Lecture 5: Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property

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University of Oregon

19 July 2016

The Second Summer School on Operator Algebras and Noncommutative Geometry 2016

East China Normal University, Shanghai

11–29 July 2016

- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 6 (20 July 2016): Applications and Problems.

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A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
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Recall the Rokhlin property (with exact permutation of the projections): Let A be a unital C*-algebra, and let $\alpha \colon G \to \text{Aut}(A)$ be an action of a finite group G on A.

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Theorem

Let A be a unital AF algebra. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. Then $C^*(G, A, \alpha)$ is AF.

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We took $D_0 = \bigoplus_{g \in G} e_g A e_g \subset A$, which is a G-invariant unital subalgebra of A containing F to within ε. Then $D = C^*(G, D_0, \alpha) \subset C^*(G, A, \alpha)$ is a unital subalgebra which contains F to within ε and exactly contains u_g for $g \in G$. Thus D contains S to within ε. Also, $D \cong M_n(e_1 A e_1)$, which is AF. So finite sets in $C^*(G, A, \alpha)$ can be approximated by AF subalgebras, hence by finite dimensional subalgebras. Thus $C^*(G, A, \alpha)$ is AF.

Crossed products of AF algebras by Rokhlin actions

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Let A be a unital AF algebra. Let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. Then $C^*(G, A, \alpha)$ is AF.

The idea of the proof: We need only consider a finite set $S \subset C^*(G, A, \alpha)$ of the form $S = F \cup \{u_g : g \in G\}$, with $F \subset A$ finite and $u_g \in C^*(G, A, \alpha)$ the standard unitary corresponding to $g \in G$. We chose a family $(e_g)_{g \in G}$ in A of Rokhlin projections for α, F, and $\delta > 0$ (depending on ε). That is,

1. $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
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The tracial Rokhlin property (continued)

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$$v_k = \operatorname{diag}\big(1,\, 1,\, \ldots,\, 1,\, -1,\, -1,\, \ldots,\, -1\big),$$

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2. e_0 and e_1 approximately commute with all elements of F.
3. $1 - e_0 - e_1$ is "small", here, the (unique) tracial state τ on A gives $\tau(1 - e_0 - e_1) < \varepsilon$.

We can assume that there is n such that $F \subset A_n = \bigotimes_{k=1}^{n} M_{3^k}$. We can increase n, so also assume $3^{-n-1} < \varepsilon$. Set

$$p_0 = \begin{pmatrix} 1_{r(n+1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0_1 \end{pmatrix} \in M_{3^{n+1}} \qquad \text{and} \qquad p_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{r(n+1)} & 0 \\ 0 & 0 & 0_1 \end{pmatrix} \in M_{3^{n}}$$

Then $w_{n+1} p_0 w_{n+1}^* = p_1$, $w_{n+1} p_1 w_{n+1}^* = p_0$,

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$$v_k = \operatorname{diag}\big(1,\, 1,\, \ldots,\, 1,\, -1,\, -1,\, \ldots,\, -1\big),$$

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In fact, it turns out to be hard to write down a product type action of $\mathbb{Z}_2$ using conjugation by matrices of the form

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Outer actions on factors of type II_1

$\|a\|_{2,\tau} = \tau(a^* a)^{1/2}$.

G is finite, and M is a II_1 factor with tracial state τ. $\alpha \colon G \to \mathrm{Aut}(M)$ has the Rokhlin property if for every finite set $F \subset M$ and every $\varepsilon > 0$, there are projections $e_g \in M$ for $g \in G$ such that:

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Take $G = \mathbb{Z}_p$ with p prime. Then there are four possibilities:

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2. α has the tracial Rokhlin property but not the Rokhlin property.
3. α is pointwise outer but does not have the tracial Rokhlin property.

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In the first two cases, $\overline{\alpha}$ is outer and has the Rokhlin property.

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2. α has the tracial Rokhlin property but not the Rokhlin property.
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4. α is inner.

In the first two cases, $\overline{\alpha}$ is outer and has the Rokhlin property. In the last two cases, $\overline{\alpha}$ is inner.

Going between C*-algebras and von Neumann algebras

Let A be a simple unital C*-algebra with tracial rank zero and a unique tracial state τ. Let G be a finite group. Then $\alpha \colon G \to \mathrm{Aut}(A)$ has the C* tracial Rokhlin property if and only if $\overline{\alpha} \colon G \to \mathrm{Aut}\big(\pi_\tau(A)''\big)$ has the von Neumann algebraic Rokhlin property.

Take $G = \mathbb{Z}_p$ with p prime. Then there are four possibilities:

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In the first two cases, $\overline{\alpha}$ is outer and has the Rokhlin property. In the last two cases, $\overline{\alpha}$ is inner. All four cases can occur.

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