Towards the classification of outer actions of finite groups on Kirchberg algebras

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9 June 2011

Conference on Structure and Classification of C*-Algebras

6–10 June 2011

Centre de Recerca Matemàtica, Universitat Autònoma de Barcelona

Introduction

Possible subtitle: Initiating the Elliott classification program for group actions.

This is work in progress. The intended main theorem has not yet been proved.

Caution: Even the results stated have not all been carefully checked. Don't quote them yet!

Rough outline

- Goal, results, and background.
  - The hoped for main theorem.
  - Some ingredients: Equivariant K-theory and E-theory.
  - Previous results.
  - Examples.
- What has been done so far, and general description of the ideas.
  - The current intermediate result.
  - How to get from there to the end.
  - How to get to the current result.
  - Why only pointwise outer actions?
- Some further details.
  - The actions on $\mathcal{O}_2$ and on $\mathcal{O}_\infty$.
  - Equivariant semiprojectivity.
  - What do we do with equivariant semiprojectivity?
  - Equivariant semiprojectivity for finite dimensional C*-algebras.
  - Equivariant semiprojectivity for certain quasifree actions.

The goal

The intended main theorem is as follows. (Some items are described afterwards.)

Conjecture

Let G be a cyclic group of prime order. Let A and B be Kirchberg algebras (purely infinite simple separable nuclear C*-algebras) which are unital and satisfy the Universal Coefficient Theorem. Let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ be pointwise outer actions of G which belong to a suitable bootstrap class (defined by Manuel Köhler). Suppose the extended K-theory of α (as defined by Köhler) is isomorphic to that of β. Then α and β are conjugate.

Conjugacy is isomorphism of dynamical systems: there exists an isomorphism $\varphi \colon A \to B$ such that $\beta_g = \varphi \circ \alpha_g \circ \varphi^{-1}$ for all $g \in G$.

The action $\alpha \colon G \to \mathrm{Aut}(A)$ is called pointwise outer if for every $g \in G \setminus \{1\}$, the automorphism α_g is not inner.

Extended K-theory

Conjecture

Let G be a cyclic group of prime order. Let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ be pointwise outer actions of G which belong to a suitable bootstrap class. Suppose the extended K-theory of α is isomorphic to that of β. Then α and β are conjugate.

The extended K-theory $EK^G(A)$ consists of three groups:

- $K_*(A)$.
- $K_*^G(A)$.
- With M being the mapping cone of the unital embedding of $\mathbb{C}$ in $C(G)$, the group $KK_G^*(M, A)$.

(See the next slides for more on equivariant K-theory and KK-theory.) $EK^G(A)$ has additional structure, given by various operations, which must be preserved by isomorphisms.

If the actions are in Köhler's bootstrap class, then the algebras automatically satisfy the UCT.

A brief summary of equivariant K-theory

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a compact group G on a unital C*-algebra A. The ordinary K_0-group of A is made from finitely generated projective modules over A. (If we use right modules, the projection $p \in M_n(A)$ corresponds to the module pA^n.)

In a similar way, the equivariant K_0-group of A, written $K_0^G(A)$, is made from finitely generated projective modules over A which carry a compatible action of G. (It is a bit more complicated than just G-invariant projections in $M_\infty(A)$.)

One generalizes to nonunital algebras and to $K_1^G(A)$ in the usual way, by unitizing and suspending. (The action of G in the suspension direction is trivial.)

The Green-Julg Theorem tells us that $K_*^G(A) \cong K_*(C^*(G, A, \alpha))$.

$K_*^G(A)$ is a module over the representation ring $R(G) = K_0^G(\mathbb{C})$, the Grothendieck group made from finite dimensional representations of G. (Tensor the A-module with the representation.)

Equivariant KK-theory and E-theory

There are also equivariant versions of KK-theory and E-theory, denoted $KK_G^*(A, B)$ and $E_*^G(A, B)$. We will use E-theory. The convenient definition is in terms of asymptotic morphisms.

