An axiomatic characterization of the position value for network situations

Anne van den Nouweland∗ Marco Slikker†

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Abstract

Network situations as introduced by Jackson and Wolinsky (1996) incorporate the influence of the architecture of a network rather than just the connectivity it provides and thereby provide a more flexible setting than communication situations, which consist of a game with transferable utility and a network. We characterize the position value for network situations along the lines of the characterization of the Shapley value by Shapley (1953). In contrast to previous attempts to provide such an axiomatization, we require no condition on the underlying network. The reason for this is that we exploit the additional flexibility of network situations.

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1 Introduction

The interface of cooperative game theory and network theory has been studied extensively in recent years. Inspired by social and economic settings dealing with, among others, friends and relatives, job search, trade and exchange, and collaborative alliances between corporations, significant progress has been made in the understanding of stability and allocation in network settings.

The literature on the interface between cooperative game theory and network theory since its inception in Myerson (1977) can be classified along two, oftentimes intertwined, dimensions – the first one being different assumptions about the basic ingredients of a model and the second one variation in allocation rules. We first address the basic ingredients. In Myerson (1977), a cooperative game with transferable utilities is supplemented with a network on the set of players of the cooperative game. This network models the lines of communication that exist between the players in the game and this connectivity of the players in the network alters the abilities of various coalitions of players to realize the possible gains from cooperation as described by the cooperative game. We will refer to such a model as a communication situation. Jackson and Wolinsky (1996) extended Myerson’s (1977) model by taking a function assigning values to networks as a basic ingredient. This allowed them to go beyond the connectivity of coalitions of players and take the architecture of a network into account as well.\footnote{Other variations on the model of communication situations are described in Slikker and van den Nouweland (2001). These variations include extensions of the communication channels to hypergraphs and probabilistic networks, among others.} We will refer to Jackson and Wolinsky’s (1996) model as a network situation. Network situations are more flexible, and, as illustrated by results in the current paper, this setting allows for results that appeared to be troublesome in communication situations.

Next we turn our attention to variation in allocation rules. In the paper that started the literature on cooperative games and networks, Myerson (1977) introduced and considered the Shapley value of a cooperative game that is derived from the communication situation by altering the worths of non-connected coalitions. This allocation rule has become known as the
Myerson value. The extension of the Myerson value to network situations is the main allocation rule studied in Jackson and Wolinsky (1996). Borm et al. (1992) introduced the position value for communication situations. Like the Myerson value, the position value is based on the Shapley value, but it stresses the role of the pairwise connections in generating value rather than the role of the players. The value of a pairwise connection is derived as the Shapley of a game on the pairwise connections and the position value equally divides the value of each pairwise connection among the pair of players who form and maintain it. The position value was extended to the setting of network situations by Slikker (2005b). Allocation rules that are not based on the Shapley value can be found in Jackson and Wolinsky (1996), Jackson and van den Nouweland (2005), Hamiache (1999), and Herings et al. (2008), among others.

The current paper focuses on the position value for network situations and provides a characterization of this allocation rule that parallels the original characterization of the Shapley value in Shapley (1953). Borm et al. (1992) were successful in finding such axiomatizations of the Myerson value and the position value for communication situations only if they restricted themselves to considering cycle-free networks. The restriction to cycle-free networks was lifted in van den Nouweland (1993) for the Myerson value, but until the current paper, attempts to do the same for the position value have been unsuccessful. We hasten to point out that characterizations of the position value for arbitrary graphs do exist in the literature – Slikker (2005a) used component efficiency and balanced link contributions, and Slikker (2005b) used potential functions to characterize the position value – but these axiomatizations do not follow the lines of Shapley’s (1953) axiomatization of the Shapley value.

Recent work related to the position value includes Caulier (2010), who concentrates on network situations in which the underlying network is complete. He does not characterize the position value, which defines payoffs to players, but considers a rule that provides payoffs to links, where these payoffs are then forced to be divided equally over players involved. Caulier (2010) follows Shapley’s characterization for this allocation procedure to the links. Recent characterizations related to the position value, can be found in
Ghintran (2010) and Ghintran et al. (2010). Ghintran (2010) characterizes a new weighted position value for communication situations with cycle-free networks. Her weighted position value divides values of links proportional to players’ a priori weights for links with positive values and proportional to the inverse of these weights for links with negative values. This approach is in line with ideas of Haeringer (2006), who considered allocation rules for cooperative games with transferable utilities. Ghintran et al. (2010) follow the approaches of Slikker (2005a) and Slikker (2005b) in a communication situation setting with probabilistic networks.

