

Lecture 4: Crossed Products by Actions with the Rokhlin Property

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- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 6 (20 July 2016): Applications and Problems.

A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.

The Rokhlin property

Definition

Let A be a unital C*-algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A . Then α has the *Rokhlin property* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are projections $e_g \in A$ for $g \in G$ such that:

- 1 $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
- 2 $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- 3 $\sum_{g \in G} e_g = 1$. (In particular, the projections e_g are orthogonal.)

Let G be a finite group. Recall from the exercises in Lecture 3:

- 1 The action of G on G by translation gives an action of G on $C(G)$ (namely $\alpha_g(f)(h) = f(g^{-1}h)$) with the Rokhlin property.
- 2 Let A be any unital C*-algebra. The action of G on $\bigoplus_{g \in G} A$ by translation of the summands has the Rokhlin property.
- 3 Let G act freely on the Cantor set X . Then the corresponding action of G on $C(X)$ has the Rokhlin property.

$\alpha: G \rightarrow \text{Aut}(A)$ has the Rokhlin property if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are projections $e_g \in A$ for $g \in G$ such that:

- ① $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
- ② $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- ③ $\sum_{g \in G} e_g = 1$.

Exercise: Let $T \subset A$ be dense. Suppose that we prove the conditions above for every finite subset $F \subset T$. Then α has the Rokhlin property.

Exercise: More generally, prove the following lemma.

Lemma

Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group G on a unital C^* -algebra A . Let $T \subset A$ generate A as a C^* -algebra. Suppose that for every finite set $F \subset T$ and every $\varepsilon > 0$, there are projections $e_g \in A$ for $g \in G$ such that:

- ① $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
- ② $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- ③ $\sum_{g \in G} e_g = 1$.

Then α has the Rokhlin property.

A Rokhlin action on a simple C^* -algebra

The conditions in the definition of the Rokhlin property, for $\varepsilon > 0$ and a finite set $F \subset A$:

- ① $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
- ② $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- ③ $\sum_{g \in G} e_g = 1$.

We want an example in which A is simple. Thus, we won't be able to satisfy condition (2) by choosing e_g to be in the center of A .

Set

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Recall: $\text{Ad}(v)(a) = v a v^*$. Let α be the product type action of \mathbb{Z}_2 generated by

$$\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

We will show that this action has the Rokhlin property.

Using a generating set

Exercise: Prove the following lemma.

Lemma

Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group G on a unital C^* -algebra A . Let $T \subset A$ generate A as a C^* -algebra. Suppose that for every finite set $F \subset T$ and every $\varepsilon > 0$, there are projections $e_g \in A$ for $g \in G$ such that:

- ① $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
- ② $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- ③ $\sum_{g \in G} e_g = 1$.

Then α has the Rokhlin property.

Hint 1: The $*$ -algebra generated by T is dense.

Hint 2: F only appears in condition (2). If, say, a and b approximately commute with e_g , then ab approximately commutes with e_g because

$$\begin{aligned} \|a b e_g - e_g a b\| &= \|a(b e_g - e_g b) + (e_g a - a e_g)b\| \\ &\leq \|a\| \cdot \|b e_g - e_g b\| + \|e_g a - a e_g\| \cdot \|b\|. \end{aligned}$$

An example (continued)

We had

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The action α of \mathbb{Z}_2 is generated by

$$\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

Define projections $p_0, p_1 \in M_2$ by

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$w p_0 w^* = p_1, \quad w p_1 w^* = p_0, \quad \text{and} \quad p_0 + p_1 = 1.$$

The action $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \text{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, in M_2 , $wp_0w^* = p_1$, $wp_1w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property, specialized to $G = \mathbb{Z}_2$. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_0 and e_1 such that:

- 1 $\|\beta(e_0) - e_1\| < \varepsilon$ and $\|\beta(e_1) - e_0\| < \varepsilon$.
- 2 $\|e_0a - ae_0\| < \varepsilon$ and $\|e_1a - ae_1\| < \varepsilon$ for all $a \in F$.
- 3 $e_0 + e_1 = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense in A , we can assume $F \subset A_n$ for some n . (See above.)

