Lecture 1. Union (Addition) and Product Rules.

At least for the time being we will focus on finite sets. So to simplify our discussion we will assume that (unless we state otherwise) all sets are finite. And for each (finite) set S we write $|S|$ for the number of elements in the set S. So a set S is empty iff $|S| = 0$. We begin, then, with the simple rule of counting unions of two sets:

Basic Union Rule. *Given two sets S and T,*

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

In particular,

(a) *If S and T are disjoint (that is, $S \cap T = \emptyset$), then*

$$|S \cup T| = |S| + |T|,$$

(b) *If $S \subseteq T$, then*

$$|T \setminus S| = |T| - |S|.$$

A formal proof here would be overkill. Instead we just note that to count the number of elements in $S \cup T$ we simply count those in S and then continue by counting the number in T. If they are disjoint, then we're done; if not, then we have double counted those in both sets and so for the total we must subtract the number in $S \cap T$. The final statement follows at once from the observation that T is the disjoint union of S and $T \setminus S$.

Example1.1. Recall Example 0.1 where Professor Jones is teaching two courses this term, Math 251(Calculus I) and Math 231 (Discrete Math I). There are 32 students in his Calculus section and 21 in his Discrete Math section. So how many students are in at least one of Jones classes? Let's denote by S the students in the calculus class and by T those in the discrete math class. So the first question is just what is $|S \cup T|$.

Well, if no one is enrolled in both sections, then S and T are disjoint, so

$$|S \cup T| = |S| + |T| = 32 + 21 = 53.$$

On the other hand, if 7 students are in both, then $|S \cap T| = 7$ and

$$|S \cup T| = |S| + |T| - |S \cap T| = 32 + 21 - 7 = 46.$$

Example 1.2. Let V be the set of integers between 1 and 2000, let S be the elements of V divisible by 5, and let T the set of elements of V divisible by 7. Then we know that

$$|S| = \lfloor 2000/5 \rfloor = 400 \qquad \text{and} \qquad |T| = \lfloor 2000/7 \rfloor = 285.$$

Then also the number of elements of V that are divisible by *both* 5 and 7 is just

$$|S \cap T| = \lfloor 2000/35 \rfloor = 57.$$

So the number of elements in V divisible by at least one of 5 or 7 is

$$|S \cup T| = |S| + |T| - |S \cap T| = 400 + 285 - 57 = 628.$$

Finally, the number of elements of V divisible by *neither* 5 nor 7 is

$$|V \setminus (S \cup T)| = |V| - |S \cup T| = 2000 - 628 = 1372.$$

(Now you should find it easy to count the number divisible by 5 but not 7, and the number divisible by 7 but not 5.)

There are a couple of useful extensions of the Union Rules. The first

Corollary. *If $S_1, S_2, \ldots, S_n$ are pairwise disjoint sets, then*

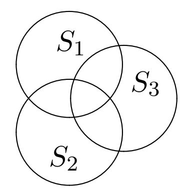
$$|S_1 \cup S_2 \cup \cdots \cup S_n| = |S_1| + |S_2| + \cdots + |S_n|.$$

The proof of this, that we'll leave as an exercise, is an easy induction. The Induction Step follows from the fact that since the sets are pairwise disjoint, $S_n \cap (S_1 \cup \cdots \cup S_{n-1}) = \emptyset$. The second extension, the three set version of the very important "Inclusion-Exclusion Principle" treated more fully later, deals with the case where the sets need not be pairwise disjoint. It states:

If S_1, S_2, S_3 are three sets, then

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|.$$

A "proof" of this is that in this formula each element of the union is counted once for each of the three sets it is in, once more for each pair it is in, and still once more if it is in all three. If you check out the cases, the above formula counts each element exactly once!



Example 1.3. This time suppose that Professor Jones is teaching Calculus with 32 students, Discrete Math with 21 students, and also, Linear Algebra with 30 students. If no student is in any two of these (the class enrollments are pairwise disjoint), then the number of students in at least one of Jones' classes is simply

$$32 + 21 + 30 = 83.$$

But suppose, instead, that 7 are in calculus and discrete, 10 are in calculus and linear algebra, 12 are in discrete and linear algebra, and finally, 5 hardy souls are in all three. In this case the number of students in at least one of Jones' classes is

$$32 + 21 + 30 - 7 - 10 - 12 + 5 = 59.$$

Challenge. You might try to find the analogous formula for counting the number of elements in the union of four, *not necessarily pairwise disjoint*, sets. But you should first try to do this without peeking ahead to Section 5.3.

