8.2. (a) \( \lim a_n = \lim \frac{n}{n^2 + 1} = 0. \)

**Proof.** Let \( \varepsilon > 0 \) Then
\[
|a_n - 0| = \frac{n}{n^2 + 1} \leq \frac{n}{n^2} = \frac{1}{n} < \varepsilon \text{ if } n > \frac{1}{\varepsilon}. 
\]

(c) \( \lim c_n = \lim \frac{4n + 3}{7n - 5} = \frac{4}{7}. \)

**Proof.** Let \( \varepsilon > 0. \) Then
\[
|c_n - \frac{4}{7}| = \left| \frac{4n + 3}{7n - 5} - \frac{4}{7} \right| = \left| \frac{21 + 20}{49n - 35} \right| \leq \frac{41}{49(n - 1)} \leq \frac{1}{n - 1} < \varepsilon
\]
if \( n > 1 + \frac{1}{\varepsilon} = \frac{1 + \varepsilon}{\varepsilon}. \)

(e) \( \lim s_n = \lim \frac{1}{n} \sin n = 0. \)

**Proof.** Let \( \varepsilon > 0. \) Then
\[
|s_n - 0| = \left| \frac{\sin n}{n} \right| \leq \frac{|\sin n|}{n} \leq \frac{1}{n} < \varepsilon
\]
if \( n > \frac{1}{\varepsilon}. \)

8.4. Given \( |t_n| < M \) for all \( n \in \mathbb{N} \) and \( \lim s_n = 0, \) then \( \lim (s_n t_n) = 0. \)

Let \( \varepsilon > 0, \) so \( \frac{\varepsilon}{M} > 0. \) Thus, \( \exists N \) such that \( n > N \implies |s_n - 0| = |s_n| < \varepsilon/M. \)
So \( n > N \implies |s_n t_n - 0| = |s_n||t_n| \leq M|s_n| < M\frac{\varepsilon}{M} = \varepsilon. \)

8.5. (a) Given \( a_n \leq s_n \leq b_n \) for all \( n \in \mathbb{N} \) and \( \lim a_n = \lim b_n = s, \) then \( \lim s_n = s. \)

**Proof.** Let \( \varepsilon > 0. \) Then there exists \( N \) such that \( n > N \implies s - \varepsilon < a_n < s + \varepsilon \) and \( s - \varepsilon < b_n < s + \varepsilon. \) So \( n > N \implies s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon \) and so \( |s_n - s| < \varepsilon. \)

(b) If \( |s_n| \leq t_n \) for all \( n \in \mathbb{N} \) and \( \lim t_n = 0, \) then \( \lim s_n = 0. \)

**Proof.** We have \( 0 \leq |s_n| \leq t_n \) for all \( n \in \mathbb{N}, \) so by part (a), \( \lim |s_n| = 0. \) But \( -|s_n| \leq s_n \leq |s_n| \) for all \( n \in \mathbb{N}, \) so again by part (a) \( \lim s_n = 0. \)
9.4. (a) \(1, \sqrt{2}, \sqrt{2} + 1, \sqrt{\sqrt{2} + 1} + 1\).

(b) If \(\lim s_n = s\), then since \(s_n > 0\) for all \(n \in \mathbb{N}\), we have \(s \geq 0\) and
\[
s^2 = \lim s_{n+1}^2 = \lim (\sqrt{s_n} + 1)^2 = (\sqrt{\lim s_n} + 1)^2 = s + 1,
\]
so \(s^2 - s - 1 = 0\) and \(s = \frac{1}{2}(1 + \sqrt{5})\).

9.5. If \(\lim t_n = t\), then
\[
t = \lim t_n = \lim \left(\frac{t_n^2 + 2}{2t_n}\right) = \frac{t^2 + 2}{2t},
\]
so \(2t^2 = t^2 + 2\) or \(t^2 = 2\) and \(t = \sqrt{2}\).

9.10. (a) \(\lim s_n = +\infty\) and \(k > 0 \implies \lim ks_n = +\infty\).

**Proof.** Let \(M > 0\). Then \(\exists N\) such that \(n > N \implies s_n > M/k \implies ks_n > k \left(\frac{M}{k}\right) = M\).