For $g = 0, 1 \in \mathbb{Z}_2$, take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$

Clearly $e_0 + e_1 = 1$. Check that $\beta(e_0) = e_1$ and $\beta(e_1) = e_0$, and that e_0 and e_1 actually commute with everything in F . (Proofs: See the next slide.) This proves the Rokhlin property.

Some other actions with the Rokhlin property

Let G be a finite group, and set $n = \text{card}(G)$. Let $g \mapsto v_g$ be the left regular representation of G on $\ell^2(G)$, identify $L(\ell^2(G))$ with M_n , and let $A = \bigotimes_{k=1}^{\infty} M_n$ be the n^∞ UHF algebra. Then the action

$$g \mapsto \alpha_g = \bigotimes_{n=1}^{\infty} \text{Ad}(v_g)$$

of G on A has the Rokhlin property.

The example we just did is the case $G = \mathbb{Z}_2$, and the proof in the general case is the same.

Exercise: Write down a detailed proof that this action has the Rokhlin property.

An example (continued)

The projections e_0 and e_1 actually commute with everything in F , essentially because the nontrivial parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_{2^{n+1}}$, which we identify with $M_{2^n} \otimes M_2$. In this tensor factorization,

$$e_g = 1 \otimes p_g,$$

and elements of F have the form

$$a \otimes 1.$$

Clearly these commute.

For $\beta(e_0) = e_1$: we have $\beta|_{A_{n+1}} = \text{Ad}(w^{\otimes n} \otimes w)$, so

$$\beta(e_0) = (w^{\otimes n} \otimes w)(1 \otimes p_0)(w^{\otimes n} \otimes w)^* = 1 \otimes wp_0w^* = 1 \otimes p_1 = e_1.$$

The proof that $\beta(e_1) = e_0$ is the same. We are done.

Yet more actions with the Rokhlin property

Let G be a finite group, and set $n = \text{card}(G)$.

Let \mathcal{O}_n be the Cuntz algebra. (Cuntz algebras, and some actions on them, are discussed in the appendix to Lecture 3.) However, call its generators s_g for $g \in G$. The relations are thus

$$s_g^* s_g = 1$$

for all $g \in G$, and

$$\sum_{g \in G} s_g s_g^* = 1.$$

There is an action $\gamma: G \rightarrow \text{Aut}(\mathcal{O}_n)$ such that

$$\gamma_g(s_h) = s_{gh}$$

for $g, h \in G$. This action is a special case of the quasifree actions on Cuntz algebras in the appendix to Lecture 3. It turns out to have the Rokhlin property (Izumi).

Actions on Cuntz algebras (continued)

Take $G = \mathbb{Z}_2$ on the previous slide. The resulting action γ of \mathbb{Z}_2 on \mathcal{O}_2 is generated by the order 2 automorphism determined by $s_1 \mapsto s_2$ and $s_2 \mapsto s_1$.

The action of \mathbb{Z}_2 on \mathcal{O}_2 generated by $s_1 \mapsto s_1$ and $s_2 \mapsto -s_2$ is conjugate to the one gotten using $G = \mathbb{Z}_2$ above, so also has the Rokhlin property.

Exercise (if you know about Cuntz algebras): Prove this conjugacy. Hint: Use an automorphism of \mathcal{O}_2 of the same sort as those that appeared in the definition of quasifree actions in Cuntz algebras as in the appendix to Lecture 3. (It will come from a unitary operator on \mathbb{C}^2 .)

The quasifree action of \mathbb{Z}_2 on \mathcal{O}_2 generated by $s_1 \mapsto -s_1$ and $s_2 \mapsto -s_2$ turns out to be pointwise outer but *not* to have the Rokhlin property.

Exactly permuting the projections (continued)

In the definition of the Rokhlin property, one can replace “ $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$ ” with “ $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$ ”.

The proof uses methods (equivariant semiprojectivity) unrelated to those here. This result simplifies some proofs by replacing some approximate equalities by equalities, so we will assume it, but it makes no real difference.

(This simplification has not been made in the crossed product notes—proving the theorem is more complicated than doing without it.)

Exactly permuting the projections

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_g such that:

- 1 $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
- 2 $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- 3 $\sum_{g \in G} e_g = 1$.

