, $M_{P_1}(v^P) = v^P(P_1, P_2) - v^P(P_2) = 10$ and $M_{P_2}(v^P) = v^P(P_1, P_2) - v^P(P_1) = 9$. Computation of the minimal rights is more involved. $m_{P_1}(v^P) = \max\{v^P(P_1), v^P(P_1, P_2) - M_{P_2}(v^P)\} = \max\{1, 1\} = 1$ and $m_{P_2}(v^P) = \max\{v^P(P_2), v^P(P_1, P_2) - M_{P_1}(v^P)\} = \max\{0, 0\} = 0$. This shows that $v^P$ is

\[\text{For clarity of exposition, we sometimes omit the brackets and write } v(1) \text{ instead of } v(\{1\}), v(2, 3) \text{ instead of } v(\{2, 3\}), \text{ and so on.} \]
quasi balanced and we derive that \( \tau(v^P) = m(v^P) + \alpha (M(v^P) - m(v^P)) = (1, 0) + \frac{1}{2}((10, 9) - (1, 0)) = (5.5, 4.5) \).

We now proceed to consider the game from the level of the players rather than the unions. We compute \( M(N, v, P) = (0, 9, 9) \) and \( m(N, v, P) = (0, 1, 1) \). Together with the results that we obtained before, this leads to
\[
\tau(v, P) = m(v, P) + \alpha M(v, P) - m(v, P) = (0, 1) + \frac{4.5}{8}((0, 9) - (0, 1)) = (0, 5.5) \text{ and } \tau_3(v, P) = 4.5. \text{ Hence, } \tau(v, P) = (0, 5.5, 4.5).
\]

We point out that the coalitional \( \tau \)-value is a generalization of the standard \( \tau \)-value, as explained below. There are two trivial systems of unions, \( P^n := \\{\{1\}, \{2\}, \ldots, \{n\}\} \), in which the players in the grand coalition have organized themselves into \( n \) one-person coalitions, and \( P^N := \{N\} \), which consists of the grand coalition only. A game with one of these two systems of unions is equivalent to a game in which there is no system of unions, because with one of these systems of unions the negotiations either within unions or between unions are trivial. Now, let \( v \in QBG(N) \) be a quasi balanced game. We supplement this game with a trivial system of unions, \( P^n \) or \( P^N \). It is easily seen that this does not change the utopia payoffs or the minimal rights of the players. Hence, we have \((v, P^n), (v, P^N) \in QBU(N)\) and, moreover, \( \tau(v, P^n) = \tau(v, P^N) = \tau(v) \).

Now that we have defined the class of quasi balanced games with a priori unions, we turn our attention to showing that we have defined a reasonably large class of games. In theorem 1, we prove that there exists a class of TU-games that satisfy the property that for every game in this class and every partition of its player set it holds that the corresponding game with a priori unions is quasi balanced. The class of games that we study in theorem 1 is the class of exact games. A TU-game \( v \in G(N) \) is exact if for every coalition \( S \subseteq N \) there exists a core element \( x \in C(v) \) such that \( \sum_{i \in S} x_i = v(S) \). It is well known that the class of exact games includes other classes of games that are often studied, such as the class of convex games (see Shapley (1971)), and that the class of exact games is a subclass of the class of quasi balanced games.

**Theorem 1** Let \( v \in G(N) \) be an exact game and \( P \in P(N) \) a partition of the player set \( N \). Then \((v, P) \in QBU(N)\).

**Proof.** It easily follows from exactness of \( v \) that the quotient game \( v^P \) is also exact and, hence, quasi balanced. We also know that \( M(v, P) = M(v) \)
and $m(v, P) \leq m(v)$. To show that $(v, P)$ is quasi balanced, the only thing left to prove is that for all $P_k \in P$

$$\sum_{i \in P_k} m_i(v, P) \leq \tau_k(v^P) \leq \sum_{i \in P_k} M_i(v, P).$$

Let $P_k \in P$ and $i \in P_k$. Let $x \in C(v)$ be such that $x_i = v(i)$ (such an $x$ exists because $v$ is exact) and let $S \subset N$. Then the following chain of (in)equalities holds.

$$v(i) = x_i = \sum_{j \in S} x_j - \sum_{j \in S \setminus i} x_j = \sum_{j \in S} x_j - \sum_{j \in S \setminus i} \left( v(N) - \sum_{k \in N \setminus j} x_k \right) \geq
$$

$$v(S) - \sum_{j \in S \setminus i} \left( v(N) - v(N \setminus j) \right) = v(S) - \sum_{j \in S \setminus i} M_j(v, P).$$

This shows that $m_i(v) \leq v(i)$, i.e., the minimal right of player $i$ is at most the worth that he can obtain alone. Because, obviously, player $i$ can actually guarantee himself the worth $v(i)$ by indeed staying alone, we conclude that $m_i(v) = v(i)$. This readily implies that $m_i(v, P) \leq m_i(v) = v(i)$. Also, because $\{i\} \in P(k)$, it holds that $v(i) \leq m_i(v, P)$. We conclude that $m_i(v, P) = v(i)$.

Note that we have shown that for every exact game it holds that the minimal right of a player is equal to the worth that this player can obtain by staying alone. We apply this to the exact quotient game $v^P \in G(M)$ and obtain $m_k(v^P) = v^P(k) = v(P_k)$ for each union $P_k \in P$. Fix $P_k \in P$. Because $v$ is exact, we can choose a $y \in C(v)$ with the property that $\sum_{i \in P_k} y_i = v(P_k)$. We then derive

$$\sum_{i \in P_k} m_i(v, P) = \sum_{i \in P_k} v(i) \leq \sum_{i \in P_k} y_i = v(P_k) = m_k(v^P).$$

Choosing a $z \in C(v)$ such that $\sum_{i \in N \setminus \{P_k\}} z_i = v(N \setminus \{P_k\})$, we derive

$$M_k(v^P) = v^P(M) - v^P(M \setminus k) = \sum_{i \in N} z_i - \sum_{i \in N \setminus \{P_k\}} z_i = \sum_{i \in P_k} \left( \sum_{j \in N} z_j - \sum_{j \in N \setminus i} z_j \right) \leq \sum_{i \in P_k} \left( v(N) - v(N \setminus i) \right) = \sum_{i \in P_k} M_i(v, P).$$
Using quasi balancedness of $v^P$, we now obtain

$$\sum_{i \in P_k} m_i(v, P) \leq m_k(v^P) \leq \tau_k(v^P) \leq M_k(v^P) \leq \sum_{i \in P_k} M_i(v, P).$$

This concludes the proof. \qed

3 Properties and axiomatic characterization

In this section we take a closer look at the $\tau$-value for games with a priori unions that we defined in the previous section. We start by providing a list of properties that are satisfied by this $\tau$-value. Later on in the section, we will show that a few of these properties can be used to characterize the $\tau$-value for games with a priori unions axiomatically.

Consider the following properties, which are defined for an allocation rule $\gamma$ for games with a priori unions.

**Efficiency**: For all $(v, P) \in U(N)$

$$\sum_{i \in N} \gamma_i(v, P) = v(N).$$

**Individual rationality**: For all $(v, P) \in U(N)$ and all $i \in N$

$$\gamma_i(v, P) \geq v(i).$$

**Null player property**: For all $(v, P) \in U(N)$ and all $i \in N$

if $v(T \cup i) = v(T)$ for all $T \subset N \setminus i$, then $\gamma_i(v, P) = 0$.

**Quotient game property**: For all $(v, P) \in U(N)$ and all $P_k \in P$

$$\sum_{i \in P_k} \gamma_i(v, P) = \gamma_k(v^P, P^M),$$

where $P^M = \{M\}$ denotes the trivial system of unions on the set of all unions.
Symmetry within the unions\(^3\): For all \((v, P) \in U(N)\), all \(P_k \in P\) and all \(i, j \in P_k\) such that \(v(T \cup i) = v(T \cup j)\) for all \(T \subset N \setminus \{i, j\}\) it holds that
\[
\gamma_i(v, P) = \gamma_j(v, P).
\]

Covariance: Let \((v, P)\) and \((w, P) \in U(N)\) be two games with a priori unions that are covariant, i.e., there exist \(c > 0\) and \((a_i)_{i \in N} \in \mathbb{R}^N\) such that
\[
w(S) = cv(S) + \sum_{i \in S} a_i
\]
for each \(S \subset N\). Then for such \(c\) and \((a_i)_{i \in N}\) we have
\[
\gamma_i(w, P) = c\gamma_i(v, P) + a_i
\]
for each \(i \in N\).