The contribution of the current paper is a Shapley-like axiomatic characterization of the position value for network situations without restrictions on the underlying network. As mentioned before, such a characterization in the setting of Myerson (1977) has required a restriction to cycle-free networks. However, network situations are a richer, more flexible setting than communication situations because in a network situation the specific way in which a coalition of players is connected can lead to a difference in value for the coalition and this variation is impossible in a communication situation. Though perhaps counterintuitive at first sight, it is exactly the richer setting that allows us to lift the restriction to cycle-free networks.

The setup of this paper is as follows. Section 2 contains preliminaries on networks and value functions. Section 3 contains the definition of the position value and introduces and discusses the axioms that are used in the axiomatization of the position value, which also appears in this section. Section 4 contains two closely related alternative axiomatizations of the position value that are obtained by considering variations in the domain and efficiency requirements.

2 Preliminaries

In this section we present notation and definitions regarding networks, value functions, network situations, and allocation rules. Additionally, we provide some basic results.

Throughout this paper, we consider a fixed set of players $N = \{1, \ldots, n\}$ and we will not express this underlying player set in our notations.
For any coalition of players $S \subseteq N$, we denote the set of all possible links between them by $g^S = \{\{i,j\} \mid \{i,j\} \subseteq S, i \neq j\}$. A network consists of a pair $(N, g)$, where $g \subseteq g^N$ is the set of links that are present between the players. Suppressing the fixed player set $N$, we identify a network with its links and the set of all possible networks is denoted $G = \{g \mid g \subseteq g^N\}$. For a network $g \in G$ and coalition of players $S \subseteq N$, we denote by $g(S) = g \cap g^S$ the links in $g$ that are between players in $S$. We denote the set of players that are involved in at least one link in network $g$ by $N(g)$. Alternatively, $N(g)$ is the intersection of all sets of players $S$ such that $g(S) = g$.

Two players $i$ and $j$ are connected in $g$ if there exists a path in $g$ that connects them, i.e., there exists $t \in \mathbb{N}$ and a sequence of players $\{i_0, \ldots, i_t\}$ with $i_0 = i$ and $i_t = j$ such that $\{i_{k-1}, i_k\} \in g$ for all $k \in \{1, \ldots, t\}$. The notion of connectedness induces for any network $g \in G$ a partition $\Pi(g)$ of the player set $N$ into coalitions of mutually connected players where these coalitions are maximal with respect to set-inclusion. The components of $g$ are $C(g) = \{(S, g(S)) \mid S \in \Pi(g)\}$.

A value function is a function $v : G \to \mathbb{R}$ with $v(\emptyset) = 0$ that assigns a value to each network in $G$. The set of all value functions is denoted by $V$. The condition $v(\emptyset) = 0$ is a normalization that has the interpretation that we only consider the possible gains generated by connections (links) between the players. A value function $v$ is anonymous if $v(g^\sigma) = v(g)$ for all networks $g \in G$ and all permutations $\sigma$ of the player set $N$, where $(\sigma(i), \sigma(j)) \in g^\sigma$ if and only if $(i, j) \in g$. A value function $v$ is component additive if the value of a network is always equal to the sum of the values of the links in each of its components: $v(g) = \sum_{(S, g(S)) \in C(g)} v(g(S))$ for all $g \in G$. The set of all component additive value functions is denoted $V^{ca}$.

A network situation is a pair $(g, v)$ consisting of a network $g$ and a value function $v$. An allocation rule on a domain $NS \subseteq G \times V$ of network situations is a function $Y : NS \to \mathbb{R}^N$ that assigns a value $Y_i(g, v)$ to each player $i \in N$ in a network situation $(g, v) \in NS$. We will consider 2 different domains of allocation rules in this paper, namely the domain $G \times V$ that contains all network situations, and the domain $G \times V^{ca}$ that contains only network situations with a component additive value function. When we consider the domain of network situations with component additive value functions, we
need to be careful to check that the axioms on allocation rules that we are
going to be using do not take us outside this domain. To this end, we can
use Corollary 1 below, in which we identify the set of all component additive
unanimity value functions as a basis of $V^{ca}$.

