1. Using the “$\varepsilon$-$\delta$” characterization of continuity prove that $f(x) = x(x-1)$ is continuous at $x_0 = 2$.

   **Proof.** Let $\varepsilon > 0$. Choose $\delta = \min\{1, \varepsilon/4\}$. Then $|x-2| < \delta \implies 1 < x < 3 \implies |x+1| = x + 2 < 4$. So
   
   
   $|x-2| < \delta \implies |f(x) - f(2)| = |x^2 - x - 2| = |x+1||x-2| < 4|x-2| < 4\delta \leq \varepsilon.$

2. Given that $f$ is a continuous function on $\mathbb{R}$ with $f(5) = 3$, show that there exists some $a > 0$ such that $f(x) > 2$ for all $x \in (5-a, 5+a)$.

   **Proof.** Let $\varepsilon = 1$. Then $\exists \delta = a > 0$ such that $|x-5| < a \implies |f(x) - f(5)| = |f(x) - 3| < 1 \implies f(x) - 3 > -1$. So $x \in (5-a, 5+a) \implies f(x) > 3-1 = 2$.

3. Prove that the equation $2^x = x^2$ has at least one solution in $\mathbb{R}$.

   **Proof.** Let $f(x) = 2^x - x^2$. Then $f$ is continuous on $\mathbb{R}$. But
   
   
   $f(-1) = 1/2 - 1 < 0$ and $f(1) = 2 - 1 > 0$,

   so by the IVT $\exists x_0 \in (-1, 1)$ with $f(x_0) = 0$ or $2^{x_0} = x_0^2$.

4. Consider the infinite series

   $$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2n-1}} + \frac{1}{3^{2n}} + \cdots$$

   (a) Show that the ratio test gives no information about the convergence of the series.

   Observe that $\frac{a_{n+1}}{a_n} = \left\{ \begin{array}{ll} \frac{1}{3} \left(\frac{2}{3}\right)^{2k-1}, & \text{if } n = 2k, \\ \frac{1}{2} \left(\frac{3}{2}\right)^{2k}, & \text{if } n = 2k + 1. \end{array} \right.$

   So

   $$\lim \inf \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1 < \lim \sup \left| \frac{a_{n+1}}{a_n} \right| = +\infty,$$

   and the ratio test fails.

   (b) Show that the series converges.

   Since $a_n \leq \frac{1}{2^n} \forall n \in \mathbb{N}$, and the geometric series $\sum 2^{-n}$ converges, we have from the Comparison Test that $\sum a_n$ converges.