Dummy out property: Let \((v, P) \in U(N)\) be a game with a priori unions and let \(D \subset N\) be the set of dummy players of the game, i.e. the set of players \(i \in N\) such that \(v(T \cup i) = v(T) + v(i)\) for all \(T \subset N \setminus i\). Define a new game with a priori unions \((v_{-D}, P')\), which has player set \(N \setminus D\), whose characteristic function \(v_{-D}\) is the restriction of \(v\) to \(N \setminus D\), and whose system of unions \(P'\) is \(P\) restricted to \(N \setminus D\), i.e.,
\[
P' = \{P_k \cap (N \setminus D) \mid P_k \in P\text{ with } P_k \cap (N \setminus D) \neq \emptyset\}.
\]
Then for all \(i \in N \setminus D\)
\[
\gamma_i(v_{-D}, P') = \gamma_i(v, P).
\]

\(M\)-proportionality: Let \((v, P) \in U(N)\) be a game with a priori unions such that \(m_i(v, P) = 0\) for all \(i \in N\). Then for each union \(P_k \in P\) there exists a constant \(c_k\) such that for all players \(i \in P_k\)
\[
\gamma_i(v, P) = c_k M_i(v, P).
\]

In the following theorem, we prove that the coalitional \(\tau\)-value satisfies all the properties listed above.

\(^3\)This property can be replaced by a stronger but perhaps more intuitive property called anonymity, which states that for all \((v, P) \in U(N)\) and all permutations \(\pi\) of \(N\) it holds that \(\gamma_{\pi(i)}(v^\pi, P^\pi) = \gamma_i(v, P)\), for all \(i \in N\). Here, \(P^\pi = \{\pi(P_k) \mid P_k \in P\}\) and the game \(v^\pi\) is defined by \(v^\pi(\pi(S)) = v(S)\) for all \(S \in 2^N\).
Theorem 2 On the class of quasi balanced games with a priori unions, the coalitional $\tau$-value satisfies the properties efficiency, individual rationality, null player property, quotient game property, symmetry within the unions, covariance, dummy out property, and $M$-proportionality.

Proof. It follows readily from the definition of the coalitional $\tau$-value that on the class of quasi balanced games with a priori unions this allocation rule satisfies the properties efficiency, individual rationality, null player property, quotient game property, and $M$-proportionality. Because the proof of the dummy out property is very long, we will not include it here. We now set out to prove that the coalitional $\tau$-value satisfies the remaining two properties.

Symmetry within the unions: Let $(v, P) \in QBU(N)$ and let $i, j \in P_k$ be two symmetric players, i.e., $v(T \cup i) = v(T \cup j)$ for all $T \subset N \setminus \{i, j\}$. Then, $M_i(v, P) = v(N) - v(N \setminus i) = v(N) - v(N \setminus j) = M_j(v, P)$. Also,

$$m_i(v, P) = \max_{S \in P(k) | i \in S} \{v(S) - \sum_{h \in S \setminus i} M_h(v, P)\} =$$

$$\max \left\{ \max_{S \in P(k) | i \in S} \{v(S) - \sum_{h \in S \setminus i} M_h(v, P)\} \right\},$$

$$\max \left\{ \max_{S \in P(k) | i \in S, j \in S} \{v(S) - \sum_{h \in S \setminus i} M_h(v, P)\} \right\} =$$

$$\max \left\{ \max_{S \in P(k) | i, j \in S} \{v(S \cup i \cup j) - \sum_{h \in S} M_h(v, P) - M_j(v, P)\}, \right\} =$$

$$\max \left\{ \max_{S \in P(k) | i, j \in S} \{v(S \cup i) - \sum_{h \in S} M_h(v, P)\} \right\} =$$

$$\max \left\{ \max_{S \in P(k) | i, j \in S} \{v(S \cup i \cup j) - \sum_{h \in S} M_h(v, P) - M_i(v, P)\}, \right\}.$$

\footnote{The proof can be obtained from the authors upon request.}
where the fourth equality follows using that players $i$ and $j$ are symmetric and that $M_i(v, P) = M_j(v, P)$, and the last equality follows by following the first three steps in reverse. Now we have shown that players $i$ and $j$ have the same utopia payoffs and the same minimal rights. Because they are also in the same union $P_k$, it now follows from the definition of the coalitional $\tau$-value that $\tau_i(v, P) = \tau_j(v, P)$.

Covariance:
Let $(v, P) \in QBU(N)$ and let $(w, P) \in U(N)$ be a game with a priori unions that is covariant with $(v, P)$. Let $c > 0$ and $(a_i)_{i \in N} \in \mathbb{R}^N$ such that

$$w(S) = cv(S) + \sum_{i \in S} a_i$$

for each $S \subset N$.

It is straightforward to show that for every union $P_k$ we have $M_k(w^P) = cM_k(v^P) + \sum_{i \in P_k} a_i$ and $m_k(w^P) = cm_k(v^P) + \sum_{i \in P_k} a_i$. Using these equalities and $v^P \in QBG(M)$, it easily follows that $w^P \in QBG(M)$ as well. In $\mathbb{R}^M$, the vectors $M(w^P)$ and $m(w^P)$ are obtained by performing a simple translation of the vectors $M(v^P)$ and $m(v^P)$, respectively. The $\tau$-value of $w^P$ is the unique compromise between $M(w^P)$ and $m(w^P)$ that is efficient for $w^P$, i.e. such that $\sum_{k \in M} \tau_k(w^P) = w^P(M) = w(N) = cv(N) + \sum_{i \in N} a_i$, while the $\tau$-value of $v^P$ is the unique compromise between $M(v^P)$ and $m(v^P)$ that is efficient for $v^P$, i.e. such that $\sum_{k \in M} \tau_k(v^P) = v^P(M)$. From this, we easily obtain that the $\tau$-value of $w^P$ is the translation of the $\tau$-value of $v^P$, i.e.,

$$\tau_k(w^P) = c\tau_k(v^P) + \sum_{i \in P_k} a_i.$$ 

We now proceed to the level of the players. Like at the level of the unions, it is straightforward to show that for every player $i \in N$ it holds that $M_i(w, P) = cM_i(v, P) + a_i$ and $m_i(w, P) = cm_i(v, P) + a_i$. Using these equalities and the fact that $(v, P)$ is quasi balanced, we obtain for each union $P_k$

$$\sum_{i \in P_k} m_i(w, P) = c \sum_{i \in P_k} m_i(v, P) + \sum_{i \in P_k} a_i \leq c\tau_k(v^P) + \sum_{i \in P_k} a_i = \sum_{i \in P_k} m_i(v, P).$$
\[ \tau_k(w^P) \leq c \sum_{i \in P_k} M_i(v, P) + \sum_{i \in P_k} a_i = \sum_{i \in P_k} M_i(w, P). \]

This shows that \((w, P)\) is quasi balanced as well. Arguments similar to those we used at the level of the unions show that \(\tau_i(w, P) = c\tau_i(v, P) + a_i\) for each \(i \in N\).

In theorem 3 we use four of its properties to provide an axiomatic characterization of the coalitional \(\tau\)-value.

**Theorem 3** On the class of quasi balanced games with a priori unions, the coalitional \(\tau\)-value is the unique solution satisfying efficiency, the quotient game property, covariance, and \(M\)-proportionality.

**Proof.** In view of theorem 2, it remains to show that the coalitional \(\tau\)-value is the only solution satisfying the four properties. So, let \(\gamma\) be a solution satisfying efficiency, the quotient game property, covariance, and \(M\)-proportionality. We will prove that \(\gamma\) coincides with the coalitional \(\tau\)-value.

Let \((v, P) \in QBU(N)\) be a quasi balanced game. Consider the quotient game \(v^P\) together with the trivial structure of unions \(P^M = \{M\}\). Then, \((v^P, P^M)\) is a game with a priori unions on the player set \(M\). It follows directly from the fact that \((v, P)\) is quasi balanced that \((v^P, P^M) \in QBU(M)\).