For every network $g \in G$, $g \neq \emptyset$, we define a unanimity value function
$u_g : G \rightarrow \mathbb{R}$ by

$$u_g(h) = \begin{cases} 1 & \text{if } g \subseteq h; \\ 0 & \text{otherwise.} \end{cases}$$

It is well known (see, for example, Shapley (1953)) that every value function
can be written as a linear combination of unanimity value functions $u_g$ in a
unique way:

$$v = \sum_{g \in G, g \neq \emptyset} \lambda_g(v) u_g,$$

where $\lambda_g(v)$, $g \in G \setminus \{\emptyset\}$, are the unanimity coefficients.

In the following lemma, we demonstrate that a component additive value
function is a linear combination of unanimity value functions on networks
that have at most one component containing more than one player.$^2$

**Lemma 1** Let $v$ be a component additive value function. Then $\lambda_g(v) = 0$
for any network $g \in G$ that has two links that are not in the same component.

**Proof.** Let $g \in G$ be a network that has two links that are not in the same
component. We show that $\lambda_g(v) = 0$ by induction to the number of links in
$g$.

Suppose $|g| = 2$, the minimum size possible. Then $g = \{l_1, l_2\}$ and
$C(g) = \{(N(l_1), \{l_1\}), (N(l_2), \{l_2\})\}$. By definition of the unanimity coeffi-
cients we have

$$v(g) = \sum_{g' \subseteq g} \lambda_{g'}(v) = \lambda_{\{l_1\}}(v) + \lambda_{\{l_2\}}(v) + \lambda_g(v)$$

and using component additivity of $v$ we derive

$$v(g) = \sum_{(T, g(T)) \in C(g)} v(g(T)) = v(\{l_1\}) + v(\{l_2\}) = \lambda_{\{l_1\}}(v) + \lambda_{\{l_2\}}(v)$$

$^2$Note that such a network may have several isolated players.
Combining (1) and (2), we obtain $\lambda_g(v) = 0$.

Now, let $k \geq 2$ and suppose that $|g| = k + 1$ and that we have shown that $\lambda_{g'}(v) = 0$ for all networks $g'$ with $|g'| \leq k$ that have two links that are not in the same component. Let $(S, g(S)) \in C(g)$ be a component of $g$ with $|g(S)| \geq 1$. Then, by assumption, $g \cap g(S) \neq \emptyset$ and $g \setminus g(S) \neq \emptyset$. Also, if $(T, g(T)) \in C(g)$, then either $g(T) \subseteq g(S)$ or $g(T) \subseteq g \setminus g(S)$. Using this fact in conjunction with component additivity of $v$, we obtain

$$v(g) = \sum_{(T, g(T)) \in C(g)} v(g(T))$$

$$= \sum_{(T, g(T)) \in C(g) : g(T) \subseteq g(S)} v(g(T)) + \sum_{(T, g(T)) \in C(g) : g(T) \subseteq g \setminus g(S)} v(g(T))$$

$$= \sum_{(T, g(T)) \in C(g \cap g(S))} v(g(T)) + \sum_{(T, g(T)) \in C(g \setminus g(S))} v(g(T))$$

$$= v(g \cap g(S)) + v(g \setminus g(S))$$

(3)

Also,

$$v(g) = \sum_{g' \subseteq g} \lambda_{g'}(v)$$

$$= \sum_{g' \subseteq g \cap g(S)} \lambda_{g'}(v) + \sum_{g' \subseteq g \setminus g(S)} \lambda_{g'}(v) + \sum_{g' \subseteq g : g' \not\subseteq \emptyset \land g' \not\subseteq g(S) \neq \emptyset} \lambda_{g'}(v)$$

$$= v(g \cap g(S)) + v(g \setminus g(S)) + \sum_{g' \subseteq g : g' \not\subseteq g(S) \neq \emptyset \land g' \not\subseteq g(S) \neq \emptyset} \lambda_{g'}(v)$$

(4)

Combining (3) and (4), we obtain

$$0 = \sum_{g' \subseteq g : g' \not\subseteq g(S) \neq \emptyset \land g' \not\subseteq g(S) \neq \emptyset} \lambda_{g'}(v)$$

$$= \sum_{g' \subseteq g : g' \not\subseteq g(S) \neq \emptyset \land g' \not\subseteq g(S) \neq \emptyset} \lambda_{g'}(v) + \lambda_g(v)$$

$$= 0 + \lambda_g(v),$$

where the last equality is obtained from the induction hypothesis. Thus, $\lambda_g(v) = 0$. □

An important implication of Lemma 1 is that the component additive unanimity value functions form a basis of the space of component additive value functions.
Corollary 1 Every component additive value function is a linear combination of unanimity value functions that are themselves component additive.