Theorem (2011)

Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group G on A . Then α has the Rokhlin property if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

- 1 $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
- 2 $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- 3 $\sum_{g \in G} e_g = 1$.

The difference is that in (1) we require exact equality.

AF algebras

We give a very brief summary of AF algebras, restricted for convenience to the unital case, and refer to the lectures of Zhuang Niu for more.

Definition

Let A be a unital C^* -algebra. Then A is an *AF algebra* if there is an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A$$

of unital finite dimensional subalgebras of A such that $\overline{\bigcup_{n=0}^{\infty} A_n} = A$.

Convention: When we refer to a unital subalgebra C of a unital C^* -algebra A , we mean that $1_A \in C$.

Some examples: The UHF algebras we have already seen; $K(H)^+$ (the unitization); $C(X)$ for the Cantor set X .

AF algebras are a basic set of examples (going back to Bratteli), and work on them continues to this day. They are the subject of the original Elliott classification theorem.

AF algebras (continued)

Recall:

Definition

Let A be a unital C^* -algebra. Then A is an *AF algebra* if there is an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A$$

of unital finite dimensional subalgebras of A such that $\overline{\bigcup_{n=0}^{\infty} A_n} = A$.

Theorem (Bratteli)

Let A be a separable unital C^* -algebra. Then the following are equivalent:

- 1 A is a AF algebra.
- 2 For every finite set $F \subset A$ and every $\varepsilon > 0$, there is a unital finite dimensional subalgebra $D \subset A$ such that $\text{dist}(a, D) < \varepsilon$ for all $a \in F$.

Actions on AF algebras (continued)

A unital C^* -algebra A is AF if there is an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A$$

of finite dimensional unital subalgebras of A such that $\overline{\bigcup_{n=0}^{\infty} A_n} = A$.

Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group G on a unital AF algebra A . One might hope that $C^*(G, A, \alpha)$ would again be AF. If one uses algebraic direct limits, this is in fact true.

The C^* version was open for some time, but turns out to be false. (The hint in the exercise on the previous slide doesn't work.) If A is AF, then $K_1(A) = 0$, $K_0(A)$ is torsion free, and A has real rank zero (definition omitted). There are (separate) examples of actions of \mathbb{Z}_2 on simple AF algebras such that the crossed product has nonzero K_1 (Blackadar), does not have real rank zero (Elliott), and has torsion in K_0 .

Actions on AF algebras

A unital C^* -algebra A is AF if there is an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A$$

of finite dimensional unital subalgebras of A such that $\overline{\bigcup_{n=0}^{\infty} A_n} = A$.

Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group G on a unital AF algebra A . One might hope that $C^*(G, A, \alpha)$ would again be AF.

The algebraic version of this is true. That is, if a complex $*$ -algebra A can be written as $A = \bigcup_{n=0}^{\infty} A_n$ for finite dimensional C^* -algebras $A_0 \subset A_1 \subset \cdots$, and if $\alpha: G \rightarrow \text{Aut}(A)$ is an action of a finite group G on A , then the algebraic crossed product is again an increasing union of the same type.

The idea is to replace $A_0 \subset A_1 \subset \cdots$ with finite dimensional C^* -algebras $B_0 \subset B_1 \subset \cdots$ such that $\alpha_g(B_n) \subset B_n$ for all $g \in G$ and $n \in \mathbb{N}$.

Exercise: Carry it out. Hint: To start, the subalgebra generated by $\bigcup_{g \in G} \alpha_g(A_0)$ is contained in A_n for some n .

Crossed products by actions with the Rokhlin property

A structure theorem for crossed products by actions with the Rokhlin property:

Theorem

Let A be a unital AF algebra. Let G be a finite group, and let $\alpha: G \rightarrow \text{Aut}(A)$ have the Rokhlin property. Then $C^*(G, A, \alpha)$ is AF.

Crossed products by actions of finite groups with the Rokhlin property preserve many other structural properties of C^* -algebras. (See below.)

Crossed products by actions with the Rokhlin property

Let A be a unital AF algebra. Let G be a finite group, and let $\alpha: G \rightarrow \text{Aut}(A)$ have the Rokhlin property. We claim $C^*(G, A, \alpha)$ is AF.