(b) \(\lim s_n = +\infty \iff (\forall M > 0, \exists N\) such that \(n > N \implies s_n > M\) \iff (\forall M > 0 \exists N\) such that \(n > N \implies -s_n < -M) \iff \lim(-s_n) = -\infty\).

(c) \(\lim s_n = +\infty\) and \(k < 0 \implies -k > 0 \implies (a) \lim(-k)s_n = \infty \implies (b) \lim ks_n = -\infty\).

9.15. \(\lim \frac{a^n}{n!} = 0\).

**Proof.** By Archimedes \(\exists m \in \mathbb{N}\) with \(|a| < m\). Set \(b = |a|/m\), so \(0 \leq b < 1\).
Set \(K = |a|^m/m!\). Then for all \(n > m\), \(|a|/n < b < 1\) and
\[
0 \leq \frac{|a|^n}{n!} = K \frac{|a|^m}{m!} \frac{|a|}{m+1} \frac{|a|}{m+2} \cdots \frac{|a|}{n} \leq K(b^{n-m}).
\]
But \(K \lim b^{n-m} = 0\) since \(0 \leq b < 1\). So by Exercise 8.5, \(\lim |a^n|/n! = 0\), and so \(\lim a^n/n! = 0\).

10.2. Suppose \(s_1 \geq s_2 \geq \cdots \geq s_n \geq \cdots \geq M\). Then by Completeness, \(s = \inf\{s_n : n \in \mathbb{N}\} \in \mathbb{R}\) (and \(s \geq M\)). We claim that \(\lim s_n = s\). Indeed, let \(\varepsilon > 0\). Then \(s + \varepsilon > s\), so \(\exists m \in \mathbb{N}\) with \(s + \varepsilon > s_m \geq s\). But the sequence is monotone, so for all \(n > m\), \(s + \varepsilon > s_m \geq s_n \geq s - \varepsilon\), or \(|s_n - s| < \varepsilon\).

10.4. Suppose that \(s_1 \geq s_2 \geq \cdots\) is not bounded. Then for each \(M > 0\), \(-M\) is not a lower bound, so there exists some \(m \in \mathbb{N}\) with \(s_m < -M\). Then \(n > m \implies s_n \leq s_m < -M\).

10.7. Suppose that \(S \neq \emptyset\) is bounded. Then by Completeness \(\exists s = \sup S \in \mathbb{R}\). If \(s \in S\), then the constant sequence \((s)\) is non-decreasing in \(S\) with limit \(s\). If \(s \notin S\), then we define a non-decreasing sequence \((s_n)\) in \(S\) by recursion:
(i) $s - 1 < s$ so $\exists s_1 \in S$ with $s - 1 < s_1 < s$;

(ii) Assume that $s_1 \leq s_2 \leq \cdots \leq s_n$ in $S$ with each $s_k > s - 1/k$. If we set $t = \max\{s_n, s - 1/(n+1)\}$, then $t < s$, so there is some $s_{n+1} \in S$ with $s - 1/(n+1) \leq t < s_{n+1} < s$ and $s_n \leq s_{n+1}$.

Then $(s_n)$ is non-decreasing sequence in $S$ with $s - 1/n \leq s_n < s$ for all $n \in \mathbb{N}$, so (by Exercise 8.5), $\lim s_n = s$.

10.10. Given $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for all $n \geq 1$.

(a) $s_2 = 2/3, s_3 = 5/9$ and $s_4 = 14/27$.

(b) (B) $s_1 = 1 > 1/2$.

(IS) Let $n \geq 1$ and assume $s_n > 1/2$. Then

\[ s_{n+1} = \frac{1}{3}(s_n + 1) \overset{(IH)}{> \frac{1}{3}(\frac{1}{2} + 1)} = \frac{1}{2}. \]

(c) $s_n - s_{n+1} = s_n - \frac{1}{3}(s_n + 1) = \frac{2}{3}s_n - \frac{1}{3} > \frac{1}{3} - \frac{1}{3} = 0$, so $s_n > s_{n+1}$.

(d) So $(s_n)$ is decreasing and bounded by $1/2$, so $\lim s_n = s$ exists by Exercise 10.2. But then

\[ s = \lim s_{n+1} = \frac{1}{3}(\lim s_n + 1) = \frac{1}{3}(s + 1), \]

whence $\frac{2}{3}s = \frac{1}{3}$, and $s = \frac{1}{2}$.