Definition

Let A and B be separable C*-algebras. An *asymptotic morphism* from A to B is a family $\varphi = (\varphi_t)_{t \in [0,\infty)}$ of functions $\varphi_t \colon A \to B$ such that:

- $t \mapsto \varphi_t(a)$ is continuous for all $a \in A$.
- For all $a, b \in A$ and $\lambda \in \mathbb{C}$, as $t \to \infty$ the quantities

$$\varphi_t(a+b) - \varphi_t(a) - \varphi_t(b), \qquad \varphi_t(\lambda a) - \lambda \varphi_t(a),$$

$$\varphi_t(ab) - \varphi_t(a)\varphi_t(b), \qquad \text{and} \qquad \varphi_t(a^*) - \varphi_t(a)^*$$

  all converge to zero.

Equivariant KK-theory and E-theory (continued)

An asymptotic morphism from A to B is a family of functions $\varphi_t \colon A \to B$ such that:

- $t \mapsto \varphi_t(a)$ is continuous for all $a \in A$.
- For all $a, b \in A$ and $\lambda \in \mathbb{C}$, as $t \to \infty$ the quantities
$$\varphi_t(a+b) - \varphi_t(a) - \varphi_t(b), \qquad \varphi_t(\lambda a) - \lambda \varphi_t(a),$$
$$\varphi_t(ab) - \varphi_t(a)\varphi_t(b), \qquad \text{and} \qquad \varphi_t(a^*) - \varphi_t(a)^*$$
all converge to zero.

In other words, $(\varphi_t)_{t \in [0,\infty)}$ is asymptotically a homomorphism.

If G is compact second countable and $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ are actions of G on A and B, for an equivariant asymptotic morphism we ask in addition that

$$\beta_g \circ \varphi_t(a) - \varphi_t \circ \alpha_g(a) \to 0$$

for all $a \in A$ and $g \in G$.

The set of homotopy classes of equivariant asymptotic morphism from A to B is written $[[A, B]]_G$.

Equivariant KK-theory and E-theory (continued)

For an equivariant asymptotic morphism we ask in addition that

$$\beta_g \circ \varphi_t(a) - \varphi_t \circ \alpha_g(a) \to 0$$

for all $a \in A$ and $g \in G$.

$[[A, B]]_G$ is the set of homotopy classes of equivariant asymptotic morphisms from A to B.

We take $SA = C_0((0,1), A)$, with the trivial action of G on $C_0((0,1))$. We let G act on $K = K(l^2 \otimes L^2(G))$ by conjugation by the direct sum of infinitely many copies of the regular representation of G. Then define $E^G(A, B) = [[SA, K \otimes SB]]_G$.

We take $E_0^G(A, B) = E^G(A, B)$ and $E_1^G(A, B) = E^G(SA, B)$. One has Bott periodicity, so take $E_*^G(A, B) = E_0^G(A, B) \oplus E_1^G(A, B)$.

Equivariant KK-theory and E-theory (continued)

$[[A, B]]_G$ is the set of homotopy classes of equivariant asymptotic morphisms from A to B.

$E^G(A, B) = [[SA,\ K \otimes SB]]_G$.

With some care, one can compose homotopy classes of equivariant asymptotic morphisms from A to B and from B to C. This gives a product $E^G(A, B) \times E^G(B, C) \to E^G(A, C)$. In particular, $E_0^G(A, A)$ is a ring, and one can make $E_*^G(A, A)$ into a ring.

$E_0^G(\mathbb{C}, \mathbb{C}) = R(G)$, the representation ring mentioned above.

For G cyclic of prime order, one can view $EK_*^G(A)$ as $E_*^G(\mathbb{C} \oplus C(G) \oplus M,\ A)$, which is a module over

$$R_G = E_*^G\big(\mathbb{C} \oplus C(G) \oplus M,\ \mathbb{C} \oplus C(G) \oplus M\big).$$

Note that $R(G)$ is contained in R_G, but R_G is much bigger.