We define \(a_k = m_k(v^P, P^M)\) for each \(k \in M\) and define the game \(w^P \in G(M)\) by \(w^P(L) = v^P(L) - \sum_{k \in L} a_k\) for each \(L \subset M\). Note that the game \((w^P, P^M)\) is covariant with a quasi balanced game and is therefore quasi balanced as well. Using the covariance property, we obtain

\[ \gamma_k(w^P, P^M) = \gamma_k(v^P, P^M) - m_k(v^P, P^M) \]

for each \(k \in M\). Also, it is easily checked that \(M_k(w^P, P^M) = M_k(v^P, P^M) - m_k(v^P, P^M)\) and \(m_k(w^P, P^M) = 0\) for all \(k \in M\). Hence, we can apply the property \(M\)-proportionality to the game \((w^P, P^M)\) and conclude that there exists a constant \(\alpha\) such that

\[ \gamma(w^P, P^M) = \alpha M(w^P, P^M). \]

Combining all this information, we obtain

\[ \gamma_k(v^P, P^M) = m_k(v^P, P^M) + \alpha (M_k(v^P, P^M) - m_k(v^P, P^M)) \]
for all \( k \in M \). Together with efficiency applied to the game \((v^P, P^M)\), this leads to the conclusion that

\[
\gamma(v^P, P^M) = \tau(v^P, P^M).
\]

We now turn our attention to the game \((v, P)\). We define \( b_i = m_i(v, P) \) for each \( i \in N \) and define the game \( u \in G(N) \) by \( u(S) = v(S) - \sum_{i \in S} b_i \) for each \( S \subset N \). In a way similar to that which we demonstrated above, we find that for each union \( P_k \) there exists a constant \( \beta_k \) such that

\[
\gamma_i(v, P) = m_i(v, P) + \beta_k (M_i(v, P) - m_i(v, P))
\]

for all \( i \in P_k \). Also, by the quotient game property, we know that

\[
\sum_{i \in P_k} \gamma_i(v, P) = \gamma_k(v^P, P^M) = \tau_k(v^P, P^M)
\]

for each \( k \in M \). This leads us to conclude that for each \( k \in M \) it holds that \( \gamma_i(v, P) = \tau_i(v, P) \) for all \( i \in P_k \). We conclude that \( \gamma(v, P) = \tau(v, P) \). \( \square \)

We conclude this section by remarking that we can follow the axiomatic characterization of the \( \tau \)-value in Tijs (1987) and weaken the property covariance in theorem 3 to a property called the minimal rights property.

4 The coalitional \( \tau \)-value for bankruptcy games

A bankruptcy problem exists when a person, firm, or institution does not have sufficient funds to meet the claims of all its creditors. In such a situation, decisions need to be made on how to divide the available funds among the creditors.

We model a bankruptcy problem to be a pair \((E, d)\), in which \( E \in \mathbb{R} \) represents the available funds of the debtor (“the estate”) and \( d = (d_1, d_2, \ldots, d_n) \in \mathbb{R}_+^N \) is the vector of claims (or demands) of the \( n \) creditors. Clearly, each creditor \( i \) has a non-negative claim \( (d_i \geq 0 \) for each \( i \in N \)) and there are insufficient funds to meet all these claims \( (0 \leq E \leq \sum_{i \in N} d_i) \).

An allocation rule for bankruptcy problems is a function \( f \) that assigns to every bankruptcy problem \((E, d)\) a vector \( f(E, d) \in \mathbb{R}^N \) such that \( 0 \leq f_i(E, d) \leq d_i \) for all \( i \in N \) and \( \sum_{i \in N} f_i(E, d) = E \). Hence, an allocation rule
proposes a possible division of the available funds $E$ among the creditors, where the amount $f_i(E,d)$ that creditor $i$ gets is no more than his claim.

O’Neill (1982) defined for each bankruptcy problem $(E,d)$ a corresponding game $(N,v_{E,d})$, the bankruptcy game. The characteristic function of this game assigns to every set of creditors $S$ the part of the available funds that they can guarantee themselves after paying off all the other creditors, i.e.,

$$v_{E,d}(S) := \max\{E - \sum_{i \in N \setminus S} d_i, 0\}$$

for all $S \subset N$.

Curiel et al (1987) studied these bankruptcy games. They showed that such games are convex ($v(S)+v(T) \leq v(S \cup T)+v(S \cap T)$ for all $S, T \subset N$) and that the $\tau$-value of a bankruptcy game is in the core of the game. They also showed that the $\tau$-value for bankruptcy games coincides with the so-called adjusted proportional rule, or $AP$-rule, which is defined using minimal rights. The minimal right of creditor $i \in N$ is

$$m_i(E,d) := \max\{E - \sum_{j \in N \setminus i} d_j, 0\}.$$ 

In words, a creditor’s minimal right is the amount that is left after the claims of the other creditors have been paid out if this amount is nonegative, and otherwise the creditor’s minimal right is zero. A creditor’s minimal right is a lower bound on how much is allocated to him. After each creditor is allotted his minimal right, the $AP$-rule allocates the amount $E' := E - \sum_{i \in N} m_i(E,d)$ that is left to the creditors in proportion to their reduced claims $d_i'(E,d) := \min\{d_i - m_i(E,d), E'\}$. The reduced claim of a creditor is his original claim lowered by his minimal right, which is how much he is claiming after receiving his minimal right, and truncated at $E'$, because claims higher than $E'$ are considered excessive. Hence,

$$AP(E,d) := m(E,d) + E' \frac{d'(E,d)}{\sum_{i \in N} d'_i(E,d)}.$$ 

The following theorem was proven by Curiel et al (1987).

**Theorem 4** Let $(E,d)$ be a bankruptcy problem and $(N,v_{E,d})$ the associated bankruptcy game. Then

$$\tau(N,v_{E,d}) = AP(E,d) \in C(N,v_{E,d}).$$
Note that the bankruptcy problem as defined above does not allow us to model the fact that groups of creditors may consolidate their claims to try and get a larger share of the available funds. In order to be able to take this into account, we use a priori unions. Suppose that the creditors have organized themselves into unions $P_1, P_2, \ldots, P_m$. The bankruptcy problem with unions is the triple $(E, d, P)$, where $E \in \mathbb{R}$ and $d = (d_1, d_2, \ldots, d_n) \in \mathbb{R}^N$ are as in the bankruptcy problem defined above and $P = \{P_1, P_2, \ldots, P_m\}$ is a partition of the set of creditors. The associated bankruptcy game with unions is the game $(v_E, d, P)$, whose player set consists of all creditors. We want to apply the coalitional $\tau$-value for games with a priori unions that we introduced in section 2 to determine an allocation of the available funds among the creditors that takes their organization into unions into account. We will show momentarily that this allocation coincides with the one that we obtain using the generalization of the AP-rule that we describe below.

Let $(E, d, P)$ be a bankruptcy problem with unions. With this problem, we straightforwardly associate a bankruptcy problem $(E, d^P)$ among the unions, where $E$ represents the funds that are available for distribution, and $d^P = (d_1^P, d_2^P, \ldots, d_m^P)$ is the vector of consolidated claims of the unions; $d_k^P := \sum_{i \in P_k} d_i$ for each union $P_k$ of creditors. Obviously, $d_k^P \geq 0$ for each $k \in M$ and $0 \leq E \leq \sum_{k \in M} d_k^P = \sum_{i \in N} d_i$, so that $(E, d^P)$ is indeed a bankruptcy problem.

We apply the AP-rule to the bankruptcy problem $(E, d^P)$ among the unions to obtain an allocation of the available funds among the unions. The minimal right of union $P_k$ is $m_k(E, d^P) = \max\{E - \sum_{i \in M \setminus k} d_i^P, 0\}$ for each $k \in M$. It is the maximum of zero and the amount not claimed by the other unions. After giving each unions its minimal right, the amount $E'^P := E - \sum_{k \in M} m_k(E, d^P)$ is left for distribution and the adjusted claims of the unions are $d_k'(E, d^P) = \min\{d_k^P - m_k(E, d^P), E'^P\}$, $k \in M$. According to the AP-rule, $E'^P$ is allocated among the unions in proportion to the adjusted claims. Hence, the amount allocated to union $P_k$ is given by

$$AP(E, d^P) = m(E, d^P) + E'^P \frac{d_k'(E, d^P)}{\sum_{k \in M} d_k'(E, d^P)}.$$
zero and the amount not claimed by the other creditors. We will first show that the amount assigned to the union $P_k$ is sufficient to cover its members' minimal rights. This follows from chain of (in)equalities below.

\[
\sum_{i \in P_k} m_i(E, d, P) = \sum_{i \in P_k} v_{E,d}(i) \leq v_{E,d}(P_k) = v_{P}(k) \leq AP(E, d^P),
\]

where the equalities follow from the definitions of the bankruptcy games $(N, v_{E,d})$ and $(M, v_{E,d^P})$, the first inequality follows from convexity of the bankruptcy game $(N, v_{E,d})$, and the second inequality follows because $AP(E, d^P)$ is in the core of the bankruptcy game $(M, v_{E,d^P})$ (see theorem 4).

After each creditor in union $P_k$ has received his minimal right, the amount $E_k' := AP_k(E, d^P) - \sum_{i \in P_k} m_i(E, d, P)$ is left to be divided among the members of this union and the adjusted claim of creditor $i \in P_k$ is $d'_i(E, d, P) := \min\{d_i - m_i(E, d, P), E - \sum_{j \in N} m_j(E, d, P)\}$. Then, applying the principle of the adjusted proportional rule to the creditors in union $P_k$, we obtain

\[
AP_i(E, d, P) := m_i(E, d, P) + E_k' \frac{d'_i(E, d, P)}{\sum_{j \in P_k} d'_j(E, d, P)}
\]

for each $i \in P_k$.