The basic idea (details later). Set $n = \text{card}(G)$. Recall that

$$C^*(G, A, \alpha) = A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g : c_g \in A \text{ for } g \in G \right\}.$$

and $(a \cdot u_g)(b \cdot u_h) = (a\alpha_g(b)) \cdot u_{gh}$.

Let $e_g \in A$, for $g \in G$, be Rokhlin projections, with $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$. Then $\text{span}(\{e_g : g \in G\})$ is a G -invariant subalgebra isomorphic to $C(G)$ with the action from translation of G on G . Let $u_g \in C^*(G, A, \alpha)$ be the canonical unitary implementing the automorphism α_g . Then $v_{g,h} = e_g u_{gh^{-1}}$ defines a system of matrix units in $C^*(G, A, \alpha)$. (This is essentially the same formula as was used in the proof that $C^*(G, C(G)) \cong M_n$.) Using the homomorphism $M_n \otimes e_1 A e_1 \rightarrow C^*(G, A, \alpha)$ given by $v_{g,h} \otimes d \mapsto v_{g,1} d v_{1,h}$, one can approximate $C^*(G, A, \alpha)$ by matrix algebras over corners of A .

Idea of the proof (continued)

A is an AF algebra, G is a finite group, and $\alpha: G \rightarrow \text{Aut}(A)$ has the Rokhlin property. Our finite set is $S = F \cup \{u_g : g \in G\} \subset C^*(G, A, \alpha)$, with $F \subset A$ finite. We will approximate S by an AF algebra.

Apply the Rokhlin property to the finite set F . Use the version in which the projections are exactly permuted by the group. Thus, we get projections $e_g \in A$ for $g \in G$ such that:

- ① $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
- ② $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
- ③ $\sum_{g \in G} e_g = 1$. (In particular, the projections e_g are orthogonal.)

Informally: $e_g a \approx a e_g$ for all $g \in G$ and all $a \in F$.

In particular, for $g \neq h$ and $a \in F$, $e_g a e_h \approx a e_g e_h = 0$. Therefore, if $a \in F$,

$$a = \sum_{g, h \in G} e_g a e_h \approx \sum_{g \in G} e_g a e_g.$$

That is, a is approximately in $D_0 = \sum_{g \in G} e_g A e_g \subset A$.

Idea of the proof

A is an AF algebra, G is a finite group, and $\alpha: G \rightarrow \text{Aut}(A)$ has the Rokhlin property. We want to approximate a finite set $S \subset C^*(G, A, \alpha)$ by a finite dimensional subalgebra.

It turns out that it suffices to consider finite subsets of some generating set. (The argument is easier than the corresponding argument for the Rokhlin property. Exercise: Do it.) So we assume $S = F \cup \{u_g : g \in G\}$, with $F \subset A$ finite and $u_g \in C^*(G, A, \alpha)$ the standard unitary corresponding to $g \in G$.

We will find an AF algebra $D \subset C^*(G, A, \alpha)$ which approximately contains S . It is not hard to see that this is enough. (Exercise: check this!) We give a sketch first, and then a careful proof (with some steps left as exercises).

Preliminary exercise: Let B be a C^* -algebra and let $q \in B$ be a projection. Show that qBq is a C^* -algebra, with identity q .

Idea of the proof (continued)

A is an AF algebra, G is a finite group, and $\alpha: G \rightarrow \text{Aut}(A)$ has the Rokhlin property. Our finite set is $S = F \cup \{u_g : g \in G\} \subset C^*(G, A, \alpha)$, with $F \subset A$ finite. We will approximate S by an AF algebra. We chose Rokhlin projections $e_g \in A$ for $g \in G$.

We have found that F is approximately contained in the unital subalgebra (justification for subalgebra and direct sum below)

$$D_0 = \sum_{g \in G} e_g A e_g = \bigoplus_{g \in G} e_g A e_g \subset A.$$

The sum is direct because the projections e_g are orthogonal, and D_0 is unital because $\sum_{g \in G} e_g = 1$. Exercise: Prove that if B is a C^* -algebra and $p_1, p_2, \dots, p_n \in B$ are mutually orthogonal projections, then $\sum_{k=1}^n p_k B p_k$ is a C^* subalgebra of B isomorphic to $\bigoplus_{k=1}^n p_k B p_k$.

