Solutions.

1. (a) \( \lim_{n \to \infty} \frac{\cos n}{2^n} = 0 \) since \(|(\cos n)/2^n| \leq 2^{-n} \) and \( \lim 2^{-n} = 0; \)
(b) \( \lim_{n \to \infty} (2n)^{1/n} = \lim(2)^{1/n} \lim n^{1/n} = 1^2 = 1. \)

2. \( \sup S = \sqrt[3]{\pi} \) and \( \inf S = -\infty. \)

3. Define a sequence recursively by: \( s_1 = 1/2 \) and \( s_{n+1} = \left( \frac{n}{n+1} \right) s_n \) \( \forall n \in \mathbb{N}. \)
Claim: \( s_n = 1/(2n) \). Then (B) \( s_1 = 1/2 = 1/(2 \cdot 1) \) so true for \( n = 1. \) (IS) Let \( n \geq 1 \) and assume \( s_n = 1/(2n) \). Then \( s_{n+1} = \left( \frac{n}{n+1} \right) s_n = \left( \frac{n}{n+1} \right) \frac{1}{2n} = \frac{1}{2(n+1)}, \) so true for \( n+1. \) Thus, the claim is true for all \( n \in \mathbb{N}. \)

4. (a) We claim: \( 0 < s_n < 5/2^{n-1} \) for all \( n \in \mathbb{N}. \) By induction. Then (B) \( 0 < s_1 < 5 = 5/2^{1-1}, \) so true for \( n = 1. \) (IS) Let \( n \geq 1 \) and assume the claim is true for \( n. \) Then \( 0 < s_{n+1} < 1/2 \cdot s_n < 1/2 \cdot 5/2^{n-1} = 5/2^n, \) so true for \( n+1. \) Thus, the claim is true for all \( n \in \mathbb{N}. \)
(b) Since \( 0 < s_n < 5/2^{n-1} \) \( \forall n \in \mathbb{N}, \) and \( \lim 5/2^{n-1} = 0, \) we have \( 0 < \lim s_n \leq 0, \) so \( \lim s_n = 0. \)

5. Since \( a + 1/n > a, \) \( a \) is a lower bound of \( S = \{a + \frac{1}{n} : n \in \mathbb{N}\}. \) Let \( b > a. \) By Archimedes there is some \( n \in \mathbb{N} \) with \( n(b-a) > 1 \) or \( b > a + 1/n. \) Thus, \( b \) is not a lower bound of \( S, \) so \( a \) is the greatest lower bound inf \( S. \)

6. (a) \( x \in (-S)^- \iff x \leq -s \forall s \in S \iff -x \geq s \forall s \in S \iff -x \in S^+ \iff x \in -(S^+). \)
(b) \( -(S^+) = -[\sup S, \infty) = (-\infty, -\sup S] \) and \( -(S)^- = (-\infty, \inf(-S)]. \) But by (a) these are equal, so \( \inf(-S) = -\sup S. \)

7. \( \emptyset \neq S \subseteq T \implies T^+ \subseteq S^+ \implies T^+ \implies [\sup T, \infty) \subseteq [\sup S, \infty) = S^+ \implies \sup S \leq \sup T. \)

8. Let \( \varepsilon > 0. \) Then \[ \left| \frac{5n}{n+2} - 5 \right| = \left| \frac{-10}{n+2} \right| = \frac{10}{n+2} \leq \frac{10}{n} < \varepsilon \iff n > \frac{10}{\varepsilon}. \] So \( n > N = \frac{10}{\varepsilon} \implies \left| \frac{5n}{n+2} - 5 \right| < \varepsilon. \)

9. Let \( M > 0 \) with \( |s_n| < M \) for all \( n \in \mathbb{N}. \) Let \( \varepsilon > 0 \) and set \( N = M/\varepsilon. \) Then \( n > N \implies |s_n/n - 0| = |s_n|/n \leq M/n < M/N = \varepsilon. \)

10. Let \( \varepsilon > 0. \) Then there exist \( N_1, N_2 \) such that for all \( n > N_1 \) and \( n > N_2, \) both \( |s_n - s| < \varepsilon/2 \) and \( |t_n - t| < \varepsilon/2. \) So if \( N = \max\{N_1, N_2\}, \) then \( n > N \) implies that \( |(s_n + t_n) - (s + t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \)
11. Let $M > 0$. Then $\log n > M \iff n > e^M$ (since $e^x$ is increasing), so if $N = e^M$, then $n > N \implies \log n > M$. Thus, $\lim \log n = +\infty$.

12. Let $a = 0$ and $b = 1/2$. Then infinitely many terms of the sequence lie below $a$, and infinitely many lie above $b$. So no tail lies above $a$ or below $b$, and hence the sequence diverges.

13. Since $(s_n)$ is bounded, by Completeness there exists $s = \inf\{s_n : n \in \mathbb{N}\} \in \mathbb{R}$. Claim $\lim s_n = s$. Let $\varepsilon > 0$. Thus, there is some $m \in \mathbb{N}$ with $s + \varepsilon > s_m$. But $(s_n)$ is monotone non-increasing, so for all $n > m$, we have $s - \varepsilon < s_n \leq s_m < s + \varepsilon$.

14. Since $\sqrt{2}a < \sqrt{2}b$, by the denseness of $\mathbb{Q}$ there is some rational $0 \neq r \in \mathbb{Q}$ with $\sqrt{2}a < r < \sqrt{2}b$, so $a < r/\sqrt{2} < b$.

15. A sequence $(s_n)$ is defined recursively by: $s_1 = 1$ and $s_{n+1} = \frac{s_n + 1}{4}$ for all $n \geq 1$. Prove that the sequence converges and find its limit.

Claim $s_n > 1/3$ for all $n \in \mathbb{N}$. By induction. (B) $s_1 = 1 > 1/3$. (IS) Let $n \geq 1$ and assume that $s_n > 1/3$. Then $s_{n+1} = (s_n + 1)/4 > (1/3 + 1)/4 = 1/3$, so $s_{n+1} > 1/3$. Then, by induction, $s_n > 1/3$ for all $n$.

Then $s_n - s_{n+1} = (3s_n - 1)/4 > 0$, so $(s_n)$ is decreasing and bounded, so it converges to some $s \in \mathbb{R}$ with $s \geq 1/3$.

Finally $s = \lim s_{n+1} = \frac{\lim s_n + 1}{4} = \frac{s + 1}{4}$ so $4s = s + 1$ and $s = 1/3$.

16. (a) True. By Archimedes $\exists n \in \mathbb{N}$ with $2^n(b - a) > 1$, so $\exists m \in \mathbb{N}$ with $2^n a < m < 2^n b$.

(b) False. Let $S = \{1\}$ and $T = \{1 - 1/n : n \in \mathbb{N}\}$.

(c) False. $|2 - (-1)| = 3 \neq 1 = |2| - |-1|$. 