We will refer to the generalization of the $AP$-rule to bankruptcy problems with unions as defined above as the coalitional $AP$-rule. Note that the coalitional $AP$-rule is indeed a generalization of the standard $AP$-rule because for every bankruptcy problem $(E, d)$ it holds that $AP(E, d) = AP(E, d, P^n) = AP(E, d, P^N)$, where $P^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ and $P^N = \{N\}$ are the two trivial systems of unions.

The coalitional $AP$-rule for bankruptcy problems with unions as defined above coincides with the coalitional $\tau$-value for the associated bankruptcy games with a priori unions. We establish this result in theorem 5.

**Theorem 5** Let $(E, d, P)$ be a bankruptcy problem with unions and let $(v_{E,d}, P)$ be the associated bankruptcy game with a priori unions. Then $\tau(v_{E,d}, P) = AP(E, d, P)$.

**Proof.** We start by considering what happens at the level of the unions. The coalitional $\tau$-value of $(v_{E,d}, P)$ is defined using its quotient game $(M, v_{E,d}^P)$ with $v_{E,d}^P(L) = v_{E,d}(\cup_{k \in L} P_k)$ for all $L \subset M$. This quotient game actually
coincides with the bankruptcy game \((M, v_{E,d})\) that is associated with the bankruptcy problem \((E, d^P)\) among the unions. This holds because for each \(L \subseteq M\) we have
\[
v_{E,d^P}(L) = \max\{E - \sum_{k \in M \setminus L} d_k^P, 0\},
\]
where \(d_k^P = \sum_{i \in P_k} d_i\) for each \(k \in M\), and
\[
v_{E,d}^P(L) = v_{E,d}(\cup_{k \in L} P_k) = \max\{E - \sum_{i \in N \setminus \cup_{k \in L} P_k} d_i, 0\}.
\]
Because the allocation obtained by applying the \(AP\)-rule to a bankruptcy problem coincides with the \(\tau\)-value of the corresponding bankruptcy game (see theorem 4), we can now conclude that
\[
\tau(M, v_{E,d}) = AP(E, d^P).
\]

We now turn our attention to the level of the individual creditors. We first prove that for each creditor \(i \in N\) his minimal right in bankruptcy situation \((E, d, P)\) is equal to his minimal right in bankruptcy game \((v_{E,d}, P)\).

Because the bankruptcy game \((N, v_{E,d})\) is convex (see theorem 4), it is known that \(m_i(v_{E,d}) = v_{E,d}(i)\) (see Driessen (1988) pp. 159-162). Also, it follows from the definition of \(m_i(v_{E,d}, P)\) that \(m_i(v_{E,d}, P) \leq m_i(v_{E,d})\) (the maximum is taken over a smaller set). Because in the game \((v_{E,d}, P)\) creditor can clearly guarantee himself a payoff of \(v_{E,d}(i)\) by forming a coalition all by himself, we conclude that \(m_i(v_{E,d}, P) = m_i(v_{E,d}) = v_{E,d}(i)\). Further, it follows straightforwardly from their respective definitions that \(m_i(E, d, P) = v_{E,d}(i)\). We now have shown that
\[
m_i(E, d, P) = m_i(v_{E,d}, P).
\]

We proceed by showing that
\[
M_i(v_{E,d}, P) - m_i(v_{E,d}, P) = d_i^*(E, d, P).
\]
This follows because Driessen (1988, pp. 159-162) proves that \(M_i(v_{E,d}) - m_i(v_{E,d}) = d_i^*(E, d)\) for every bankruptcy problem \((E, d)\) and because \(M_i(v_{E,d}) = M_i(v_{E,d}, P)\), \(m_i(v_{E,d}) = m_i(v_{E,d}, P)\), and \(d_i^*(E, d) = d_i^*(E, d, P)\).
Now, fix a union \( P_k \). Then, for each \( i \in P_k \),
\[
\tau_i(E, d, P) = m_i(E, d, P) + \alpha_k \left( M_i(E, d, P) - m_i(E, d, P) \right),
\]
where \( \alpha_k \) is such that \( \sum_{i \in P_k} \tau_i(E, d, P) = \tau_k(v_{E,d}^P) \). Also, for each \( i \in P_k \),
\[
AP_i(E, d, P) = m_i(E, d, P) + \sum_{j \in P_k} \frac{d_i'(E, d, P)}{d_j'(E, d, P)}
\]
and \( \sum_{i \in P_k} AP_i(E, d, P) = AP_k(E, d^P) = \tau_k(v_{E,d}^P) \). Hence, we can represent \( AP_i(E, d, P) \) as
\[
AP_i(E, d, P) = m_i(E, d, P) + \beta_k d_i'(E, d, P),
\]
where \( \beta_k \) is such that \( \sum_{i \in P_k} AP_i(E, d, P) = \tau_k(v_{E,d}^P) \). Together with \( m_i(E, d, P) = m_i(E, d, P) \) and \( M_i(E, d, P) - m_i(E, d, P) = d_i'(E, d, P) \) for each \( i \in P_k \), this shows that \( AP_i(E, d, P) = \tau_i(v_{E,d}, P) \).

The following example illustrates the coalitional AP-rule.

**Example 2** Consider the 3-creditor bankruptcy problem \( (E, d) \) with \( E = 400 \), and \( d = (100, 200, 300) \). Suppose that creditors 1 and 2 form a union and consolidate their claims to try and get a bigger share of the pie when their claims are comparable to that of the third creditor, i.e., \( P = \{ \{1, 2\}, \{3\} \} \).

To find the AP-value of the bankruptcy problem with unions \( (E, d^P) \), we first consider the bankruptcy problem \( (E, d^P) \) among the unions. In this problem, union \( \{1, 2\} \) has a claim \( d_{12} = 300 \) and union \( \{3\} \) has a claim \( d_3 = 300 \). It is then easily seen that each of the unions gets half of the available amount, so that \( AP_{12}(E, d^P) = AP_3(E, d^P) = 200 \).

We now immediately can conclude that \( AP_3(E, d, P) = 200 \). To determine the allocation of \( AP_{12}(E, d^P) \) among creditors 1 and 2, we compute the minimal rights \( m_1(E, d, P) = m_2(E, d, P) = 0 \) and the adjusted claims \( d_1'(E, d, P) = 100 \) and \( d_2'(E, d, P) = 200 \). Creditor 2 then gets twice as much as creditor 1, so that \( AP_1(E, d, P) = \frac{200}{3} \) and \( AP_2(E, d, P) = \frac{400}{3} \). Hence,
\[
AP(E, d, P) = (200/3, 400/3, 200).
\]

Note that it is advantageous for creditors 1 and 2 to consolidate their claims, because
\[
AP(E, d) = (60, 120, 220).
\]
This follows from \( m(E, d) = (0, 0, 100) \) and \( d'(E, d) = (100, 200, 200) \).
In the remainder of this section, we will provide an axiomatic characterization of the coalitional $\tau$-value that is tailored toward bankruptcy games with unions. Because we have shown that the coalitional $AP$-rule for bankruptcy problems coincides with the coalitional $\tau$-value for the corresponding bankruptcy games, and because the coalitional $AP$-rule is, in our opinion, easier to compute, we will write everything that follows in the format of the $AP$-rule.

An allocation rule for bankruptcy problems with unions is a function $f$ that assigns to every bankruptcy problem with unions $(E, d, P)$ a vector $f(E, d, P) \in \mathbb{R}^N$ such that $0 \leq f_i(E, d, P) \leq d_i$ for all creditors $i \in N$ and $\sum_{i \in N} f_i(E, d, P) = E$. Consider the following properties, which are defined for an allocation rule $f$ for bankruptcy problems with unions.

**Independence of irrelevant claims:** For all bankruptcy problems with unions $(E, d, P)$

$$f(E, d, P) = f(E, (\min\{d_i, E\})_{i \in N}, P).$$

This property is also known as the truncation property or invariance under claims truncation.

**Covariance of minimal rights:** For all bankruptcy problems with unions $(E, d, P)$

$$f(E, d, P) = m(E, d, P) + f(E - \sum_{i \in N} m_i(E, d, P), d - m(E, d, P), P).$$

**Equal treatment within the unions:** For all bankruptcy problems with unions $(E, d, P)$ and all unions $P_k \in P$ and creditors $i, j \in P_k$ with equal claims ($d_i = d_j$) it holds that

$$f_i(E, d, P) = f_j(E, d, P).$$

This property is also known as symmetry.