The Universal Coefficient Theorem

$EK_*^G(A) = E^G(\mathbb{C} \oplus C(G) \oplus M, \, A)$, which is a module over $R_G = E_*^G\big(\mathbb{C} \oplus C(G) \oplus M, \ \mathbb{C} \oplus C(G) \oplus M\big)$.

Without the group: for A in a suitable class (the bootstrap class), there is a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}_1^{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow E_*(A, B) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)) \longrightarrow 0.$$

(Rosenberg-Schochet.) The first map has degree 1.

Köhler proved a similar result for C*-algebras with an action of a cyclic group G of prime order, computing $E_*^G(A, B)$, using $EK_*^G(-)$ in place of $K_*(-)$, and using extensions over R_G instead of over $\mathbb{Z}$. For actions of G on A in a suitable bootstrap class, and arbitrary actions on B:

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}_1^{R_G}\big(EK_*^G(A), \, EK_*^G(B)\big) &\longrightarrow E_*^G(A, B) \\ &\longrightarrow \mathrm{Hom}_{R_G}\big(EK_*^G(A), \, EK_*^G(B)\big) \longrightarrow 0. \end{aligned}$$

Previous result: Classification of Rokhlin actions

The following is Theorem 4.2 of M. Izumi, *Finite group actions on C^*-algebras with the Rohlin property. II*, Adv. Math. **184**(2004), 119–160.

Theorem

Let A be a unital UCT Kirchberg algebra, and let G be a finite group. Let $\alpha, \beta \colon G \to \mathrm{Aut}(A)$ be actions with the Rokhlin property. Then α is conjugate to β if and only if the actions of G they induce on $K_*(A)$ are equal.

We omit the definition of the Rokhlin property.

Interpreted as a theorem about conjugacy of dynamical systems, the invariant involved includes A, equivalently, it includes $K_*(A)$ and $[1_A] \in K_0(A)$.

There are severe restrictions on the possible actions of G on $K_*(A)$.

Previous result: Classification of actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$

The following is essentially a restatement of part of Theorem 4.8 of M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. I*, Duke Math. J. **122**(2004), 233–280.

$\mathcal{O}_2$ is the Cuntz algebra.

Theorem

Let $\alpha, \beta \colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2)$ be actions which are pointwise outer but strongly approximately inner. Then α is conjugate to β if and only if $K_*^{G,\alpha}(\mathcal{O}_2) \cong K_*^{G,\beta}(\mathcal{O}_2)$ via an isomorphism which sends [1] to [1].

$K_*^{G,\alpha}(A)$ is the equivariant K-theory of A with respect to the group action α.

$K_*(\mathcal{O}_2)$ isn't needed in the invariant, since it is zero.

We omit the definition of strong approximate innerness.

Previous result: Quasifree actions on $\mathcal{O}_\infty$

Quasifree actions are described starting on the next slide.

Theorem

Let G be a finite group. Then any two quasifree actions of G on $\mathcal{O}_\infty$ coming from injective representations of G are conjugate.

This is in a recent preprint of Goldstein and Izumi, written after I started this project but based on work done earlier.

Cuntz algebras

Let $d \in \{2, 3, \ldots\}$. Recall that the Cuntz algebra $\mathcal{O}_d$ is the universal C*-algebra generated by elements $s_1, s_2, \ldots, s_d$ satisfying

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1.$$

The first relation says that $s_1, s_2, \ldots, s_d$ are isometries, and the second says that they have orthogonal ranges which add up to 1.

The infinite Cuntz algebra $\mathcal{O}_\infty$ is the universal C*-algebra generated by elements $s_1, s_2, \ldots$ satisfying

$$s_j^* s_j = 1 \quad \text{for } j \in \mathbb{Z}_{>0} \qquad \text{and} \qquad s_j s_k^* = 0 \quad \text{for distinct } j, k \in \mathbb{Z}_{>0}.$$

The s_j are isometries with pairwise orthogonal ranges.

Examples: Quasifree actions on Cuntz algebras

Relations: $s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1$ and $s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1$.

