1. For each of the following series decide whether or not it converges, and give reasons for your answers.

(a) \( \sum_{n=1}^{\infty} \frac{n!}{2^n} \),  
(b) \( \sum_{n=2}^{\infty} \frac{(-1)^n}{\log n} \),  
(c) \( \sum_{n=2}^{\infty} \frac{1}{n \log n} \),  
(d) \( \sum_{n=1}^{\infty} \left[ \frac{\sqrt{n+1} - \sqrt{n}}{n} \right] \).

2. A series \( \sum a_n \) of positive terms satisfies \( \lim \sup(n^2 a_n) = 4 \). Prove that the series \( \sum a_n \) converges. [Hint: What can you say about \( n^2 a_n \) compared to 5.]

3. A series \( \sum a_n \) satisfies the condition that for each \( m, n \in \mathbb{N} \)

\[
\sum_{k=n}^{n+m} |a_k| < \frac{1}{n}.
\]

Show that the series \( \sum a_n \) must converge.

4. Using an \( \varepsilon-\delta \) argument prove that \( f(x) = 5x^2 \) is continuous at \( x = 3 \).

5. Prove that if \( f(x) \) is continuous on \([0, 1]\) with \( 1 \geq f(0) > f(1) \geq 0 \), then \( f(c) = c^2 \) for some \( c \) in \([0, 1]\).

6. Let \( f \) be nonnegative and continuous on \([0, 1]\). Prove that \( f \) is bounded above on \([0, 1]\).

7. Prove that if \( \sum a_n \) converges, then \( \lim_{n \to \infty} \left( \sum_{k=n}^{\infty} a_k \right) = 0 \).

8. A function \( f \) on \( \mathbb{R} \) satisfies \( f(2^{-n}) = (-1)^n \) for every integer \( n \). Prove that \( f \) cannot be continuous.

9. Prove that if a sequence \( (s_n) \) has \( \lim \sup s_n = s \in \mathbb{R} \), then \( s \) is a subsequential limit of \( (s_n) \).

10. Let \( S = \{ m/2^n : m, n \in \mathbb{N} \text{ and } 2^n < m \} \). Explain why there exists an enumeration \( (s_n) \) of the elements of \( S \).

(a) Find the set of subsequential limits of \( (s_n) \).

(b) Find \( \lim \sup \left[ \left( \sum_{k=0}^{n} \frac{5}{2^k} \right) s_n \right] \) and justify your answer.

11. Prove that \( f(x) = \sqrt{x} \) is uniformly continuous on \([2, +\infty)\).

12. For each of the following statements decide whether it is true or false and give a brief reason for your decision.

(a) If every rational number in \((0, 1)\) is a subsequential limit of \( (s_n) \), then every number in \([0, 1]\) is a subsequential limit of \( (s_n) \).

(b) For all bounded sequences \( (s_n), (t_n) \), \( \lim \sup (s_n t_n) = (\lim \sup s_n)(\lim \sup t_n) \).

(c) If \( f \) is continuous on \([0, 1]\), then it is uniformly continuous on \((0, 1)\).
Solutions.


2. There is some $N$ with $n > N \implies n^2a_n < 5 \implies a_n < 5/n^2$. But $\sum 5/n^2$ converges, so by the Comparison Test, $\sum a_n$ converges.

3. Let $\varepsilon > 0$. Set $N = 1/\varepsilon$. Then $m \geq n > N \implies \left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| < 1/n \leq \varepsilon$, so by the Cauchy Criterion, $\sum a_n$ converges.

4. First, note that $|x-3| < 1 \iff 2 < x < 4 \iff 5 < x + 3 < 7$. So let $\varepsilon > 0$. Then set $\delta = \min\{1, \varepsilon/35\}$. So $|x-3| < \delta \implies |f(x) - f(3)| = 5|x + 3||x - 3| < 5 \cdot 7 \cdot \varepsilon/35 = \varepsilon$.

5. Let $g(x) = f(x) - x^2$. Then $g(x)$ is continuous on $[0, 1]$. But $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$. So by the Intermediate Value Theorem, there is some $c \in [0, 1]$ with $g(c) = 0$. That is, $f(c) = c^2$.

6. Suppose not. Then for each $n \in \mathbb{N}$, $\exists x_n \in [0, 1]$ with $f(x_n) > n$. Thus, by Bolzano-Weierstrass the bounded sequence $(x_n)$ has a convergent subsequence $(x_{\sigma(n)})$. Say $x_0 = \lim_{n \to \infty} x_{\sigma(n)}$. Since $0 \leq x_n \leq 1$ for all $n$, $0 \leq x_0 \leq 1$. And since $f$ is continuous, $f(x_0) = \lim_{n \to \infty} f(x_{\sigma(n)}) = \infty$, a contradiction.

7. Since $\sum a_n$ converges, each series $\sum_{k=n}^{\infty} a_k$ converges. Let $\varepsilon > 0$. By the Cauchy Criterion, $\exists N$ such that $m \geq n > N \implies \left| \sum_{k=n}^{m} a_k \right| < \varepsilon/2$, so $\sum_{k=n}^{\infty} a_k = \lim_{m \to \infty} \sum_{k=n}^{m} a_k \leq \varepsilon/2 < \varepsilon$.

8. $\lim 2^{-n} = 0$, but $\lim f(2^{-n}) = \lim(-1)^n$ does not exist.

9. For every $\varepsilon > 0$ there is a tail of $(s_n)$ in $(s - \varepsilon, s + \varepsilon)$; hence there are infinitely many $s_n$ in $(s - \varepsilon, s + \varepsilon)$.

10. Since $S \subseteq \mathbb{Q}$, and there exists an enumeration of $\mathbb{Q}$, some subsequence is an enumeration of $S$.

   (a) Let $0 \leq a < b \leq 1$. By Archimedes, $\exists n \in \mathbb{N}$ with $2^n(b-a) > 1$, so $\exists m \in \mathbb{N}$ with $2^ma < m < 2^nb$ and $a < m/2^n < b$. So $S$ is dense in $[0, 1]$. Thus, for each $x \in [0, 1]$, every neighborhood of $x$ contains infinitely many elements of $S$. Therefore $[0, 1]$ is the set of ssl’s of the enumeration $(s_n)$.

   (b) The sequence is $(5/2(1 - 2^{-n-1})s_n)$, and $\lim 5/2(1 - 2^{-n-1}) = 5/2$. So $5/2[0, 1] = [0, 5/2]$ is the set of ssl’s of the sequence $(5/2(1 - 2^{-n-1})s_n)$. Thus the desired limsup is $5/2$.

11. Let $\varepsilon > 0$. Then set $\delta = \varepsilon$. So $\forall x, y \in [2, \infty)$, $|x - y| < \delta \implies |\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x-y|}{2\sqrt{2}} < \delta = \varepsilon$.

12. (a) True. The rationals are dense in $[0, 1]$.

   (b) False. Consider $(s_n) = (1, 0, 1, 0, 1, 0, \ldots)$ and $(t_n) = (0, 1, 0, 1, 0, 1, \ldots)$, so $\lim \sup s_n t_n = 0$ and $(\lim \sup s_n)(\lim \sup t_n) = 1$.

   (c) True. Given $\varepsilon > 0$ the $\delta > 0$ that works on $[0, 1]$ works as well on $(0, 1)$.