**The quotient problem property:** For all bankruptcy problems with unions $(E, d, P)$ and for each union $P_k \in P$

$$\sum_{i \in P_k} f_i(E, d, P) = f_k(E, d^P, P^M),$$

where $P^M = \{M\}$, the trivial system of unions.
Additivity of claims within the unions: Let \((E, d, P)\) be a bankruptcy problem with unions such that \(m_i(E, d, P) = 0\) and \(d_i \leq E\) for all \(i \in N\) and fix a player \(j \in N\). Let \((E, d', P')\) be a bankruptcy problem with unions that is obtained form \((E, d, P)\) by replacing player \(j\) by several smaller players \(j_1, j_2, \ldots, j_t\) with nonnegative claims \(d_{j_1}, d_{j_2}, \ldots, d_{j_t}\), respectively, such that \(d_j = d_{j_1} + d_{j_2} + \ldots + d_{j_t}\). Let \(P_k \in P\) be the union of which player \(j\) is a member and define \(P' = P \setminus P_k \cup (P_k \setminus \{j_1, j_2, \ldots, j_t\})\). Hence, player \(j\) in union \(P_k\) is replaced by the smaller players and these smaller players are all members of the same union \(P_k\). Then

\[
f_i(E, d, P) = f_i(E, ((d_i)_{i \in N, i \neq j}, d_{j_1}, d_{j_2}, \ldots, d_{j_t}), P')
\]

for all \(i \in N \setminus j\). This property is also known as restricted non-manipulability by merging or splitting.

It is straightforward to verify that the coalitional \(AP\)-rule satisfies the five properties that we described above. Therefore, we state the following theorem without a proof.

**Theorem 6** The coalitional \(AP\)-rule satisfies independence of irrelevant claims, covariance of minimal rights, equal treatment within the unions, the quotient problem property and additivity of claims within the unions.

The following proposition shows that the properties are not logically independent, as equal treatment within the unions follows from three of the other properties.\(^5\)

**Proposition 1** Let \(f\) be an allocation rule for bankruptcy problems with unions that satisfies independence of irrelevant claims, covariance of minimal rights, and additivity of claims within the unions. Then \(f\) also satisfies equal treatment within the unions.

**Proof.** Note that it follows using independence of irrelevant claims and covariance of minimal rights that it suffices to consider bankruptcy problems with unions \((E, d, P)\) such that \(m_i(E, d, P) = 0\) and \(d_i \leq E\) for all \(i \in N\). So, let \((E, d, P)\) be a bankruptcy problem with unions and \(i, j \in P_k \in P\) two creditors with \(d_i = d_j\). We will prove that \(f_i(E, d, P) = f_j(E, d, P)\).

\(^5\)We point the reader to de Frutos (1999), in which a related result is proved, namely that the property of non-manipulability by merging or splitting implies equal treatment.
Consider four new players, players $i_1, i_2, i_3,$ and $i_4$ with claims $d_{i_1} = d_{i_2} = d_{i_3} = d_{i_4} = \frac{d}{4}$.

Let $(E, d', P')$ be the bankruptcy problem that we obtain from $(E, d, P)$ by replacing player $i$ by the two smaller players $i_1$ and $i_2$, like we explained in the definition of the property additivity of claims within the unions. Because $f$ satisfies this property, we know $f_j(E, d, P) = f_j(E, d', P')$. We then replace in the bankruptcy problem $(E, d', P')$ player $j$ by the two smaller players $i_3$ and $i_4$ to obtain the bankruptcy problem $(E, d^*, P^*)$ (note that $d_{i_3} + d_{i_4} = d_i = d_j$). Because $f$ allocates $E$ among the players in both problems $(E, d', P')$ and $(E, d^*, P^*)$, and because it satisfies additivity of claims within the unions, we can conclude that $f_j(E, d', P') = f_{i_3}(E, d^*, P^*) + f_{i_4}(E, d^*, P^*)$.

We now have

\[ f_j(E, d, P) = f_{i_3}(E, d^*, P^*) + f_{i_4}(E, d^*, P^*). \]

Let $(E, d'', P'')$ be the bankruptcy problem that we obtain from $(E, d, P)$ by replacing player $j$ by the two smaller players $i_1$ and $i_2$. Because $f$ satisfies additivity of claims within the unions, we know $f_i(E, d, P) = f_i(E, d'', P'')$. We then replace in the bankruptcy problem $(E, d'', P'')$ player $i$ by the two smaller players $i_3$ and $i_4$ to obtain the bankruptcy problem $(E, d^*, P^*)$. Because $f$ allocates $E$ among the players in both problems $(E, d'', P'')$ and $(E, d^*, P^*)$, and because it satisfies additivity of claims within the unions, we can conclude that $f_i(E, d'', P'') = f_{i_3}(E, d^*, P^*) + f_{i_4}(E, d^*, P^*)$. We now have

\[ f_i(E, d, P) = f_{i_3}(E, d^*, P^*) + f_{i_4}(E, d^*, P^*). \]

The proposition now follows by noting that $(E, d^*, P^*)$ and $(E, d^*, P^*)$ are the same bankruptcy problem and, hence, $f_{i_3}(E, d^*, P^*) = f_{i_3}(E, d^*, P^*)$ and $f_{i_4}(E, d^*, P^*) = f_{i_4}(E, d^*, P^*)$. 

\[ \square \]

In the following theorem we axiomatically characterize the coalitional AP-rule.

**Theorem 7** The coalitional AP-rule is the unique allocation rule for bankruptcy problems with unions that satisfies independence of irrelevant claims, covariance of minimal rights, the quotient problem property, and additivity of claims within the unions.

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Proof. In view of theorem 6, we only need prove that the coalitional AP-rule is the only allocation rule satisfying the four properties. So, let $f$ be an allocation rule that satisfies independence of irrelevant claims, covariance of minimal rights, the quotient problem property, and additivity of claims within the unions. By proposition 1 we know that $f$ satisfies equal treatment within the unions.

Let $(E, d, P^N)$ be a bankruptcy problem with a trivial system of unions. We will start by considering bankruptcy problems of this form and will later consider more general bankruptcy problems with unions. Because $f$ satisfies independence of irrelevant claims and covariance of minimal rights, we can assume that $m_i(E, d, P^N) = 0$ and $d_i \leq E$ for all $i \in N$. Because $f$ is an allocation rule, we know that $0 \leq f_i(E, d, P^N) \leq d_i$ for all creditors $i \in N$ and $\sum_{i \in N} f_i(E, d, P^N) = E$. Therefore, we can, without loss of generality, choose two creditors $i, j \in N$ with $d_i / c_0 \leq d_j$ and let $c \in \mathbb{N}$ with $c \geq c_0$. Then, obviously, $d_i / c \leq d_j$.

Let $(E, d', P')$ be the bankruptcy problem that we obtain from $(E, d, P^N)$ by replacing creditor $i$ by $c$ smaller creditors $i_1, i_2, \ldots, i_c$ with a claim of $d_i / c$ each. Because $f$, being an allocation rule, divides $E$ among the creditors, and because it satisfies additivity of claims within the unions and equal treatment within the unions, we know that $f_{i_1}(E, d', P') = f_i(E, d, P^N) / c$ for $t = 1, 2, \ldots, c$.

We define $q$ to be the integer part of $d_i / c$. Let $(E, d'', P'')$ be the bankruptcy problem with unions that we obtain from $(E, d', P')$ by replacing creditor $j$ by $q$ creditors $j_1, j_2, \ldots, j_q$ with a claim of $d_i / c$ each and one creditor $j_{q+1}$ with a claim of $d_{i,c} = d_j - q d_i / c$. Because $f$ satisfies additivity of claims within the unions and equal treatment within the unions, every agent with a claim of $d_i / c$ receives $f_i(E, d, P^N) / c$ in bankruptcy problem $(E, d', P')$. Note that $d_i / c - 1 < q$, so that $d_{i,c} < d_i / c$. Because $f$, being an allocation rule, divides $E$ among the creditors, and because it satisfies equal treatment within the unions and additivity of claims within the unions, we can now derive $0 \leq f_{j_{q+1}}(E, d'', P'') \leq f_i(E, d, P^N) / c$.

We use additivity of claims within the unions to obtain $f_{j_1}(E, d, P^N) = f_{i_1}(E, d', P') = q f_i(E, d, P^N) / c +$.

\footnote{Note that we could replace creditor $j_1$ by two smaller creditors, one with a claim of $d_{i,c}$, who would then receive $f_{j_{q+1}}(E, d'', P'')$, and the other with a claim of $d_i / c - d_{i,c}$, who would receive a nonnegative amount. The two of them together would receive $f_i(E, d, P^N) / c$.}
We use this to conclude that $\frac{d_i}{d_i} - \frac{1}{c} < \frac{f_j(E,d,P^N)}{f(E,d,P^N)} \leq \frac{d_i}{d_i} + \frac{1}{c}$ (note that $q \leq \frac{d_i}{d_i}$).

