Introduction. This is intended as something of a review and/or summary of many of the main topics that we have covered this term. Students often fail to appreciate the importance of stating definitions and results in clear precise mathematical language. Vague or even slightly inaccurate statements are essentially worthless and usually lead to real disasters. Thus much of this review is given in the form of questions asking you to state the answers in correct mathematical form. The level of your success with this enterprise should serve as a pretty good measure of your understanding of the material. And, incidentally, most of these “questions” would be entirely appropriate for the final exam!

1. Let \( \emptyset \neq S \subseteq \mathbb{R} \).

(a) State what is meant, if it exists, by the minimum/maximum of \( S \).

(b) Define the infimum/supremum of \( S \). (Be careful; there is more than one case.) Need these exist? Explain.

(c) Give at least two alternate characterizations of \( \inf S \), of \( \sup S \).

2. (a) State

The Induction Axiom (= (N5)); The Completeness Axiom; The Archimedean Property.

(b) If \( S \subseteq T \subseteq \mathbb{R} \), state what it means for \( S \) to be dense in \( T \). Give some examples of density and non-density.

3. (a) For a sequence \((s_n)\) of real numbers define what it means for

\[(s_n) \text{ to converge to } L \in \mathbb{R}; \quad \lim s_n = \{+\infty; \quad (s_n) \text{ to diverge}; \quad (s_n) \text{ to be Cauchy}.\]

(b) Give at least two alternate characterizations of: “\((s_n)\) converges”.

4. (a) For a sequence \((s_n)\) of real numbers define what it means

for \((s_n)\) to be monotone; \quad for \((s_n)\) to be bounded \quad to be a subsequence of \((s_n)\).

(b) State the limiting behavior of a monotone sequence.

(c) For a sequence \((s_n)\) of real numbers define

a subsequential limit of \((s_n)\); \quad \lim \inf s_n; \quad \lim \sup s_n;

(d) State a characterization of a subsequential limit.

(e) State the Bolzano-Weierstrass Theorem.

(e) State the relation of \(\lim \inf s_n\) and \(\lim \sup s_n\) to the set of ssl’s of a sequence \((s_n)\).

(d) For a sequence \((s_n)\) state the limiting behavior of the sequences \(|s_n|^{1/n}\) and \(|s_{n+1}/s_n|\).

5. (a) For a sequence \((f_n)\) of functions on \( S \) define what it means for \((f_n)\) to

converge pointwise to \( f \) on \( S \); \quad converge uniformly to \( f \) on \( S \); \quad be uniformly Cauchy on \( S \).

(b) If a sequence \((f_n)\) of functions continuous on \( S \) converge pointwise or uniformly to \( f \) on \( S \), state what can be said about the continuity of \( f \). [There are various possibilities.]


6. (a) Let $f$ be a function with $\text{dom } f \subseteq \mathbb{R}$ and let $a \in \mathbb{R} \cup \{\pm \infty\}$. Define what it means for $f$ to be continuous at $a$; 
\[ \lim_{x \to a^-} f(x) = L \in \mathbb{R}; \quad \lim_{x \to a^+} f(x) = -\infty. \]
(b) Give an $\varepsilon/M-\delta/M$ characterization of each of the properties of part (a).
(c) State the Intermediate Value Theorem, the Extreme Value Theorem.

7. (a) Let $f$ be a function on $S \subseteq \mathbb{R}$. Define what it means for $f$ to be uniformly continuous on $S$.
(b) If $S$ is a bounded interval, characterize what it means for $f$ to be uniformly continuous on $S$.

8. (a) Define what it means for a series $\sum a_n$ of real numbers to converge; converge absolutely; diverge; to satisfy the Cauchy criterion.
(b) State all of the tests you can for the convergence or absolute convergence or divergence of a series $\sum a_n$ of real numbers.

9. (a) Define what is meant by a power series.
(b) For a power series define what is meant by its radius of convergence and by its interval of convergence.
(c) State various ways to compute the radius of convergence of a power series.
(d) Given a power series state what you can about the continuity, differentiability, and integrability of the series on its interval of convergence. [For example, state Abel’s Theorem.]

10. (a) For a series $\sum f_n$ of functions on $S \subseteq \mathbb{R}$, define what it means for the series to converge uniformly to $f$ on $S$; to satisfy the uniform Cauchy criterion on $S$.
(b) State the Weierstrass test for the uniform convergence of a series $\sum f_n$ of functions on some $S \subseteq \mathbb{R}$.
(c) A useful test for the divergence of a series $\sum a_n$ of real numbers is that convergence implies that $\lim |a_n| = 0$. State the analogue of this property for series $\sum f_n$ of functions.

Post-Mortem. In no way is this a comprehensive overview of what we’ve done this term. For example, most of the results we saw this term, properties of the real numbers, limits, continuity, etc., etc., have not been included. So as part of your review of the course, we strongly suggest that you put together a similar summary of the results and techniques (e.g., induction) that we’ve dealt with. And as far as cementing your mastery of these ideas and facts, you should create whole bunches of illustrative examples. One thing that usually proves very valuable in getting thoroughly comfortable with some area of mathematics is to be sure you have plenty of examples of the results. And, of course, for every theorem of the form $H \implies C$ you should know and understand the status of its converse.