Let $\rho \colon G \to L(\mathbb{C}^d)$ be a unitary representation of G. Write

$$\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix}$$

for $g \in G$. Then there exists a unique action $\beta^{\rho} \colon G \to \mathrm{Aut}(\mathcal{O}_d)$ such that

$$\beta_g^{\rho}(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g) s_j$$

for $j = 1, 2, \ldots, d$. (This can be checked by a computation.)

Examples:

- For $G = \mathbb{Z}_n$, choose n-th roots of unity $\zeta_1, \zeta_2, \ldots, \zeta_d$ and let a generator of the group multiply s_j by ζ_j.
- Take $d = \mathrm{card}(G)$, and label the generators s_g for $g \in G$. Then define $\beta^G \colon G \to \mathrm{Aut}(\mathcal{O}_d)$ by $\beta_g^G(s_h) = s_{gh}$ for $g, h \in G$.

Examples: Quasifree actions (continued)

Relations: $s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1$ and $s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1$.

$\rho\colon G \to L(\mathbb{C}^d)$ is a unitary representation.

$$\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix} \quad \text{and} \quad \beta_g^{\rho}(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g) s_j.$$

An analogous construction gives actions on $\mathcal{O}_\infty$.

Example: Label the generators of $\mathcal{O}_\infty$ as $s_{g,j}$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$. Define $\iota\colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ by $\iota_g(s_{h,j}) = s_{gh,j}$ for $g \in G$ and $j \in \mathbb{Z}_{>0}$.

This is the quasifree action coming from the direct sum of infinitely many copies of the regular representation. One can compute its equivariant K-theory, getting $K_0^G(\mathcal{O}_\infty) \cong R(G)$ (recall that this is the representation ring of G), with $[1] \mapsto 1$, and $K_1^G(\mathcal{O}_\infty) = 0$.

Example: The tensor flip

Define $\varphi \colon \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ by $\varphi(a \otimes b) = b \otimes a$ for $a, b \in \mathcal{O}_\infty$. Using $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$, this defines an action of $\mathbb{Z}_2$ on $\mathcal{O}_\infty$, the tensor flip.

Is this action conjugate to the action ι above? (It is equivariantly strongly selfabsorbing. I don't yet know the equivariant K-theory, but I suspect it is $R(G)$.)

More generally, subgroups of the symmetric group S_n act on $(\mathcal{O}_\infty)^{\otimes n}$.

Methods

Recall the classification conjecture:

Conjecture

Let G be a cyclic group of prime order. Let A and B be unital UCT Kirchberg algebras. Let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be pointwise outer actions of G in Köhler's bootstrap class. Suppose the extended K-theory of α (as defined by Köhler) is isomorphic to that of β. Then α and β are conjugate.

Three basic methods go into the work:

- Reduction to known results in the case in which there is no group.
- Imitating known arguments from the case in which there is no group.
- New arguments.

The current status

We don't have equivariant classification yet. We nearly have:

Conjecture

Let G be a finite group, let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ be pointwise outer actions on unital Kirchberg algebras, and let $\gamma \colon G \to \mathrm{Aut}(C)$ be any action on a unital C*-algebra C. Let $t \mapsto \varphi_t$ and $t \mapsto \psi_t$ be full equivariant asymptotic morphisms from A to $K \otimes B \otimes C$. Suppose φ and ψ are homotopic (as equivariant asymptotic morphisms). Then they are equivariantly asymptotically unitarily equivalent.

Equivariant asymptotic unitary equivalence means that there is a continuous path $t \mapsto u_t$ of G-invariant unitaries in B such that $\lim_{t \to \infty} \big(u_t \varphi_t(a) u_t^* - \psi_t(a) \big) = 0$ for all $a \in A$.

We omit the definition of fullness.

We can use the trivial action of G on K. (This is because we assume the action on B is pointwise outer.)

To get the rest of the way

Conjecture

Let G be a finite group, let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(B)$ be pointwise outer actions on unital Kirchberg algebras, and let $\gamma \colon G \to \mathrm{Aut}(C)$ be any action on a unital C*-algebra. Then homotopic full equivariant asymptotic morphisms from A to $K \otimes B \otimes C$ are equivariantly asymptotically unitarily equivalent.