We now have shown that $\frac{d_i}{d_i} - \frac{1}{c} < f(E,d,P^N) \leq \frac{d_i}{d_i} + \frac{1}{c}$ for each $c \in \mathbb{N}$ with $c \geq c_0$. Taking the limit for $c \to \infty$, we derive that $d_j / d_i = f_j(E,d,P^N)$. This means that $f$ divides $E$ among the creditors (who each have a minimal right of zero) in proportion to their claims, which are all lower than or equal to $E$. Hence, $f(E,d,P^N) = AP(E,d,P^N)$.

Now, let $(E,d,P)$ be a bankruptcy problem with any system of unions. We can apply what we found to the quotient bankruptcy problem $(E,d,P)$ and find $f_k(E,d,P^M) = AP_k(E,d,P^M)$ for every union $P_k \in P$. Fix a specific union $P_k \in P$. Because $f$ is an allocation rule, and because $f$ satisfies the quotient problem property, we know that $f$ divides $f_k(E,d,P^M) = AP_k(E,d,P^M)$ among the creditors in $P_k$ and also $0 \leq f_i(E,d,P^N) \leq d_i$ for all creditors $i \in P_k$. Therefore, we can, without loss of generality, choose two creditors $i,j \in P_k$ with $d_i \neq 0$ and $d_j \neq 0$. Proceeding as we demonstrated above, we can now show that $f(E,d,P) = AP(E,d,P)$. □

Remark. Theorem 7 is inspired by a result in Curiel et al. (1987). There the (standard) $AP$-rule is axiomatically characterized by the properties independence of irrelevant claims, covariance of minimal rights, additivity of claims, and equal treatment. Our proposition 1 shows that the property equal treatment is redundant in this characterization.

5 The coalitional $\tau$-value for airport games

Airport games arise when considering the problem of allocating the fixed costs of constructing a runway to the airplanes that use it, from a game theoretic angle. This class of games was studied by Littlechild and Thompson (1977), who showed that these games have a very specific structure because the cost of building a suitable runway depends on the "largest" aircraft for which it is designed.

Suppose that the airplanes that use the runway are classified into $T$ types. A movement by an airplane is defined as a take-off or a landing. For each type $t = 1, 2, \ldots, T$, let $N_t$ be the (finite) set of movements by airplanes of type $t$ and let $n_t = |N_t|$ be the number of movements by such airplanes. The
The set of all movements is denoted by $N = N_1 \cup N_2 \cup \ldots \cup N_T$. Let $c_t$ be the cost of constructing a runway that accommodates movements by airplanes of type $t$. Without loss of generality, we assume that $0 < c_1 < c_2 < \ldots < c_T$.

The airport game $(N, c)$ assigns to each set of movements the cost of building a runway suitable for all these movements. Because these costs depend solely on the "largest" aircraft in the set, we obtain

$$c(S) = c_{t(S)}$$

where $t(S) = \max \{t \mid S \cap N_t \neq \emptyset \}$

for each $S \subset N$, $S \neq \emptyset$, and $c(\emptyset) = 0$. We exclude trivial games and assume throughout the following discussion that there are at least two movements, i.e., $|N| \geq 2$, and that $n_t > 0$ for each type $t$.

Tijs and Driessen (1986) and Driessen (1988) applied the $\tau$-value in the context of airport games. Because airport games are concave ($c(S) + c(T) \geq c(S \cup T) + c(S \cap T)$ for all $S, T \subset N$) rather than convex, they have to compute the so-called reverse $\tau$-value $\tau^*(N, c) = -\tau(N, -c)$. Tijs and Driessen (1986) show that this has a nice interpretation in terms of separable costs and non-separable costs, which we will outline below.

The separable cost of movement $i \in N$ in the cost game $(N, c)$ is

$$SC(i, c) := c(N) - c(N \setminus \{i\}).$$

Note that this cost is non-negative if $(N, c)$ is an airport game. It is the share of the total cost $c(N)$ that is fully attributable to movement $i$, because the total costs will drop by this amount if movement $i$ is cancelled. Therefore, $SC(i, c)$ is a lower bound on the contribution to the cost $c(N)$ that is to be paid by $i$. Now, for each coalition $S \subset N$, the non-separable cost is given by

$$NSC(S, c) := c(S) - \sum_{i \in S} SC(i, c).$$

Driessen (1988) proves that $NSC(S, c) \geq 0$ for all $S \subset N$ in any airport game $(N, c)$.

An upper bound on player $i$'s contribution to the non-separable cost $NSC(N, c)$ is $w(i, c) := \min_{S; i \in S} NSC(S, c)$. By forming a coalition $S$ containing $i$ that minimizes $NSC(S, c)$, player $i$ can guarantee himself a maximum contribution of $SC(i, c) + NSC(S, c)$ by letting all the other players in $j \in S$ pay their separable cost $SC(j, c)$ and covering the rest of the cost.
to coalition $S$ himself. Note that the other players $j \in S$ will agree to this because they will be required to pay only the lower bound to their contribution. The vector $w(c) = (w(i, c))_{i \in N}$ is called the weight vector of the game $(N, c)$. Driessen (1988) shows that for any airport game $(N, c)$ it holds that $\sum_{i \in N} w(i, c) \geq NSC(N, c)$, so that the sum of the maximal contributions of all movements to the non-separable cost $NSC(N, c)$ is enough to cover these costs.

With the notations described above, we obtain for each movement $i \in N$ a lower bound $SC(i, c)$ and an upper bound $SC(i, c) + w(i, c)$ on the cost contribution of this movement. Taking the weighted average of these two bounds, where the weight is such that the sum of the contributions of the movements exactly covers the total costs $c(N)$, we obtain the reverse $\tau$-value as

$$
\tau^r_i(N, c) = \begin{cases} 
SC(i, c) & \text{if } NSC(N, c) = 0 \\
SC(i, c) + NSC(N, c) \frac{w(i, c)}{\sum_{j \in N} w(j, c)} & \text{if } NSC(N, c) > 0.
\end{cases}
$$

Driessen (1988) obtained an exact expression for the reverse $\tau$-value of an airport game in terms of the costs $c_t$, $1 \leq t \leq T$. We describe his result in the following theorem.

**Theorem 8** Let $(N, c)$ be an airport game.

1. If $n_T \geq 2$, then $\tau^r_i(N, c) = \frac{c_T}{\sum_{t=1}^{T} n_t c_t} c_k$ for each $i \in N_k$ and $k = 1, 2, \ldots, T$.

2. If $n_T = 1$ and $T \geq 2$, then

$$
\tau^r_i(N, c) = \frac{c_T - 1}{\sum_{t=1}^{T-1} n_t c_t + c_{T-1}} c_k \text{ for each } i \in N_k \text{ and } k = 1, 2, \ldots, T - 1, \text{ and }
$$

$$
\tau^r_i(N, c) = \tau^r_j(N, c) + c_T - c_{T-1} \text{ for each } i \in N_T \text{ and } j \in N_{T-1}.
$$

Hence, if there are at least two movements by airplanes of type $T$, then the reverse $\tau$-value assigns to every movement a landing fee that is proportional to the cost of building a runway suitable to accommodate it. If there is only one movement by the largest type of airplane, then in a first step its separable cost $c_T - c_{T-1}$ is charged to this movement, and in a second step the remaining cost $c_{T-1}$ is allocated to all movements in proportion to their cost. Note that in the second step the movement of type $N_T$ is regarded as
one of type \( N_{T-1} \). Note that theorem 8 covers all airport games with at least two players (movements).

Note that the airport game as defined above does not take into account the fact that the movements in an airport are grouped according to the airlines to which they belong. Generally, negotiations about landing fees will take place between airlines and the airport authorities. In order to be able to take this into account, Vázquez-Brage et al. (1997) used the model of games with a priori unions. Suppose that \( A \) airlines use the airport, such that we have a system of a priori unions \( P = \{ P^1, P^2, \ldots, P^A \} \), where \( P^a \) denotes the set of movements of airline \( a \) for each \( a = 1, 2, \ldots, A \). Without loss of generality, we assume that \( P^a \neq \emptyset \) for each \( P^a \in P \). The airport cost allocation problem is then modeled as a game with a priori unions \( (N, c, P) \), where \( N \) is the set of movements, \( c \) is the characteristic function of the cost game \( (N, c) \) as defined above, and \( P \) is the partition of the set of movements according to airlines. Because \( P^a \subset N \), \( c(P^a) \) is well-defined for each \( a = 1, 2, \ldots A \). Without loss of generality, we assume that \( c(P^1) \leq c(P^2) \leq \ldots \leq c(P^A) \). We will use the coalitional \( \tau \)-value for games with a priori unions that we introduced in section 2 to determine aircraft landing fees that take the organization of movements into airlines into account. The method using separable costs and non-separable costs that we described above is easily extended to compute the reverse \( \tau \)-value for (cost) games with a priori unions.

