Introduction to Sequences and Limits.

Strange as it may seem the key to understanding limits is to recognize when two real numbers are equal. Well, it’s pretty much like the situation with two people or two houses, or two numbers. They are the same if they occupy the same space. So let’s see how we determine if two numbers occupy the same space.

Let \( r \in \mathbb{R} \) and let \( \varepsilon > 0 \) be a positive real number. then the \( \varepsilon \)-neighborhood of \( r \) is the open interval

\[
(r - \varepsilon, r + \varepsilon) = \{ x \in \mathbb{R} : r - \varepsilon < x < r + \varepsilon \} = \{ x \in \mathbb{R} : |x - r| < \varepsilon \}.
\]

These neighborhoods characterize the real number \( r \); that is, they define the space that \( r \) occupies. More formally,

**Lemma 1.** If \( x \) and \( r \) are two real numbers, then \( x = r \iff x \) is in every \( \varepsilon \)-neighborhood of \( r \).

**Proof.** Trivially, \( r \) lies in each \( \varepsilon \)-neighborhood of itself. But \( \{ r - \varepsilon : \varepsilon > 0 \} = \{ s \in \mathbb{R} : s < r \} = \{ r \}^- \) and, similarly, \( \{ r + \varepsilon : \varepsilon > 0 \} = \{ s \in \mathbb{R} : s > r \} = \{ r \}^+ \). But \( \sup \{ r \}^- = \inf \{ r \}^+ = r \), so

\[
\bigcap_{\varepsilon > 0} (r - \varepsilon, r + \varepsilon) = \{ r \}^- \cap \{ r \}^+ = \{ r \}.
\]

That is, \( x \) is in every neighborhood of \( r \) iff \( x = r \).

We shall return to this simple but important fact later. But now let’s turn to sequences. A **sequence** is just a function \( s \) with domain \( D \) in \( \mathbb{Z} \) and with output set in \( \mathbb{R} \). If the domain \( D \) is finite, we say that the sequence is finite; here we will concentrate on infinite sequences and will for now pretty well restrict the domains to be sets like

\[
D_m = \{ n \in \mathbb{Z} : n \geq m \}
\]

for some \( m \in \mathbb{Z} \). Moreover, we will adopt standard notation. That is, if \( s \) is a sequence on the set \( D_m \) of integers, then we will usually denote the value of \( s \) at the integer \( n \geq m \) by \( s_n \) and will often denote the sequence itself by any of the following, including simply listing the values of the sequence:

\[
(s_n)_{n \geq m} = (s_m, s_{m+1}, s_{m+2}, \ldots, s_n, \ldots)
\]
or we may specify the sequence by the generic value of \( s_n \) at \( n \). Here are some examples

**Examples 2.** First, a couple with nice formulas for their \( n \)th terms.

\[
(s_n)_{n \geq 1} = (-1, 1/2, -1/3, 1/4, -1/5, \ldots, (-1)^n / n, \ldots);
\]

\[
(t_n)_{n \geq 0} = (1, 0, -1, 0, 1, \ldots, \cos n \pi / 2, \ldots).
\]
And here’s one without a simple formula:

\[ u_n = \text{the } n\text{th decimal digit of } \pi, \text{ so } (u_n) = (1, 4, 1, 5, 9, \ldots). \]

If \( a \) is a real number, then the constant sequence \((a)\) is just the sequence

\[ (a)_{n \geq m} = (a, a, a, \ldots, a, \ldots). \]

One of the first things one wants to know about an infinite sequence \((s_n)\) is its long-term behavior. In particular, what exactly does that mean? Well, we get some clues from the above Examples 2. In the long term the values of the sequence \((s_n)\) become very small, so much so that after a few billion terms, it’s going to be very tough to distinguish them from the terms in the constant zero sequence. In fact, this sequence \((s_n)\) “converges” to 0. Then the sequence \((t_n)\) just seem to cycle through the numbers 0, 1, \(-1\), round and round. In the long run it really goes nowhere and we’ll say it “diverges”. Of course, the sequence \((u_n)\) has no regularity at all it just flops around apparently randomly among the numbers 0, 1, \ldots 9. This sequence also “diverges”. The long range behavior of the constant sequence \((a)\) is too transparent to mention further. Finally, here is a curious case, the sequence

\[ (v_n)_{n \geq 0} = (n)_{n \geq 0} = (0, 1, 2, 3, 4, 5, 6, \ldots, n, \ldots) \]

Opposite the sequence \((s_n)\) the terms in this one just get bigger and bigger without limit. We also say this sequence “diverges”, but it seems to have some goal in life, and so we say that it “diverges to \(+\infty\)”.

Since it is the long-range behavior that we’re trying to pin down, we are not at all interested in the first few terms of a sequence. In fact, in determining that behavior we can just ignore the first 100, million, zillion, \(10^{100}\) terms of the sequence; they are just a drop in the bucket. It’s the stuff that’s left over that counts. Thus if \((s_n)\) is a sequence and if \(N \in \mathbb{R}\) is a real number (and in practice we will often, but not always, have \(N\) a natural number), the \(N^{th} - \text{tail}\) of the sequence \((s_n)\) is the “subsequence”

\[ N^{th}-\text{tail of } (s_n) = (s_n)_{n \geq N} = (s_m, s_{m+1}, s_{m+2}, \ldots) \]

where \(m\) is the smallest integer greater than \(N\).