Proof. Let \( v \) be a component additive value function. By Lemma 1 we find that \( v \) is a linear combination of unanimity value functions on networks in which all links are in one component.

Let \( g \in G \) be a network in which all links are in one component. Then \( (N(g), g) \in C(g) \). Thus, for every network \( h \in G \), it holds that \( g \subseteq h \) if and only if there is a component \( (S, h(S)) \in C(h) \) such that \( g \subseteq h(S) \). Therefore, \( u_g(h) = \sum_{(S, h(S)) \in C(h)} u_g(h(S)) \).

This demonstrates that \( u_g \) is component additive. \( \square \)

3 Axiomatization of the Position Value

3.1 The Position Value

The position value (cf. Borm et al. (1992)) is an allocation rule for communication situations that is based on the Shapley value (cf. Shapley (1953)) of the links in a network. The definition of the position value was extended to network situations in Slikker (2005b) (see also Slikker and van den Nouweland (2001)). We give a definition of the position value that uses the unanimity coefficients and avoids us having to introduce additional notation for the Shapley value.

The Position Value The position value is the allocation rule \( \pi \) according to which each player \( i \in N \) in a network situation \((g,v)\) receives half of the Shapley value of each of the links in which (s)he is involved, i.e., those in \( g_i = \{l \in g \mid l = \{i,j\} \text{ for some } j \in N\} \):\(^3\)

\[
\pi_i(g, v) = \sum_{g' \subseteq g} \frac{|g'_i|}{2|g'|} \lambda_g'(v).
\]

We will consider the position value as an allocation rule on the restricted domain \( G \times V^{ca} \) of network situations with component additive value functions as well as the domain \( G \times V \) of all network situations.

\(^3\)As is usual, the empty sum is defined to be equal to 0.
3.2 Properties of Allocation Rules

In this subsection, we provide the definitions of the extensions to a setting of network situations of the properties used by Borm et al. (1992) to axiomatize the position value on the set of networks that do not contain any cycles. We start by highlighting a property that was not used by Borm et al. (1992) and that we will not use in our axiomatizations, but that we find useful to highlight and use in the proof of Lemma 3 below.

Component Decomposability An allocation rule $Y$ on $G \times V$ ($G \times V^{ca}$) is component decomposable if

$$Y_i(g, v) = Y_i(g(S), v)$$

for all network situations $(g, v) \in G \times V^{ca}$ with a component additive value function, all components $(S, g(S)) \in C(g)$, and all players $i \in S$.

**Lemma 2** The position value is component decomposable.

**Proof.** Let $v \in V^{ca}$, $g \in G$, $(S, g(S)) \in C(g)$, and $i \in S$.

If $|S| = \{i\}$, then $g_i = \emptyset$ and $\pi_i(g, v) = \pi_i(g(S), v) = 0$ by definition of the position value.

Suppose $|S| \geq 2$. Then the following chain of equalities holds.

$$\pi_i(g, v) = \sum_{g' \subseteq g} \frac{|g'_i|}{2|g'|} \lambda_{g'}(v)$$

$$= \sum_{g' \subseteq g(S)} \frac{|g'_i|}{2|g'|} \lambda_{g'}(v) + \sum_{g' \subseteq g \setminus g(S)} \frac{|g'_i|}{2|g'|} \lambda_{g'}(v) + \sum_{g' \subseteq g: g' \cap g(S) \neq \emptyset \land g' \setminus g(S) \neq \emptyset \land g' \setminus g(S) \neq \emptyset} \frac{|g'_i|}{2|g'|} \lambda_{g'}(v)$$

$$= \sum_{g' \subseteq g(S)} \frac{|g'_i|}{2|g'|} \lambda_{g'}(v)$$

$$= \pi_i(g(S), v),$$

where the third equality follows from the facts that $|g'_i| = 0$ for all $g' \subseteq g \setminus g(S)$ (because $i \in S$ and $(S, g(S)) \in C(g)$) and $\lambda_{g'}(v) = 0$ for all $g' \subseteq g$ such that $g' \cap g(S) \neq \emptyset$ and $g' \setminus g(S) \neq \emptyset$ (Lemma 1). \qed
The following property, component efficiency, states that for every component of a network, the joint payoffs of its players equal the value of its links. Such a condition only makes sense if the value function is such that the value of the links in a component in the network is independent of the links that may or may not exist between players outside the component. Therefore, we limit the requirement in the definition of component efficiency to value functions that are component additive.\(^4\)

**Component Efficiency** An allocation rule \( Y \) on \( G \times V \) \((G \times V^{ca})\) is component efficient if
\[
\sum_{i \in S} Y_i(g, v) = v(g(S))
\]
for all network situations \((g, v) \in G \times V^{ca}\) with a component additive value function and all components \((S, g(S)) \in C(g)\).