To determine the reverse \( \tau \)-value of an airport game with a priori unions, we first compute the reverse \( \tau \)-value of each airline in the quotient game \( (P, c^P) \). To simplify notation, we denote the cost \( c(P^a) \) of airline \( a \) by \( c(a) \), and we denote the total fee charged to airline \( a \) by \( \tau_a^r(N, c, P) = \tau_{P^a}(P, c^P) \).

**Theorem 9** Let \( (N, c, P) \) an airport game with a priori unions.

1. If there are at least two airlines that have a movement by an airplane of type \( T \), then

\[
\tau^r_a(N, c, P) = \frac{c(A)}{\alpha=1 c(\alpha)} c(\alpha) \text{ for each } a = 1, 2, \ldots, A.
\]

2. If only one airline has a movement by an airplane of type \( T \) and \( A \geq 2 \), then

\[
\tau^r_a(N, c, P) = \frac{c(A-1)}{\alpha=1 c(\alpha)+c(A-1)} c(\alpha) \text{ for each } a \neq A, \text{ and}
\]

\[
\tau^r_A(N, c, P) = \tau^r_{A-1}(N, c, P) + c(A) - c(A-1).
\]
3. If there is only one airline, then
\[ \tau^*_a(N, c, P) = c(a) \] for the unique airline \( a \).

**Proof.** This theorem follows directly from theorem 8 by noting that the quotient game \((P, c^P)\) is itself an airport game. Namely, for all sets of airlines \( \hat{A} \subset A \), we have that \( c^P(\hat{A}) = c(\bigcup_{a \in \hat{A}} P^a) = \max\{c_t \mid \text{there is an } a \in \hat{A} \text{ such that } P^a \cap N_t \neq \emptyset\} = \max\{c(a) \mid a \in \hat{A}\}. \)

The interpretation of the results in theorem 9 is as follows. If there are at least two airlines that have a movement by the largest type of airplane, then the reverse coalitional \( \tau \)-value assigns to each airline a landing fee proportional to the cost of a runway needed by this airline. If only one airline has a movement by the largest type of airplane and there are at least two airlines, then airline \( A \) pays its separable cost \( c^a(A) - c(A - 1) \) and the remaining costs \( c(A - 1) \) are divided among the airlines in proportion to their costs, thereby regarding airline \( A \) as the next most expensive one \( A - 1 \). Finally, if there is only one airline, \( A = 1 \), then the quotient game has only one player, which is then naturally required to pay the entire cost \( c(A) \).

In the following theorem we address the division of the costs \( \tau_a^*(N, c, P) \) among the movements in \( P^a \), for each airline \( a = 1, 2, \ldots, A \).

**Theorem 10** Let \((N, c, P)\) an airport game with a priori unions.

1. If \( n_T \geq 2 \) then \( \tau^*_i(N, c, P) = \frac{\tau^*_a(N, c, P)}{j \in P^a c(j)} c(i) \) for each \( i \in P^a \) \( a = 1, 2, \ldots, A \).

2. If \( n_T = 1 \), then
   \[ \tau^*_i(N, c, P) = \frac{\tau^*_a(N, c, P)}{j \in P^a c(j)} c(i) \] for \( i \in P^a, a \neq A \),
   \[ \tau^*_i(N, c, P) = \frac{\tau^*_A(N, c, P) - (c_T - c_{T-1})}{j \in P^A c(j) - (c_T - c_{T-1})} c(i) \] for each \( i \in P^A, i \notin N_T \),
   \[ \tau^*_i(N, c, P) = c_T - c_{T-1} + \frac{\tau^*_A(N, c, P) - (c_T - c_{T-1})}{j \in P^A c(j) - (c_T - c_{T-1})} c_{T-1} \] for each \( i \in P^A, i \in N_T \).

\(^7\)We point out that we can avoid using theorem 8, because theorem 9 follows from theorem 10 (note that we prove theorem 10 independently of the other two theorems). This can be seen by applying theorem 10 to the airport game \((P, c^P, \{P\})\), which is the quotient game of \((N, c, P)\) supplemented with the trivial system of unions that contains one union encompassing all airlines.

\(^8\)Note that to minimize the use of brackets, we write \( c(i) \) instead of \( c(\{i\}) \).
Proof. To prove the first part of the theorem, assume that \( n_T \geq 2 \). Then, it is easily seen that for all movements \( i \in N \), it holds that \( SC(i, c) = c(N) - c(N\{i\}) = 0 \). This immediately leads to \( NSC(S, c) = c(S) - \sum_{i \in S} SC(i, c) = c(S) \) for all \( S \subseteq N \). Now, fix an airline \( a \in \{1,2,\ldots,A\} \) and let \( i \in P^a \). The weight of movement \( i \) equals \( w(i, c) = \min_{S: i \in S \in P(a)} NSC(S, c) = \min_{S: i \in S \in P(a)} c(S) = c(i) \), where the last equality follows from the fact that the smallest possible coalition has the lowest cost. Now, note that \( c(a) = c(P^a) = c_t(P^a) = c(j) \) for some \( j \in P^a \). Using individual rationality of the \( \tau \)-value, it then follows that

\[
\frac{1}{X} \sum_{j \in P^a} w(j, c) = \frac{1}{X} \sum_{j \in P^a} c(j) \geq c(a) \geq c(i),
\]

so that the sum of the maximal contributions of all movements of airline \( a \) to its cost contribution over and above the separable costs is enough to cover this airline's contribution. Because the separable cost of each movement equals 0, we obtain the reverse coalitional \( \tau \)-value of each movement in proportion to its weight, where the proportion is such that the sum of the fees charged to movements of airline \( a \) equals its contribution \( \tau_a^r(N, c, P) \), i.e.,

\[
\tau_i^r(N, c, P) = \frac{c(i)}{\sum_{j \in P^a} c(j)} \tau_a^r(N, c, P)
\]

for each \( i \in P^a \) and for each airline \( a = 1,2,\ldots,A \).

To prove the second part of the theorem, assume that \( n_T = 1 \). Then, it is easily seen that \(^9\)

\[
SC(i, c) = \begin{cases} 0 & \text{if } i \not\in N_T \\ c_T - c_{T-1} & \text{if } i \in N_T. \end{cases}
\]

Using this, we find that

\[
NSC(S, c) = c(S) - \sum_{i \in S} SC(i, c) = \begin{cases} c(S) & \text{if } N_T \cap S = \emptyset \\ c_{T-1} & \text{if } N_T \cap S \neq \emptyset \end{cases}
\]

Noting that for all \( a = 1,2,\ldots,A \) and all \( i \in P^a \) such that \( i \not\in N_T \) we have that \( \min_{S: i \in S \in P(a), N_T \cap S = \emptyset} NSC(S, c) = c(i) \) and \( \min_{S: i \in S \in P(a), N_T \cap S \neq \emptyset} NSC(S, c) = c_{T-1} \), we find

\[
w(i, c) = \min_{S: i \in S \in P(a)} NSC(S) = \min\{c(i), c_{T-1}\} = c(i).
\]

\(^9\)Note that we assumed that \( N_{T-1} \neq \emptyset \).
Further, for \( i \in \mathbb{N} \) it holds that \( i \in P \) and
\[
w(i, c) = \min_{S \in S \in P(A)} NSC(S) = c_{T-1}.
\]

We now need to check that for each airline \( a \) the sum of the maximal contributions of all its movements to its cost contribution over and above the separable costs is enough to cover this airlines contribution. For airlines \( a \neq A \) we have already shown this in proving the first part of the theorem. For airline \( A \) it follows using individual rationality of the \( \tau \)-value. Namely, denoting the unique movement in \( \mathbb{N} \) by \( i \),
\[
\sum_{j \in P^A} w(j, c) = \sum_{j \in P^A, j \neq i} c(j) + c_{T-1}
\geq c_{T-1} = c_T - (c_T - c_{T-1})
= c(A) - SC(i, c) \geq \tau^r_a(N, c, P) - \sum_{j \in P^A} SC(j, c).
\]

Taking for each airline \( a \) the weighted average of the lower bound \( (SC(i, c))_{i \in P^a} \) and the upper bound \( (SC(i, c) + w(i, c))_{i \in P^a} \) that equates the sum of the contributions of the movements in \( P^a \) to the reverse \( \tau \)-value \( \tau^r_a(N, c, P) \), we obtain

1. \( \tau^r_i(N, c, P) = \frac{c(i)}{\tau^r_a(N, c, P)} \tau^r_a(N, c, P) \) if \( i \in P^a, a \neq A \),
2. \( \tau^r_i(N, c, P) = \frac{c(i)}{\tau^r_A(N, c, P) - c_T + c_{T-1}} \left( \tau^r_A(N, c, P) - c_T + c_{T-1} \right) \) if \( i \in P^A, i \notin N_T \)
3. \( \tau^r_i(N, c, P) = \frac{c_{T-1}}{\tau^r_A(N, c, P) - c_T + c_{T-1}} \left( \tau^r_A(N, c, P) - c_T + c_{T-1} \right) + c_T - c_{T-1} \) if \( i \in P^A, i \in N_T \).