What is needed to get from here to the goal:

- Show that if $\alpha \colon G \to \mathrm{Aut}(A)$, $\beta \colon G \to \mathrm{Aut}(B)$, and $\gamma \colon G \to \mathrm{Aut}(C)$ are as in the conjecture, then $E_0^G(A, K \otimes B \otimes C)$ is the set of homotopy classes of full equivariant asymptotic morphisms from A to $K \otimes B \otimes C$. (Before, we had SA and SB for A and B.)
- An equivariant approximate intertwining argument, to show that for pointwise outer actions on unital Kirchberg algebras, E^G-equivalence implies equivariant isomorphism.
- The Universal Coefficient Theorem for actions of G.

To get the rest of the way (continued)

What is needed to get from homotopy implies equivariant asymptotic unitary equivalence to the goal:

- Show that if $\alpha \colon G \to \mathrm{Aut}(A)$, $\beta \colon G \to \mathrm{Aut}(B)$, and $\gamma \colon G \to \mathrm{Aut}(C)$ are as in the conjecture, then $E_0^G(A, \, K \otimes B \otimes C)$ is the set of homotopy classes of full equivariant asymptotic morphisms from A to $K \otimes B \otimes C$.

  The point is that one does not have to suspend. The equivariant versions of the ingredients I used here in the nonequivariant case are mostly already known.

- An equivariant Elliott approximate intertwining argument, to show that for pointwise outer actions on unital Kirchberg algebras, KK_G-equivalence implies equivariant isomorphism.

  This should be standard.

- The Universal Coefficient Theorem for actions of G.

  Use Köhler's Universal Coefficient Theorem when G is cyclic of prime order. (This is the only place we don't allow an arbitrary finite group.)

Used to prove the conjecture

We want to show homotopy implies equivariant asymptotic unitary equivalence for suitable equivariant asymptotic morphisms. Here are some things that are used.

- Computation of equivariant K-theory for quasifree actions on Cuntz algebras. (Quasifree actions were defined above.)
- Equivariant semiprojectivity for certain quasifree actions on Cuntz algebras.
- A pointwise outer action of a finite group on a unital Kirchberg algebra has the tracial Rokhlin property.
- Equivariant analogs of Kirchberg's absorption theorems.

Used to prove the conjecture (continued)

Some things used to prove that homotopy implies equivariant asymptotic unitary equivalence for suitable equivariant asymptotic morphisms:

- Computation of equivariant K-theory for quasifree actions on Cuntz algebras.
  This uses fairly standard methods, and much is already known. We omit the details.
- Equivariant semiprojectivity for certain quasifree actions on Cuntz algebras.
  This requires some new work. See the last part of the talk.
- A pointwise outer action of a finite group on a unital Kirchberg algebra has the tracial Rokhlin property.
  This follows easily from work of Nakamura. We omit the definition and details.
- Equivariant analogs of Kirchberg's absorption theorems.
  We say a little more about these below. Some of this was done independently by Goldstein and Izumi.

Why pointwise outer actions?

Here are three main ingredients in the proof of classification without the group. The first two are Kirchberg's absorption theorems; in the nonequivariant case, the third is trivial.

Theorem

Let A be a simple separable unital nuclear C*-algebra. Then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.

Theorem

Let A be a Kirchberg algebra. Then $\mathcal{O}_\infty \otimes A \cong A$.

(In fact, there is an isomorphism from A to $\mathcal{O}_\infty \otimes A$ which is asymptotically unitarily equivalent to the map $a \mapsto 1 \otimes a$.)

Theorem

Let A be a purely infinite simple C*-algebra, and let $p \in A$ be a nonzero projection such that $[p] = 0$ in $K_0(A)$. Then there exists a unital homomorphism $\mathcal{O}_2 \to pAp$.