Recall that we are assuming that $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$. Exercise: Use this to prove that D_0 is G -invariant.

Idea of the proof (continued)

A is an AF algebra, G is a finite group, and $\alpha: G \rightarrow \text{Aut}(A)$ has the Rokhlin property. Our finite set is $S = F \cup \{u_g: g \in G\} \subset C^*(G, A, \alpha)$, with $F \subset A$ finite. We will approximate S by an AF algebra. We chose Rokhlin projections $e_g \in A$ for $g \in G$, and we found that $D_0 = \bigoplus_{g \in G} e_g A e_g$ is a unital G -invariant subalgebra of A which approximately contains F .

The action of G permutes the summands. Exercise: Prove that D_0 is equivariantly isomorphic to $C(G, e_1 A e_1)$ with the action $\beta_g(b)(h) = b(g^{-1}h)$ for $g, h \in G$ and $b \in C(G, e_1 A e_1)$.

Set $n = \text{card}(G)$. We showed before that $C^*(G, C(G)) \cong M_n$. Exercise: Use the same method to prove that if B is any unital C^* -algebra, and $\beta: G \rightarrow \text{Aut}(C(G, B))$ is the action $\beta_g(b)(h) = b(g^{-1}h)$ for $g, h \in G$ and $b \in C(G, B)$, then $C^*(G, C(G, B)) \cong M_n(B)$.

Set $D = C^*(G, D_0, \alpha) \subset C^*(G, A, \alpha)$. Thus $D \cong M_n(e_1 A e_1)$.

Idea of the proof (continued)

A is an AF algebra, G is a finite group, and $\alpha: G \rightarrow \text{Aut}(A)$ has the Rokhlin property. Our finite set is $S = F \cup \{u_g: g \in G\} \subset C^*(G, A, \alpha)$, with $F \subset A$ finite. We got $D \subset C^*(G, A, \alpha)$ as $D = C^*(G, D_0, \alpha)$, in which $D_0 \subset A$ is a unital subalgebra which approximately contains F .

Since D_0 is unital, $u_g \in D$ for all $g \in G$. Therefore D approximately contains $S = F \cup \{u_g: g \in G\}$, as wanted.

All that remains is to show that D is AF. Recall that $D \cong M_n(e_1 A e_1)$.

It is a general fact that if C is an AF algebra and $q \in C$ is a projection, then qCq is an AF algebra. (Suppose $C = \overline{\bigcup_{n=0}^{\infty} C_n}$ for an increasing sequence of finite dimensional C^* -algebras $(C_n)_{n \in \mathbb{N}}$. Using methods from K-theory, show that q is unitarily equivalent to a projection in one of the C_n . Now the result is easy. Exercise: Write out the details.) Since $e_1 A e_1$ is AF, so is $D \cong M_n(e_1 A e_1)$.

A future modification of the argument

We want to approximate elements of $C^*(G, A, \alpha)$ using unital homomorphisms from $M_n \otimes e_1 A e_1$ to $C^*(G, A, \alpha)$.

In Lecture 5, we are going to need the same argument again, but under slightly weaker conditions. We will still assume that the projections e_g are orthogonal, are exactly permuted by the group action, and can be chosen to approximately commute with a given finite subset of A . However, the sum $e = \sum_{g \in G} e_g$ will no longer necessarily be equal to 1.

We can nevertheless carry out the same argument; we get unital homomorphisms from $M_n \otimes e_1 A e_1$ to $eC^*(G, A, \alpha)e$, and we just get the weaker conclusion that we can approximate a finite set in $eC^*(G, A, \alpha)e$, rather than one in $C^*(G, A, \alpha)$, by a matrix algebra over a corner of A .

Crossed products by actions with the Rokhlin property (continued)

Recall the conclusion of the theorem: $C^*(G, A, \alpha)$ is AF.

