17.7. (a) Clear

(b) [Note: This problem is not clear — the solution could be by using an $\varepsilon$-$\delta$ attack or by using sequences. We'll do both.]

(i) Let $\varepsilon > 0$ and let $x_0 \in \mathbb{R}$. If $\delta < \varepsilon$, then $|x - x_0| < \delta \implies$

$$|f(x) - f(x_0)| = ||x| - |x_0|| \overset{\text{Ex.3.3}}{\leq} |x - x_0| < \delta < \varepsilon.$$

(ii) Since $|x| = \begin{cases} x \text{ if } x > 0; \\ -x \text{ if } x \leq 0, \end{cases}$ $|x|$ is continuous for all $x_0 \neq 0$. But if $x_0 = 0$ and $(x_n)$ is a sequence with $\lim x_n = 0$, then (by Exercise 8.6) $\lim f(x_n) = 0$.

17.9. (b) Claim that $f(x) = \sqrt{x}$ is continuous at $x_0 = 0$. Let $\varepsilon > 0$ and $\delta = \varepsilon^2$. Then $\forall x > 0, |x - x_0| = x \leq \delta = \varepsilon^2 \implies |\sqrt{x} - \sqrt{0}| = |\sqrt{x}| < \sqrt{\delta} = \varepsilon$.

(c) Claim that if $f(x) = x \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$, then $f$ is continuous at $x_0 = 0$. Let $\varepsilon > 0$. Then for $\delta = \varepsilon$, $\forall x \in \mathbb{R}$

$$|x - x_0| = |x| \leq \delta \implies |f(x) - f(x_0)| = |x \sin(1/x)| \leq |x| < \delta = \varepsilon.$$

17.10. (a) $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$ is discontinuous at $x_0 = 0$. Two versions.

Proof 1. Let $\varepsilon = 1/2$. So for every $\delta > 0$, if $x = \delta/2$, then $|x - x_0| = \delta/2 < \delta$, but $|f(x) - f(x_0)| = 1 \not\leq \varepsilon$.

Proof 2. Let $x_n = 1/n$. Then $\lim x_n = x_0$, but $\lim f(x_n) = 1 \neq f(x_0)$.

(b) Claim $\text{sgn}$ is discontinuous at $x_0 = 0$.

Proof 1. Let $\varepsilon = 1/2$. So for every $\delta > 0$, if $x = \delta/2$, then $|x - x_0| = \delta/2 < \delta$, but $|f(x) - f(x_0)| = 1 \not\leq \varepsilon$.

Proof 2. Let $x_n = 1/n$. Then $\lim x_n = x_0$, but $\lim f(x_n) = 1 \neq x_0$.

17.12. (a) Let $x \in \mathbb{R}$. Then there is a sequence $(x_n)$ in $\mathbb{Q}$ that converges to $x$. So by continuity $f(x) = \lim f(x_n) = 0$.

(b) Since $f - g$ is continuous and $f(x) - g(x) = 0$ for all $x \in \mathbb{Q}$, we have that $f - g = 0$ by part (a) and so $f(x) = g(x)$ for all $x \in \mathbb{R}$.

17.13. (a) Let $x_0 \in \mathbb{R}$. Set $\varepsilon = 1/2$. Then for every $\delta > 0$ there is some $x, y \in (x_0 - \delta, x_0 + \delta)$ with $x \in \mathbb{Q}$ and $y \not\in \mathbb{Q}$. So $|x - x_0| < \delta$ and $|y - x_0| < \delta$, yet $|f(x) - f(x_0)| \geq \varepsilon$ or $|f(y) - f(x_0)| \geq \varepsilon$.

(b) For every sequence with $\lim x_n = 0$ we have $0 \leq h(x_n) \leq x_n$ for all $n$, so $\lim h(x_n) = 0 = h(x_0)$. On the other hand if $x_0 \neq 0$, then there are sequences $(x_n)$ and $(y_n)$, each converging to $x_0$, with each $x_n \in \mathbb{Q}$ and $y_n \not\in \mathbb{Q}$, so that $\lim h(x_n) \neq \lim h(y_n)$, and $h$ is not continuous at $x_0$. 

Section 18.

18.4. Consider the function \( f(x) = 1/(x - x_0) \) for all \( x \neq x_0 \). Since \( g(x) = x - x_0 \) is continuous for all \( x \) and \( g(x) = 0 \) only at \( x_0 \), \( f(x) \) is continuous for all \( x \neq x_0 \), and clearly \( \lim |f(x_n)| = +\infty \).

18.6. Let \( f(x) = x - \cos x \). Then \( f \) is continuous and \( f(0) = -1 \) and \( f(\pi/2) = 0 \), so by the IVT there is some \( x \in (0, \pi/2) \) with \( f(x) = 0 \).

18.8. Since \( f(a)f(b) < 0 \), either \( f(a) > 0 \) or \( f(b) > 0 \), say \( f(a) > 0 \). Then \( f(b) < 0 \). So by the IVT there is some \( x \) between \( a \) and \( b \) with \( f(x) = 0 \).

18.10. Consider \( g(x) = f(x + 1) - f(x) \) on \([0, 1]\). Then either \( g(0) = 0 \), so \( f(1) = f(0) \), or \( g(0) \neq 0 \) and \( g(0)g(1) < 0 \) so by Exercise 18.8 \( \exists \) some \( x \) in \([0, 1]\) with \( g(x) = 0 \), so \( x \) and \( y = x + 1 \) satisfy \( f(x) = f(y) \) with \(|x - y| = 1|\).

Section 19.

19.1. (a) and (b) are uniformly continuous since the interval is closed and bounded and the functions are continuous.
(c) and (g) are uniformly continuous on \((0, 1]\) since each is continuous on the closed bounded interval \([0, 1]\).
(d) is not uniformly continuous on \( \mathbb{R} \). This follows from Exercise 19.4, below.
(e) and (f) are not uniformly continuous on \((0, 1]\) since neither can be extended to a continuous function on \([0, 1]\).

19.2. (b) Let \( \varepsilon > 0 \) and choose \( \delta < \min\{1, \varepsilon/7\} \). Then for all \( x, y \in [0, 3] \) \( |x - y| < \delta \implies |x + y| \leq 7 \) and so
\[
|x - y| < \delta \implies |x^2 - y^2| = |x + y||x - y| \leq 7\delta < \varepsilon.
\]

19.4. (a) Assume \( f \) is unbounded on a bounded set \( S \). W.m.a. that \( f \) is unbounded above. So \( \exists \) a monotone increasing sequence \((x_n)\) in \( S \) with \( f(x_n) > n \) for each \( n \in \mathbb{N} \). Let \( \varepsilon = 1 \). Since \( S \) is bounded, the monotone sequence \((x_n)\) is convergent, and hence Cauchy. So for each \( \delta > 0 \) there is an \( N \) such that \( m > n > N \) implies \( |x_n - x_m| < \delta \). But clearly since \( \lim f(x_n) = +\infty \), there exist \( m > n > N \) with \( f(x_m) - f(x_n) > 1 = \varepsilon \).
So \( f \) is not uniform continuous on \( S \).
(b) Since \( f(x) = 1/x^2 \) is unbounded on \((0, 1]\), by part (a), it cannot be uniformly continuous there.