**Lemma 3** The position value is component efficient.

**Proof.** Let \( v \in V^{ca}, g \in G, \) and \((S, g(S)) \in C(g)\). Then
\[
\sum_{i \in S} \pi_i(g, v) = \sum_{i \in S} \pi_i(g(S), v) = \sum_{i \in S} \sum_{g' \leq g(S)} \frac{|g'_i|}{2|g'|} \lambda_{g'}(v) = \sum_{g' \leq g(S)} \lambda_{g'}(v) = v(g(S)),
\]
where the first equality follows from component decomposability of the position value (Lemma 2). \(\square\)

Additivity is a property that specifies that there are no externalities in the allocation rule when the players in a network are involved in several situations, each described by their own value function.

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\(^4\) Component efficiency is the extension of the property with that name in Borm et al. (1992). Jackson and Wolinsky (1996) define a related property, called component balance, in a setting of value functions. The property defined here is related to, but not the same as component balance. The difference between the two properties is that component balance ignores isolated players.
Additivity An allocation rule $Y$ on $G \times V$ ($G \times V^{ca}$) is additive if

$$Y(g, v_1 + v_2) = Y(g, v_1) + Y(g, v_2)$$

for all network situations $(g, v_1), (g, v_2) \in G \times V$ ($G \times V^{ca}$).

Note that if $v_1, v_2 \in V^{ca}$ then $v_1 + v_2 \in V^{ca}$ as well, so that we can safely consider additivity of an allocation rule on the domain of network situations with component additive value functions.

Lemma 4 The position value is additive.

The additivity property of the position value follows easily from the well-known fact that the unanimity coefficients are additive in value functions, and we omit the proof of this lemma.

The superfluous link property states that the presence or absence of a link that has no influence on the value of any network, also has no influence on the players’ allocations in a network. A link $l \in g$ is superfluous in network situation $(g, v)$ if

$$v(g') = v(g' \{l\})$$

for all networks $g' \subseteq g$.

Superfluous Link Property An allocation rule $Y$ on $G \times V$ ($G \times V^{ca}$) satisfies the superfluous link property if

$$Y(g, v) = Y(g \{l\}, v)$$

for all network situations $(g, v) \in G \times V$ ($G \times V^{ca}$) and all links $l$ that are superfluous in $(g, v)$.

Lemma 5 The position value satisfies the superfluous link property.

Proof. Let $v \in V$, $g \in G$, and let $l \in g$ be superfluous in $(g, v)$.

Claim $\lambda_{g'}(v) = 0$ for all $g' \subseteq g$ with $l \in g'$.

For all $g' \subseteq g$ with $l \in g'$, we have

$$v(g') = v(g' \{l\}) = \sum_{g'' \subseteq g' \{l\}} \lambda_{g''}(v)$$
and

\[ v(g') = \sum_{g'' \subseteq g'} \lambda_{g''}(v) = \sum_{g'' \subseteq g': l \notin g'} \lambda_{g''}(v) + \sum_{g'' \subseteq g': l \in g'} \lambda_{g''}(v). \]

Subtracting the former equality from the latter one, we obtain

\[ \sum_{g'' \subseteq g': l \notin g''} \lambda_{g''}(v) = 0. \]  

(5)

Now we proceed by induction.

Suppose \( g' = \{l\} \). Then it follows from (5) that \( \lambda_{g'}(v) = 0 \).

Now, let \( k \geq 1 \) and suppose that we know that \( \lambda_{g'}(v) = 0 \) for all \( g' \subseteq g \) with \( l \in g' \) and \( |g'| \leq k \). Suppose \( g' \subseteq g \) is such that \( l \in g' \) and \( |g'| = k + 1 \). Then it follows from (5) and the induction hypothesis that

\[ 0 = \sum_{g'' \subseteq g': l \notin g''} \lambda_{g''}(v) = \lambda_{g'}(v). \]

This proves the claim.