To prove the theorem, we prove that for every finite set $S \subset C^*(G, A, \alpha)$ and every $\varepsilon > 0$, there is an AF subalgebra $D \subset C^*(G, A, \alpha)$ such that every element of S is within ε of an element of D . Let $u_g \in C^*(G, A, \alpha)$ be the canonical unitary implementing the automorphism α_g . Thus, a general element has the form $\sum_{g \in G} c_g u_g$, with $c_g \in A$ for $g \in G$. It suffices to consider a finite set of the form $S = F \cup \{u_g: g \in G\}$, where F is a finite subset of A . So let $F \subset A$ be a finite subset and let $\varepsilon > 0$.

Set

$$n = \text{card}(G) \quad \text{and} \quad \delta = \frac{\varepsilon}{n(n-1)}.$$

Crossed products by actions with the Rokhlin property (continued)

We had: $S = F \cup \{u_g : g \in G\}$, with F a finite subset of A .

Apply the Rokhlin property to α with F as given and with δ in place of ε , obtaining projections $e_g \in A$ for $g \in G$ such that $\alpha_g(e_h) = e_{gh}$ for $g, h \in G$, $\|e_g a - a e_g\| < \delta$ for $g \in G$ and $a \in F$, and $\sum_{g \in G} e_g = 1$.

Define $v_{g,h} = e_g u_{gh^{-1}}$ for $g, h \in G$. In particular, $v_{g,g} = e_g$, so the $v_{g,g}$ are orthogonal projections which add up to 1.

We claim that the $v_{g,h}$ form a system of $n \times n$ matrix units in $C^*(G, A, \alpha)$. Recall for comparison: when proving that $C^*(G, C(G)) \cong M_n$, we used the matrix units $v_{g,h} = \chi_{\{g\}} u_{gh^{-1}}$. The computation here is exactly the same as there, so we don't repeat it.

Crossed products by actions with the Rokhlin property (continued)

We have to approximate elements of $S = F \cup \{u_g : g \in G\}$ by elements of $D = \varphi(M_n \otimes e_1 A e_1)$.

We first consider u_g with $g \in G$. In fact, for u_g no approximation is necessary. Recall that $v_{g,h} = e_g u_{gh^{-1}}$. We have

$$\varphi\left(\sum_{h \in G} w_{h,g^{-1}h}\right) = \varphi_0\left(\sum_{h \in G} w_{h,g^{-1}h}\right) = \sum_{h \in G} v_{h,g^{-1}h} = \sum_{h \in G} e_h u_g = u_g.$$

Crossed products by actions with the Rokhlin property (continued)

We had: $(v_{g,h})_{g,h \in G}$ is an $n \times n$ system of matrix units in $C^*(G, A, \alpha)$.

Let $(w_{g,h})_{g,h \in G}$ be a system of matrix units for M_n . There is a unital homomorphism $\varphi_0: M_n \rightarrow C^*(G, A, \alpha)$ such that $\varphi_0(w_{g,h}) = v_{g,h}$ for all $g, h \in G$. In particular, $\varphi_0(w_{g,g}) = e_g$ for all $g \in G$.

Now define a unital homomorphism $\varphi: M_n \otimes e_1 A e_1 \rightarrow C^*(G, A, \alpha)$ by $\varphi(w_{g,h} \otimes d) = v_{g,1} d v_{1,h}$ for $g, h \in G$ and $d \in e_1 A e_1$.

Exercise: Prove that φ is a *-homomorphism.

Corners of AF algebras are AF, and φ is injective, so $D = \varphi(M_n \otimes e_1 A e_1)$ is an AF subalgebra of $C^*(G, A, \alpha)$. We complete the proof by showing that every element of S is within ε of an element of D . Recall that $S = F \cup \{u_g : g \in G\}$, and F is a finite subset of A .

Approximating elements of F

We have to approximate elements of $S = F \cup \{u_g : g \in G\}$, with $F \subset A$ finite, by elements of $D = \varphi(M_n \otimes e_1 A e_1)$. Recall that $v_{g,h} = e_g u_{gh^{-1}}$, and that $\varphi: M_n \otimes e_1 A e_1 \rightarrow C^*(G, A, \alpha)$ is defined by $\varphi(w_{g,h} \otimes d) = v_{g,1} d v_{1,h}$ for $g, h \in G$ and $d \in e_1 A e_1$. We already took care of u_g .