**Example 3.** If \((s_n)_{n \geq 1}\) and \((t_n)_{n \geq 0}\) are the sequences in Example 2, then their 20\(^{th}\) tails are, respectively,

\[ (s_n)_{n \geq 20} = (1/20, -1/21, 1/22, -1/23, \ldots), \text{ and } (t_n)_{n \geq 20} = (1, 0, -1, 0, 1, 0, \ldots). \]

Of course, every tail of a constant sequence is just another constant sequence. Finally, the 20\(^{th}\) tail of the sequence \((n^2)\) is

\[ (n)_{n \geq 20} = (400, 441, 484, 529, \ldots). \]

Now here is the critical fact about the idea of the long-range behavior of a sequence.

*The long-range behavior of a sequence is precisely the same as that of any of its tails.*
For example, the sequence $(s_n) = (-1, 1/2, -1/3, 1/4, -1/5, \ldots)$ of Example 2 will have the same long-range behavior as any of its $1/n$-tails,

$$\left(\frac{\pm 1}{n}, \frac{\pm 1}{n+1}, \frac{\pm 1}{n+2}, \ldots\right).$$

Now if the idea of limit should be a function of the long-range behavior, the limit of this sequence $(s_n)$ and all of its tails should be the same and should be 0. To be more precise, as we go farther out, the tails should be more and more indistinguishable from the 0-sequence. In view of Lemma 1, this would seem to suggest that every $\varepsilon$-neighborhood of 0 should contain a tail of the sequence. And, sure enough, that’s exactly what we mean by the limit of a sequence! Indeed, let’s quit the dawdling and define what we mean by the limit of a sequence.

**Definition.** Let $(s_n)$ be a sequence and let $L \in \mathbb{R}$ be a real number. Then we say that the sequence $(s_n)$ has limit $L$ or that $(s_n)$ **converges** to $L$ in case every $\varepsilon$-neighborhood of $L$ contains all of the terms of some tail of $(s_n)$. In that case we also may write

$$\lim_{n \to \infty} s_n = L.$$

If no such number $L$ exists in $\mathbb{R}$, then we say that the sequence $(s_n)$ **diverges**.

Although this definition may appear different from Definition 7.1 in the text, it really says the same thing in slightly different words. In either case they say that if the long-range behavior of a sequence $(s_n)$ is the same as the constant sequence $(L)$, then to be consistent with Lemma 1, every $\varepsilon$-neighborhood of $L$ should contain most of the entries of the sequence. But let’s formally show that these two definitions really say the same thing.

The above Definition says that $L$ is the limit of $(s_n)$ if and only if for every $\varepsilon > 0$, the $\varepsilon$-neighborhood of $L$ contains the $N$-tail of $(s_n)$ for some $N$. That is, that there is a number $N$ such that every entry in the $N$-tail lies in the $\varepsilon$-neighborhood of $L$. But the entries in the $N$-tail are the numbers $s_n$ for $n > N$. So our Definition can be rephrased:

For every $\varepsilon > 0$ there is some $N$ such that for every $n > N$ the entry $s_n$ of the $N$-tail of $(s_n)$ lies in the $\varepsilon$-neighborhood of $L$; that is,

$$|s_n - L| < \varepsilon.$$

And, sure enough, this exactly the effect of Definition 7.1 of the text.

Now a big question is, how do we implement this is actual practice? Well, here is a sort of schematic description of what’s involved:

Suppose we have a sequence $(s_n)$ and believe that it converges to the limit $L \in \mathbb{R}$. To prove that $L$ actually is the limit, we have to test it against the Definition; that is, show that every $\varepsilon$-neighborhood of $L$ contains some entire tail of $(s_n)$. 

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So we must choose an arbitrary $\varepsilon > 0$; now this is a generic $\varepsilon$, not a specific one like $1/\pi$ or $10^{-10}$. Then we need to solve for some $N$ so that

$$|s_n - L| < \varepsilon \quad \text{for all } n > N.$$ 

It’s this last step that can be involved. Generally speaking our strategy is to manipulate $|s_n - L|$ to find some manageable function $f(n)$ of $n \in \mathbb{N}$ with $|s_n - L| \leq f(n)$, and then solve the inequality $f(n) < \varepsilon$. If the resulting solution is of the form $n > N$ for some $N$, then we’re done. Let’s try a couple of examples.

**Example 4.** Let’s consider the sequence

$$(s_n)_{n \geq 1} = \left( \frac{n+1}{n+3} \right).$$

Of course, we suspect that this sequence converges to the limit $L = 1$. Here’s how we prove that we’re right.

**Proof.** Let $\varepsilon > 0$. Then

$$|s_n - L| = \left| \frac{n+1}{n+3} - 1 \right| = \left| \frac{-2}{n+3} \right| = \frac{2}{n+3} \leq \frac{2}{n}.$$ 

(Thus, we have found a reasonably well behaved function $f(n) = 2/n$ that is at least as big as $|s_n - L|$, so if we can find when $f(n)$ is small, we’ll know at least some of the times when $|s_n - L|$ is small. That is, we want to test when $|s_n - L|$ is less than $\varepsilon$. But that will happen at least whenever $f(n) = 2/n$ is less than $\varepsilon$. So HA!, we’re in business!) But

$$\frac{2}{n} < \varepsilon \iff n > \frac{2}{\varepsilon},$$

so if we set $N = 2/\varepsilon$, then $|s_n - 1| \leq \frac{2}{n} < \varepsilon$ for all $n > N = 2/\varepsilon$. 

\[\blacksquare\]