Using the claim, we now derive

\[
\begin{align*}
\pi_i(g, v) &= \sum_{g' \subseteq g} \frac{|g'|}{2|g'|} \lambda_{g'}(v) \\
&= \sum_{g' \subseteq g: l \notin g'} \frac{|g'|}{2|g'|} \lambda_{g'}(v) + \sum_{g' \subseteq g: l \in g'} \frac{|g'|}{2|g'|} \lambda_{g'}(v) \\
&= 0 + \sum_{g' \subseteq g \setminus \{l\}} \frac{|g'|}{2|g'|} \lambda_{g'}(v) = \pi_i(g \setminus \{l\}, v).
\end{align*}
\]

\( \square \)

Link anonymity states that when all the links in a network are interchangeable for the purpose of determining the values of subnetworks, the relative allocations of the players in the network are determined by the relative number of links that each player is involved in. A value function \( v \in V \) is link anonymous on \( g \) if

\[ v(g') = v(g'') \]

for all subnetworks \( g', g'' \subseteq g \) that have the same number of links (i.e., \( |g'| = |g''| \)).\(^5\)

\(^5\)Note that link anonymity of a value function is different from anonymity of a value function. Anonymity of a value function is player-centered, whereas link anonymity is link-centered.
Link Anonymity An allocation rule \( Y \) on \( G \times V \) (\( G \times V^{ca} \)) is link anonymous if for every network \( g \in G \) and value function \( v \in V \) (\( V^{ca} \)) that is link anonymous on \( g \), there exists an \( \alpha \in \mathbb{R} \) such that

\[
Y_i(g, v) = \alpha |g_i|
\]

for all players \( i \in N \).

Lemma 6 The position value satisfies link anonymity.

Proof. Let \( g \in G \) and \( v \in V \) be link anonymous on \( g \). Then it is easily shown by induction to the number of links in a network that \( \lambda_{g'}(v) = \lambda_{g''}(v) \) for all \( g', g'' \subseteq g \) with \( |g'| = |g''| \). This, in turn, allows us to conclude that we can find a \( \beta \in \mathbb{R} \) such that

\[
\sum_{g' \subseteq g, l \in g'} \frac{\lambda_{g'}(v)}{|g'|} = \beta
\]

for all \( l \in g \). Using this, we derive

\[
\pi_i(g, v) = \sum_{g' \subseteq g} \frac{|g'|}{2|g'|} \lambda_{g'}(v) = \sum_{g' \subseteq g} \left( \sum_{l \in g'} \frac{\lambda_{g'}(v)}{2|g'|} \right) = \sum_{l \in g} \frac{1}{2} \left( \sum_{g' \subseteq g, l \in g'} \frac{\lambda_{g'}(v)}{|g'|} \right) = \sum_{l \in g} \frac{1}{2} \beta = \frac{\beta}{2} |g_i|
\]

for each \( i \in N \).

\[\square\]

3.3 Axiomatization

We are now able to provide an axiomatization of the position value analogous to the axiomatization of the Shapley value in Shapley (1953).

Theorem 1 The position value is the unique allocation rule on \( G \times V^{ca} \) that is component efficient and additive and satisfies the superfluous link property and link anonymity.
Proof. We have established in the preceding lemmas that π satisfies the 4 properties on the entire domain \( G \times V \). It follows easily from this that π satisfies these properties on the restricted domain \( G \times V^{ca} \) as well. We will now establish that there is only one allocation rule on \( G \times V^{ca} \) satisfying these 4 properties. Let \( Y : G \times V^{ca} \to \mathbb{R}^N \) be an allocation rule that is component efficient and additive and satisfies the superfluous link property and link anonymity. Let \( g \in G \) and \( v \in V^{ca} \). We know that \( v = \sum_{g' \in G, g' \neq \emptyset} \lambda_{g'}(v)u_{g'} \) and, by Lemma 1, \( \lambda_{g'}(v) = 0 \) for all networks \( g' \) that have two links that are not in the same component. Thus, additivity of \( Y \) implies that \( Y(g, v) = \sum_{g' \in G: (N(g'), g') \in C(g')} Y(g, \lambda_{g'}(v)u_{g'}) \). Therefore, it suffices to demonstrate that \( Y(g, \lambda_{g'}(v)u_{g'}) \) is uniquely determined for all \( g' \in G \) with the property that \( (N(g'), g') \in C(g') \).

Let \( g' \in G \) with \( (N(g'), g') \in C(g') \) and \( w := \lambda_{g'}(v)u_{g'} \). We distinguish two cases.