Let $a \in F$. The obvious first step in approximating a is to use

$$\sum_{g \in G} e_g a e_g.$$

In fact, one needs to (implicitly) use this approximation in the form

$$\sum_{g \in G} \alpha_g(e_1 \alpha_g^{-1}(a) e_1).$$

This happens because the definition of φ sends $w_{g,h} \otimes d$, for $d \in e_1 A e_1$, to an element obtained by using the action of the group elements g and h .

Approximating elements of F (continued)

We have to approximate elements of $S = F \cup \{u_g : g \in G\}$, with $F \subset A$ finite, by elements of $D = \varphi(M_n \otimes e_1 A e_1)$. We already took care of u_g , but we still need to deal with $a \in F$.

There are two steps:

① Show that $\left\| a - \sum_{g \in G} e_g a e_g \right\| < \varepsilon$.

② Show that $\sum_{g \in G} e_g a e_g$ is in the range of φ .

Once we have these, we are done: we have $\text{dist}(a, D) < \varepsilon$.

Step 1: Recall that $n = \text{card}(G)$. We chose $\delta > 0$ so that $n(n-1)\delta = \varepsilon$, and we chose Rokhlin projections $e_g \in A$ such that $\|e_g a - a e_g\| < \delta$ for $a \in F$ and $g \in G$. For $g \neq h$, we therefore have

$$\|e_g a e_h\| \leq \|e_g a - a e_g\| + \|a e_g e_h\| = \|e_g a - a e_g\| < \delta.$$

So

$$\left\| a - \sum_{g \in G} e_g a e_g \right\| \leq \sum_{g \neq h} \|e_g a e_h\| < n(n-1)\delta = \varepsilon.$$

This finishes step 1.

Approximating elements of F (conclusion)

To finish the proof of the theorem, we need only do step 2 from the previous slide: Show that $\sum_{g \in G} e_g a e_g$ is in the range of the map $\varphi: M_n \otimes e_1 A e_1 \rightarrow C^*(G, A, \alpha)$.

Recall that $v_{g,h} = e_g u_{gh^{-1}}$, that $(v_{g,h})_{g,h \in G}$ is a system of $n \times n$ matrix units in $C^*(G, A, \alpha)$, and that $\varphi(w_{g,h} \otimes d) = v_{g,1} d v_{1,h}$.

Set

$$b = \sum_{g \in G} w_{g,g} \otimes e_1 \alpha_g^{-1}(a) e_1 \in M_n \otimes e_1 A e_1.$$

Now (justifications given afterwards):

$$\begin{aligned} \varphi(b) &= \sum_{g \in G} v_{g,1} e_1 \alpha_g^{-1}(a) e_1 v_{g,1}^* = \sum_{g \in G} e_g u_g e_1 \alpha_g^{-1}(a) e_1 u_g^* e_g \\ &= \sum_{g \in G} e_g \alpha_g(e_1 \alpha_g^{-1}(a) e_1) e_g = \sum_{g \in G} e_g a e_g. \end{aligned}$$

The first step uses $v_{g,1}^* = v_{1,g}$ (matrix unit property). The second step is the definition of $v_{g,1}$. The third step is the fact that u_g implements α_g . The fourth step is $\alpha_g(e_1) = e_g$ and $e_g^2 = e_g$.

This completes the proof of the theorem.

Other structural consequences of the Rokhlin property

Crossed products by actions of finite groups with the Rokhlin property preserve various other classes of C^* -algebras. In many cases, the proofs are similar to what we did for AF algebras. Some examples of such classes:

- ① Simple unital C^* -algebras.
- ② Various classes of unital but not necessarily simple countable direct limit C^* -algebras using semiprojective building blocks. (With Osaka.)
- ③ Simple unital AH algebras with slow dimension growth and real rank zero. (With Osaka.)
- ④ D -absorbing separable unital C^* -algebras for a strongly self-absorbing C^* -algebra D . (Hirshberg-Winter.)
- ⑤ Separable nuclear unital C^* -algebras whose quotients all satisfy the Universal Coefficient Theorem. (With Osaka.)
- ⑥ Separable unital approximately divisible C^* -algebras. (Hirshberg-Winter.)
- ⑦ Unital C^* -algebras with the ideal property and unital C^* -algebras with the projection property. (With Pasnicu.)