This concludes the proof. \( \square \)

The cost shares in theorem 10 have an interpretation similar to those we provided for the cost shares in theorems 8 and 9. If there are at least two movements by airplanes of the largest type, the contribution of an airline is divided among its movements in proportion to their cost. The same method is used for airlines \( a \neq A \) if there is only one movement by the largest
type of airplane, which is then part of airline $A$. In this case, the unique movement by the largest type of airplane is required to pay its separable cost $c_T - c_{T-1}$. The remaining cost, $\tau_A^r(N, c, P) = (c_T - c_{T-1})$ is allocated to the movements of airline $A$ proportional to their cost, whereby the movement of type $N_T$ is now regarded as one of type $N_{T-1}$. Note that if $n_T = 1$ and in addition $P^A \cap N_{T-1} \neq \emptyset$, then for $i \in N_T$ and $j \in N_{T-1}$ we have that $\tau_i^r(N, c, P) = c_T - c_{T-1} + \tau_j^r(N, c, P)$, which brings out more clearly the analogy with theorem 8.

To illustrate the use of the reverse coalitional $\tau$-value to compute aircraft landing fees, we consider the situation at Labacolla, the airport of Santiago de Compostela, Spain, in the first three months of 1993, which was also considered in Vázquez-Brage et al. (1997). In table 1 we provide the reader with information on the different types of aircraft that used Labacolla in the time period studied, the number of movements by each type, and their cost. We also provide the allocation of the total costs according to the reverse $\tau$-value when the movements are treated as separate entities rather than as part of an airline. All costs are given in thousands of Pesetas.

<table>
<thead>
<tr>
<th>Type</th>
<th>$t$</th>
<th>Number of movements</th>
<th>Cost</th>
<th>Reverse $\tau$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>CESSNA</td>
<td>1</td>
<td>10</td>
<td>8.120</td>
<td>8.301</td>
</tr>
<tr>
<td>LEARJET-25</td>
<td>2</td>
<td>6</td>
<td>15.134</td>
<td>15.470</td>
</tr>
<tr>
<td>B-757</td>
<td>3</td>
<td>78</td>
<td>32.496</td>
<td>33.218</td>
</tr>
<tr>
<td>DC-9</td>
<td>4</td>
<td>464</td>
<td>34.265</td>
<td>35.027</td>
</tr>
<tr>
<td>B-737</td>
<td>5</td>
<td>232</td>
<td>39.494</td>
<td>40.372</td>
</tr>
<tr>
<td>B-727</td>
<td>6</td>
<td>438</td>
<td>44.850</td>
<td>45.847</td>
</tr>
<tr>
<td>DC-10</td>
<td>7</td>
<td>30</td>
<td>50.000</td>
<td>51.112</td>
</tr>
</tbody>
</table>

In the time period considered, there were 23 airlines that used Labacolla. These are listed in the first column of table 2, in which we also specify the number of movements by each type of aircraft for each airline. The last column of the table lists the reverse coalitional $\tau$-value for the movements, distinguished by type and airline. The fees are given in thousands of Pesetas.
We point out that, for the same type of aircraft, the fees per movement are lower for airlines that use the airport of Labacolla more intensively. This is because airlines with many movements can spread their costs over all these movements, while an airline with only a few movements can distribute its costs only over a few movements. Note that the total fee to be paid by an airline only depends on the types of aircraft it uses at the airport, and not on the number of movements. We remind the reader that in the airport game we are only considering the fixed cost of building the runway. The variable

<table>
<thead>
<tr>
<th>Airline</th>
<th>Number of movements</th>
<th>Type of aircraft</th>
<th>Reverse Coalitional τ-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air Europa</td>
<td>36</td>
<td>B-757</td>
<td>9.820</td>
</tr>
<tr>
<td></td>
<td>172</td>
<td>B-737</td>
<td>11.936</td>
</tr>
<tr>
<td>Aviaco</td>
<td>12</td>
<td>DC-9</td>
<td>174.002</td>
</tr>
<tr>
<td>Britannia</td>
<td>6</td>
<td>B-737</td>
<td>401.112</td>
</tr>
<tr>
<td>British Airways</td>
<td>2</td>
<td>B-757</td>
<td>990.115</td>
</tr>
<tr>
<td>Condor Flugdienst</td>
<td>2</td>
<td>B-757</td>
<td>990.115</td>
</tr>
<tr>
<td>Caledonian Airways</td>
<td>2</td>
<td>B-757</td>
<td>990.115</td>
</tr>
<tr>
<td>Eurobelgian Airlines</td>
<td>2</td>
<td>B-737</td>
<td>1203.335</td>
</tr>
<tr>
<td>Futura</td>
<td>32</td>
<td>B-737</td>
<td>75.208</td>
</tr>
<tr>
<td>Gestair Executive Set</td>
<td>2</td>
<td>CESSNA</td>
<td>247.407</td>
</tr>
<tr>
<td>Iberia</td>
<td>452</td>
<td>DC-9</td>
<td>2.668</td>
</tr>
<tr>
<td></td>
<td>438</td>
<td>B-727</td>
<td>3.492</td>
</tr>
<tr>
<td>Air Charter</td>
<td>2</td>
<td>B-737</td>
<td>1203.335</td>
</tr>
<tr>
<td>Corse Air</td>
<td>4</td>
<td>B-737</td>
<td>601.667</td>
</tr>
<tr>
<td>Air UK Leisure</td>
<td>2</td>
<td>B-737</td>
<td>1203.335</td>
</tr>
<tr>
<td>Ibertrans</td>
<td>2</td>
<td>CESSNA</td>
<td>247.407</td>
</tr>
<tr>
<td>LTE</td>
<td>36</td>
<td>B-757</td>
<td>55.006</td>
</tr>
<tr>
<td>Mac Aviation</td>
<td>6</td>
<td>LEARJET 25</td>
<td>153.705</td>
</tr>
<tr>
<td>Monarch Airlines Ltd.</td>
<td>3</td>
<td>B-737</td>
<td>802.223</td>
</tr>
<tr>
<td>Sobelair</td>
<td>6</td>
<td>B-737</td>
<td>401.112</td>
</tr>
<tr>
<td>Trabajos Aéreos</td>
<td>2</td>
<td>CESSNA</td>
<td>247.407</td>
</tr>
<tr>
<td>Tea Basel LTD</td>
<td>2</td>
<td>B-737</td>
<td>1203.335</td>
</tr>
<tr>
<td>Oleohidráulica Balear SA</td>
<td>4</td>
<td>CESSNA</td>
<td>123.704</td>
</tr>
<tr>
<td>Viasa</td>
<td>30</td>
<td>DC-10</td>
<td>262.767</td>
</tr>
<tr>
<td>Spanair</td>
<td>2</td>
<td>B-737</td>
<td>1203.335</td>
</tr>
</tbody>
</table>
cost associated with movements can be directly attributed to the various movements and constitute another part of the total fees. Hence, the total fee paid by an airline, that is, the fee for variable costs and for fixed costs, will be higher if the airline decides to make more movements at the airport.

Three of the airlines that used Labacolla in the time period that we are considering, namely Aviaco, Iberia, and Viasa, are part of the Iberia group. Suppose that the Iberia group decides to negotiate the fees for its three member-airlines as one group. Then we obtain a situation with only 21 bargaining units, in which the reverse $\tau$-value for the Iberia group is 12900.035 (in thousands of Pesetas). Note that this is more than the sum of the fees charged to the three airlines if they each negotiate their own fees, which is $12 \times 174.002 + 452 \times 2.668 + 438 \times 3.492 + 30 \times 262.767 = 12706.466$ (in thousands of Pesetas). This shows that mergers by airlines do not necessarily lower their landing fees if these are determined using the reverse coalitional $\tau$-value.

Comparing our results to those obtained by Vázquez-Brage et al. (1997), we see that the reverse coalitional $\tau$-value generates lower fees for the more expensive airplanes and higher fees for the less expensive ones as compared to the Shapley value or the Owen value. In general, the proportion of fee to cost is less variable among the different types of aircraft and different airlines when the reverse $\tau$-value or the reverse coalitional $\tau$-value is used to compute these fees than when the Shapley value or the Owen value is used. Another difference between the Owen value and the coalitional $\tau$-value is that mergers by airlines do not necessarily lower their landing fees if these are determined using the reverse coalitional $\tau$-value, whereas they will always be lowered if the Owen value is used to determine landing fees.
References