Why pointwise outer actions? (continued)

Three main ingredients for classification without the group:

1. $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for A simple separable unital nuclear.
2. $\mathcal{O}_\infty \otimes A \cong A$ for A a Kirchberg algebra.
3. If A is purely infinite and $p \in A$ is a nonzero projection such that $[p] = 0$ in $K_0(A)$, then there is a unital homomorphism $\mathcal{O}_2 \to pAp$.

We want equivariant versions of these. Suppose we allow arbitrary actions. Taking the trivial action on A in (2) forces one to use the trivial action on $\mathcal{O}_\infty$. Taking a nontrivial action on A in (1) forces one to use a nontrivial action on $\mathcal{O}_2$. These choices make (3) impossible when $A = \mathcal{O}_\infty$.

The right condition on the action is pointwise outerness.

The actions on $\mathcal{O}_2$ and on $\mathcal{O}_\infty$

Recall Kirchberg's absorption theorem for $\mathcal{O}_2$: $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for A simple separable unital nuclear. We need an action $\zeta \colon G \to \mathrm{Aut}(\mathcal{O}_2)$ such that this isomorphism holds equivariantly whenever A is purely infinite simple and the action on A is pointwise outer. By Izumi, there is a unique action (up to conjugacy) of G on $\mathcal{O}_2$ with the Rokhlin property. Since a tensor product action has the Rokhlin property if one factor does, we had better choose this action for ζ.

The equivariant absorption theorem for $\mathcal{O}_2$ follows immediately.

There is no action of G on $\mathcal{O}_\infty$ which has the Rokhlin property. Instead, we use the action $\iota \colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ above. Recall that it is the quasifree action coming from the direct sum of infinitely many copies of the regular representation.

The equivariant absorption theorem for $\mathcal{O}_\infty$ requires more work, but that is a subject for a different talk.

Equivariant semiprojectivity

For short, a *G-algebra* (G, A, α) is a C*-algebra A together with a continuous action $\alpha\colon G \to \mathrm{Aut}(A)$.

The following definition is the "right" way to get the property that approximately equivariant approximate homomorphisms are close to exactly equivariant true homomorphisms.

Definition

Let G be a topological group, and let (G, A, α) be a unital G-algebra. We say that (G, A, α) (or A, or α) is *equivariantly semiprojective* if whenever (G, C, γ) is a G-algebra, $J_0 \subset J_1 \subset \cdots$ are G-invariant ideals in C, $J = \overline{\bigcup_{n=0}^{\infty} J_n}$, and $\varphi\colon A \to C/J$ is a unital equivariant homomorphism, then there exists n and a unital equivariant homomorphism $\psi\colon A \to C/J_n$ such that the composition

$$A \xrightarrow{\psi} C/J_n \longrightarrow C/J$$

is equal to φ.

(Diagram on next slide.)

Equivariant semiprojectivity

(G, A, α) is equivariantly semiprojective if whenever (G, C, γ) is a G-algebra, $J_0 \subset J_1 \subset \cdots$ are G-invariant ideals in C, $J = \overline{\bigcup_{n=0}^{\infty} J_n}$, and $\varphi \colon A \to C/J$ is unital equivariant, then there exists n and a unital equivariant $\psi \colon A \to C/J_n$ such that the following diagram commutes:

$$\begin{tikzcd} & C \arrow[d] \\ & C/J_n \arrow[d] \\ A \arrow[r, "\varphi"'] \arrow[ur, dotted, "\psi"] & C/J \end{tikzcd}$$

Probably equivariant semiprojectivity is only interesting when G is compact, or perhaps even finite.

At least under good conditions, one has the equivariant analog of the usual relation between semiprojectivity and stable relations.

What do we do with equivariant semiprojectivity?

One easy consequence, which we use, is:

Theorem

Let G be a finite group. Let A be a unital Kirchberg algebra, let D be any unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon G \to \mathrm{Aut}(D)$ be actions of G on A and D. Equip $\mathcal{O}_{\infty}$ with the action ι above (the quasifree action coming from the direct sum of infinitely many copies of the regular representation). Then any unital equivariant asymptotic morphism from $\mathcal{O}_{\infty}$ to $A \otimes D$ is asymptotically equal to a continuous path of unital equivariant homomorphisms.