**Case 1** Suppose \( g' \setminus g \neq \emptyset \). Then \( w(g'') = 0 \) for all \( g'' \subseteq g \). Thus, every link \( l \in g \) is superfluous in \( (g'', w) \), for every \( g'' \subseteq g \) with \( l \in g'' \). Repeated application of the superfluous link property of \( Y \) allows us to conclude that \( Y(g, w) = Y(\emptyset, w) \). In the empty network, each component consists of a single isolated player. Component efficiency of \( Y \) then gives us \( Y_i(\emptyset, w) = w(\emptyset) = 0 \) for each \( i \in N \).

**Case 2** Suppose \( g' \subseteq g \). Let \( l \in g \setminus g' \) and \( g'' \subseteq g \) with \( l \in g'' \). Then

\[
w(g'') = \begin{cases} 
\lambda_{g'}(v) & \text{if } g' \subseteq g'' \\
0 & \text{if } g' \setminus g'' \neq \emptyset.
\end{cases}
\]

Thus, every link \( l \in g \setminus g' \) is superfluous in \( (g'', w) \), for every \( g'' \subseteq g \) with \( l \in g'' \). Repeated application of the superfluous link property (deleting the links is \( g \setminus g' \) one by one) of \( Y \) allows us to conclude that

\[
Y(g, w) = Y(g', w). \tag{6}
\]

The value function \( w \) is link anonymous on \( g' \). Hence, by link anonymity of \( Y \), we can find an \( \alpha \in \mathbb{R} \) such that for each \( i \in N \) it holds that

\[
Y_i(g', w) = \alpha |g_i'|. \tag{7}
\]

Because \( (N(g'), g') \in C(g') \), we know that \( \Pi(g') = \{N(g'), \{i\}_{i \in N \setminus N(g')}\} \).

We demonstrated in Corollary 1 that \( u_{g'} \) is component additive, so that we
know that \( w \) is component additive. Using component efficiency of \( Y \), we derive

\[
\sum_{i \in N(g')} Y_i(g', w) = w(g') = \lambda_{g'}(v).
\]

(8)

Taking (7) and (8) together, we derive

\[
\lambda_{g'}(v) = \sum_{i \in N(g')} Y_i(g', w) = \sum_{i \in N(g')} \alpha |g'_i| = 2\alpha |g'|,
\]

from which it readily follows that \( \alpha = \frac{\lambda_{g'}(v)}{2|g'|} \) must hold. This, together with (6) and (7), demonstrates that \( Y(g, w) \) is uniquely determined. \( \square \)

4 Alternative Axiomatizations

In this section, we provide alternative axiomatic characterizations of the position value that are obtained by requiring efficiency at the level of the grand coalition rather than at the component level. Unlike component efficiency, efficiency is a property that is more appealing to require also for value functions that are not necessarily component additive.

Efficiency An allocation rule \( Y \) on \( G \times V \) \((G \times V^{ca})\) is efficient if

\[
\sum_{i \in N} Y_i(g, v) = v(g)
\]

for all network situations \((g, v) \in G \times V \) \((G \times V^{ca})\).

Lemma 7 The position value is efficient.

Proof. Let \( v \in V \) and \( g \in G \). Then

\[
\sum_{i \in N} \pi_i(g, v) = \sum_{i \in N} \sum_{g' \subseteq g} \frac{|g'_i|}{2|g|} \lambda_{g'}(v) = \sum_{g' \subseteq g} \lambda_{g'}(v) = v(g),
\]

where the first equality follows from the definition of \( \pi \) and the second one from rearranging terms. \( \square \)

We can use efficiency to provide an axiomatization of the position value on a domain that includes all value functions.
Theorem 2 The position value is the unique allocation rule on $G \times V$ that is efficient and additive and satisfies the superfluous link property and link anonymity.

Proof. We have established in the preceding lemmas that $\pi$ satisfies the 4 properties. We will now establish that there is only one allocation rule on $G \times V$ satisfying these 4 properties. Let $Y : G \times V \rightarrow \mathbb{R}^N$ be an allocation rule that is efficient and additive and satisfies the superfluous link property and link anonymity. Let $g \in G$ and $v \in V$. We know that $v = \sum_{g' \in G, g' \neq \emptyset} \lambda_{g'}(v) u_{g'}$. Thus, additivity of $Y$ implies that $Y(g,v) = \sum_{g' \in G} Y(g,\lambda_{g'}(v) u_{g'})$. Therefore, it suffices to demonstrate that $Y(g,\lambda_{g'}(v) u_{g'})$ is uniquely determined for all $g' \in G$.