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To answer questions of the sort posed in Example 0.2 we need another general counting "rule", called the "Product Rule". Before stating it, though, recall that a **sequence** of **length** k is an ordered set, or list, $(s_1, s_2, \ldots, s_k)$ of elements. With that here is the

Product Rule. Suppose that a sequence $(s_1, s_2, \ldots, s_k)$ is chosen recursively with

(a) There are n_1 choices for s_1;

(b) For each $1 < j \le k$ once $s_1, \ldots, s_{j-1}$ have been chosen, there are n_j choices for s_j,

then there are $n_1 n_2 \cdots n_k$ choices for the sequence.

In most cases this example is much easier to apply than one might think from its statement. So let's look at a few typical examples before pressing on.

Example 1.4. Suppose that $S_1, S_2, \ldots, S_k$ are finite sets. Then their **product**

$$S_1 \times S_2 \times \cdots \times S_k$$

is the set of all sequences $(s_1, s_2, \ldots, s_k)$ with each $s_i \in S_i$. In forming the sequences in this product the choice at any point is independent of the previous choices, so the number of choices of any s_i is just $|S_i|$, the number of elements in S_i. Thus, from the Product Rule we have the familiar identity

$$|S_1 \times S_2 \times \cdots \times S_k| = |S_1||S_2| \cdots |S_k|.$$

So, in particular, for any set S the number of sequences of length k from S is

$$|S^k| = |S|^k.$$

Example 1.5. Let $S = \{a, b, c, d, e\}$ be the set of five letters including just the two vowels $V = \{a, e\}$ and three consonants $C = \{b, c, d\}$. So the 4 letter words, sequences of length 4 (with repetitions allowed), we can form using only the letters from the set S are the elements of the set

$$S^4 = S \times S \times S \times S,$$

with exactly

$$|S^4| = |S|^4 = 5^4 = 625.$$

However, if we ask how many of these **start with a vowel**, we are asking for number of elements in the set

$$V \times S \times S \times S$$

or, by the Product Rule,

$$|V||S||S||S| = |V||S|^3 = 2 \cdot 5^3 = 250.$$

On the other hand, the four letter words, **with no repetitions**, is the set of sequences (s_1, s_2, s_3, s_4) from S with no repetitions. So in forming such a sequence the choice of any letter depends on the choice of the previous ones. That is, there are all 5 letters available for s_1, but then only 4 are left for s_2, only 3 choices for s_3, and finally, there are only 2 letters left for s_4; thus, there are only

$$5 \cdot 4 \cdot 3 \cdot 2 = 120$$

such four letter words. However, if we ask for the number of four letter words with no repetitions and that **begin** with a vowel, we asking for the number of those length 4 sequences with only 2 choices for s_1, and then in turn 4, 3, and 2 choices for s_2, s_3, and s_4. So the number of such words is

$$2 \cdot 4 \cdot 3 \cdot 2 = 24.$$

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Let S be a set with $|S| = n$. If $1 \leq 1 \leq n$, then a $k-$**permutation** of S is a sequence $(s_1, s_2, \ldots, s_k)$ with no repetitions of elements from S; so note that such a k-permutation is more succinctly described as a one-to-one map $\sigma : \{1, 2, \ldots, k\} \longrightarrow S$. Simply a **permutation** of S is an n-permutation of S. We denote the number of k-permutations of S by

$$P(n, k).$$

To specify an n-permutation $(s_1, s_2, \ldots, s_n)$ of S, we have the full n choices for s_1, then only $n-1$ choices for s_2, $n-2$ choices for s_3, and so on until finally there is only one choice left for s_n; so by the Product Rule, we have

$$P(n,n) = n(n-1)(n-2)\cdots 2\cdot 1 = n!$$

is the number of permutations of S. Then note that a k-permutation, $(s_1, s_2, \ldots, s_k)$, for $1 \leq k < n$, is just the first k terms of an n-permutation, so the number of such k-permutations is

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Example 1.6. Our standard alphabet has 26 letters S with five vowels $V = \{a, e, i, o, u\}$. So the number of five letter words in this alphabet is

$$|S \times S \times S \times S \times S| = |S|^5 = 26^5.$$

Now a five letter word with no repetitions in this alphabet is just a 5-permutation of S. So there are exactly

$$P(26,5) = \frac{26!}{21!} = 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22.$$

such words.