Equivariant semiprojectivity is also needed (in a less obvious way) for:

Theorem

Let A, α, D, β, and ι be as in the previous theorem, and suppose α is pointwise outer. Then any two unital equivariant asymptotic morphisms from $\mathcal{O}_{\infty}$ to $A \otimes D$ are equivariantly asymptotically unitarily equivalent.

Equivariant semiprojectivity of finite dimensional C*-algebras

We need certain quasifree actions on Cuntz algebras to be equivariantly semiprojective. The hardest step is:

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a compact group G on a finite dimensional C*-algebra A. Then (G, A, α) is equivariantly semiprojective.

We describe some ideas of the proof.

Rather than describing the lifting problem, we describe how to show that an approximately equivariant unital approximate homomorphism from A to some G-algebra B is close to an exactly equivariant unital true homomorphism.

Finite dimensional C*-algebras (continued)

Let (G, A, α) be a finite dimensional G-algebra, let (G, B, β) be a unital G-algebra, and let $\varphi \colon A \to B$ be unital, approximately equivariant, and an approximate homomorphism. We want to find a nearby equivariant unital homomorphism $\psi \colon A \to B$.

The usual method is to "straighten out" φ step by step, using functional calculus. It seems not to be possible to get equivariance this way.

Step 1: Restrict φ to the unitary group $U(A)$. Its values are then nearly unitary, and hence at least invertible.

Step 2: Average over G. Let μ be normalized Haar measure on G, and for $u \in U(A)$ set

$$\sigma(u) = \int_G \left(\beta_g \circ \varphi \circ \alpha_g^{-1}\right)(u)\, d\mu(g).$$

Then σ is exactly equivariant but is only approximately unitary and only approximately a group homomorphism. It is close to $\varphi|_{U(A)}$.

Finite dimensional C*-algebras (continued)

So far, we have an exactly equivariant approximately unitary approximate group homomorphism $\sigma\colon U(A) \to B$ which is close to $\varphi|_{U(A)}$.

Step 3: Set $\rho_0(u) = \sigma(u)\big[\sigma(u)^*\sigma(u)\big]^{-1/2}$. Then $\rho_0\colon U(A) \to U(B)$ (its values are exactly unitary), and ρ is exactly equivariant and approximately a group homomorphism.

Steps 4 and 5 below (without the equivariance) have been independently discovered by Grove, Karcher, and Ruh (1972), and by Kazhdan (1982).

Step 4(1): Let ν be normalized Haar measure on $U(A)$. For $u \in U(A)$ set

$$\rho_1(u) = \rho_0(u) \exp\left(\int_{U(A)} \log\left(\rho_0(u)^* \rho_0(uw) \rho_0(w)^* \right) d\nu(w) \right).$$

Then ρ_1 is still exactly unitary and exactly equivariant. It is still only approximately a group homomorphism, but (see below) the error is less than before. It is also close to ρ_0.

Finite dimensional C*-algebras (continued)

Step 4(1): Let ν be normalized Haar measure on $U(A)$. For $u \in U(A)$ set

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Then ρ_1 is still exactly unitary and exactly equivariant. It is still only approximately a group homomorphism, but (see below) the error is less than before. It is also close to ρ_0.

Step 4(n): Given ρ_{n-1}, set

$$\rho_n(u) = \rho_{n-1}(u) \exp\left(\int_{U(A)} \log\left(\rho_{n-1}(u)^* \rho_{n-1}(uw) \rho_{n-1}(w)^* \right) d\nu(w) \right).$$

This is close to ρ_{n-1}. It is still exactly unitary and exactly equivariant, and is yet closer to being a true homomorphism.