Let $g' \in G$ and $w := \lambda_{g'}(v) u_{g'}$. We distinguish two cases.

Case 1 Suppose $g' \cap g \neq \emptyset$. Then $w(g'') = 0$ for all $g'' \subseteq g$. Thus, $w$ is link anonymous on $g$, and by link anonymity of $Y$ we can find an $\alpha \in IR$ such that $Y_i(g,w) = \alpha |g_i|$ for all $i \in N$. In the empty network $|g_i| = 0$ for all $i$, which gives us $Y_i(\emptyset,w) = \alpha |g_i| = 0$ for all $i \in N$. If $g \neq \emptyset$, we can use efficiency of $Y$ to derive

$$0 = \sum_{i \in N} Y_i(g,w) = \sum_{i \in N} \alpha |g_i| = 2\alpha |g|.$$ 

From this it follows that $\alpha = 0$ must hold and thus $Y_i(g,w) = \alpha |g_i| = 0$ for all $i \in N$.

Case 2 Suppose $g' \subseteq g$. We can follow the corresponding part of the proof of Theorem 1 to derive (6) and (7). Using efficiency of $Y$, we derive

$$\sum_{i \in N} Y_i(g',w) = w(g') = \lambda_{g'}(v).$$ 

Taking (7) and (9) together, we derive

$$\lambda_{g'}(v) = \sum_{i \in N} Y_i(g',w) = \sum_{i \in N} \alpha |g'_i| = 2\alpha |g'|,$$

from which it readily follows that $\alpha = \frac{\lambda_{g'}(v)}{2|g'|}$ must hold. This, together with (6) and (7), demonstrates that $Y(g,w)$ is uniquely determined. \qed
We can also use efficiency instead of component efficiency to axiomatize the position value on the domain that includes only network situations with component additive value functions. On this restricted domain efficiency clearly is a weaker axiom than component efficiency because both axioms apply to the same network situations, namely those with a component additive value function, but for these network situations component efficiency implies efficiency. Nevertheless, efficiency suffices to axiomatize the position value on the set of network situations with component additive value functions, as stated in following theorem.

**Theorem 3** The position value is the unique allocation rule on $G \times V^{ca}$ that is efficient and additive and satisfies the superfluous link property and link anonymity.

**Proof.** The proof of this Theorem is a combination of the proofs of Theorems 1 and 2 in the following way: As in the proof of Theorem 1 we can show that it suffices to establish that $Y(g, \lambda_{g'}(v)u_{g'})$ is uniquely determined for all $g' \in G$ with the property that $(N(g'), g') \in C(g')$. We can establish this statement by following Cases 1 and 2 in the proof of Theorem 2. □

Because efficiency is a weaker axiom than component efficiency, Theorem 1 is implied by Theorem 3. Note, however, that there is no such relationship between Theorems 1 and 2, because the domain of Theorem 2 is larger than that of Theorem 1 and the position value does not necessarily satisfy a condition as in component efficiency for value functions that are not component additive. This is illustrated in the following example.

**Example 1** Let $N = \{1, 2, 3, 4\}$ and consider the value function $v$ that assign a value $1$ to a network consisting of one link, a value $4$ to any network consisting of two links that do not have any players in common (i.e., any 2-link network on $N$ that has two components - each consisting of two players connected by one link), and a value $0$ to all other networks. This value function is not component additive because, for example, $v(\{12, 34\}, v) \neq$ we adopt the common practice of denoting a link between any two players $i$ and $j$ by $ij$ rather than $\{i, j\}$. 
\(v\{12\} + v\{34\}\). The value function is anonymous, however, and the position value of each player \(i \in N\) in the network \(g = \{12,34\}\) equals \(\pi_i(g,v) = 1\). This violates the condition in component efficiency because, for instance, for \(S = \{1,2\} \in \Pi(g)\) we have \(g(S) = \{12\}\) and in the component \((S,g(S))\), it holds that \(\pi_1(g,v) + \pi_2(g,v) = 2 > 1 = v(\{12\})\).

Note that for this non-component additive value function, the position value also does not satisfy the condition in component decomposability because \(\pi_1(g(S),v) = \frac{1}{2} \neq 1 = \pi_1(g,v)\).

References