Finite dimensional C*-algebras (continued)

Step 4(n): Given ρ_{n-1}, set

$$\rho_n(u) = \rho_{n-1}(u) \exp\left(\int_{U(A)} \log\left(\rho_{n-1}(u)^* \rho_{n-1}(uw) \rho_{n-1}(w)^* \right) d\nu(w) \right).$$

Step 5: The maps ρ_n are exactly unitary and exactly equivariant. They form a Cauchy sequence, uniformly in $u \in U(A)$, and as $n \to \infty$, the errors $\|\rho_n(uv) - \rho_n(u)\rho_n(v)\|$ converge uniformly to zero. (See the next slide.) Therefore $\rho(u) = \lim_{n \to \infty} \rho_n(u)$ is an exactly equivariant homomorphism from $U(A)$ to $U(B)$. Moreover, ρ is uniformly close to $\varphi|_{U(A)}$.

Step 6: Since ρ and $\varphi|_{U(A)}$ are uniformly close, they are unitarily equivalent. It follows that ρ extends to a unital homomorphism from A to B. This is the equivariant homomorphism which is close to φ.

Finite dimensional C*-algebras (continued)

Recall:

$$\rho_n(u) = \rho_{n-1}(u) \exp\left(\int_{U(A)} \log\left(\rho_{n-1}(u)^* \rho_{n-1}(uw) \rho_{n-1}(w)^* \right) d\nu(w) \right).$$

Why is ρ_n better than ρ_{n-1}?

- One can check that if everything commutes, then ρ_n will in fact be exactly a homomorphism. (This comes from group cohomology.)
- There are constants C_1 and C_2 such that, if $\|a\|, \|b\| \leq r$ with r small enough, then

$$\big\| \exp(a+b) - \exp(a)\exp(b) \big\| \leq C_1 r^2$$

  and

$$\big\| \log\big((1+a)(1+b)\big) - \big(\log(1+a) + \log(1+b)\big) \big\| \leq C_2 r^2.$$

  (The linear terms in the power series cancel out.)
- Even with these estimates, things must work out just right.

Theorem

Let G be a finite group. Set $d = \mathrm{card}(G)$, and label the generators of $\mathcal{O}_d$ as s_g for $g \in G$. Then the quasifree action $\beta_g(s_h) = s_{gh}$ is equivariantly semiprojective.

Sketch of proof.

Let C, J_n, and J be as before (so G acts on everything and $J = \overline{\bigcup_{n=0}^{\infty} J_n}$), and let $\varphi\colon \mathcal{O}_d \to C/J$ be unital and equivariant.

The elements $s_g s_g^*$ generate a unital copy of $C(G)$ in $\mathcal{O}_d$, on which G acts by translation. Choose n such that one can lift $\varphi|_{C(G)}$ equivariantly to $\psi_0\colon C(G) \to C/J_n$. Increasing n, we may assume that $\psi_0(s_1 s_1^*)$ is Murray-von Neumann equivalent to 1. That is, there exists $t \in C/J_n$ such that $t^* t = 1$ and $t t^* = \psi_0(s_1 s_1^*)$. Increasing n further and modifying t, we may assume its image in C/J is $\varphi(s_1)$. Set $t_g = (\gamma_n)_g(t)$ for $g \in G$.

Equivariance of ψ_0 implies that $t_g t_g^* = \psi_0(s_g s_g^*)$ for all $g \in G$. Thus $\sum_{g \in G} t_g t_g^* = 1$. We can define an equivariant unital homomorphism $\psi\colon \mathcal{O}_d \to C/J_n$ by $\psi(s_g) = t_g$ for $g \in G$, and ψ lifts φ. $\square$

Equivariant semiprojectivity of further quasifree actions

Theorem

Let G be a finite group. Set $d = \mathrm{card}(G)$, and label the generators of $\mathcal{O}_d$ as s_g for $g \in G$. Then the quasifree action $\beta_g(s_h) = s_{gh}$ is equivariantly semiprojective.

One can use similar methods to get equivariant semiprojectivity for the quasifree action coming from the direct sum of finitely many copies of the regular representation of G, for the corresponding quasifree actions on the Cuntz-Toeplitz algebras, and, following Blackadar, for the quasifree action on $\mathcal{O}_\infty$ coming from the direct sum of infinitely many copies of the regular representation of G.