Lecture 2: Crossed Products by Finite Groups; the Rokhlin Property

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- Lecture 1 (26 July 2014): Actions of Finite Groups on C*-Algebras and Introduction to Crossed Products.
- Lecture 2 (27 July 2014): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 3 (28 July 2014): Crossed Products by Actions with the Rokhlin Property.
- Lecture 4 (29 July 2014): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 5 (30 July 2014): Examples and Applications.

A rough outline of all five lectures

- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.

Crossed products by finite groups

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all finite formal linear combinations of elements in G with coefficients in A. We conventionally write u_g instead of g for the element of $A[G]$. Thus, a general element of $A[G]$ has the form $c = \sum_{g \in G} c_g u_g$ with $c_g \in A$ for $g \in G$. The multiplication and adjoint are given by:

$$(a u_g)(b u_h) = (a[u_g b u_g^{-1}]) u_{gh} = (a \alpha_g(b)) u_{gh}$$

$$(a u_g)^* = u_g^* a^* = (u_g^{-1} a^* u_g) u_g^{-1} = \alpha_g^{-1}(a^*) u_{g^{-1}}.$$

for $a, b \in A$ and $g, h \in G$, extended linearly. In particular, $u_g^* = u_{g^{-1}}$.

Exercise: Prove that these definitions make $A[G]$ a *-algebra over $\mathbb{C}$.

There is a unique norm which makes this a C*-algebra. (See below.)

Crossed products by finite groups (continued)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. To keep as elementary as possible, assume that A is unital. We construct a C* norm on the skew group ring $A[G]$.
Recall:

$$(au_g)(bu_h) = (a\alpha_g(b))u_{gh} \quad \text{and} \quad (au_g)^* = \alpha_g^{-1}(a^*)u_{g^{-1}}.$$

Fix a unital faithful representation $\pi \colon A \to L(H_0)$ of A on a Hilbert space H_0. Set $H = l^2(G, H_0)$, the set of all $\xi = (\xi_g)_{g \in G}$ in $\bigoplus_{g \in G} H_0$, with the scalar product

$$\big\langle (\xi_g)_{g \in G}, \, (\eta_g)_{g \in G} \big\rangle = \sum_{g \in G} \langle \xi_g, \eta_g \rangle.$$

Then define $\sigma \colon A[G] \to L(H)$ as follows. For $c = \sum_{g \in G} c_g u_g$,

$$(\sigma(c)\xi)_h = \sum_{g \in G} \pi(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h})$$

for all $h \in G$. (Some explanation is on the next slide.)

Crossed products by finite groups (continued)

Recall:

$$(au_g)(bu_h) = a\alpha_g(b)u_{gh} \quad \text{and} \quad (au_g)^* = \alpha_g^{-1}(a^*)u_{g^{-1}}.$$

Also, for $c = \sum_{g \in G} c_g u_g$,

$$(\sigma(c)\xi)_h = \sum_{g \in G} \pi(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h}).$$

For $a \in A$ and $g \in G$, identify a with au_1 and get

$$(\sigma(a)\xi)_h = \pi(\alpha_{h^{-1}}(a))(\xi_h) \quad \text{and} \quad (\sigma(u_g)\xi)_h = \xi_{g^{-1}h}.$$

One can check that σ is a *-homomorphism. We will just check the most important part, which is that $\sigma(u_g)\sigma(b) = \sigma(\alpha_g(b))\sigma(u_g)$. We have

$$\big[\sigma(\alpha_g(b))\sigma(u_g)\xi\big]_h = \pi\big(\alpha_{h^{-1}}(\alpha_g(b))\big)(\sigma(u_g)\xi)_h = \pi(\alpha_{h^{-1}g}(b))(\xi_{g^{-1}h})$$

and

$$\big(\sigma(u_g)\sigma(b)\xi\big)_h = \big(\sigma(b)\xi\big)_{g^{-1}h} = \pi\big(\alpha_{h^{-1}g}(b)\big)\xi\big)_{g^{-1}h}.$$

Exercise: Prove in detail that σ, as defined above, is a *-homomorphism.

Crossed products by finite groups (continued)

Recall: $\pi \colon A \to L(H_0)$ is an isometric representation of A, $H = \bigoplus_{g \in G} H_0$, and $\sigma \colon A[G] \to L(H)$ is given, for $c = \sum_{g \in G} c_g u_g \in A[G]$ and $\xi = (\xi_g)_{g \in G} \in H$, by

$$(\sigma(c)\xi)_h = \sum_{g \in G} \pi(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h}).$$

It is easy to check that

$$\|\sigma(c)\| \leq \sum_{g \in G} \|c_g\|.$$

Exercise: Prove this.

Exercise: Prove that $\|\sigma(c)\| \geq \max_{g \in G} \|c_g\|$.

Hint: Look at $\sigma(c)\xi$ for ξ in just one of the summands of H_0 in H, that is, $\xi_k = 0$ for all but one $k \in G$.

The norms on the right hand sides are equivalent, so $A[G]$ is complete in the norm $\|c\| = \|\sigma(c)\|$.

Crossed products by finite groups (continued)

We are still considering an action $\alpha \colon G \to \mathrm{Aut}(A)$ of a finite group G on a C*-algebra A.

We started with a faithful representation $\pi \colon A \to L(H_0)$ of A on a Hilbert space H_0. Then we constructed a representation $\sigma \colon A[G] \to L(l^2(G, H_0))$, given, for $c = \sum_{g \in G} c_g u_g \in A[G]$ and $\xi = (\xi_g)_{g \in G} \in H$, by

$$(\sigma(c)\xi)_h = \sum_{g \in G} \pi(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h}).$$

We found that $A[G]$ is complete in the norm $\|c\| = \|\sigma(c)\|$. By standard theory, the norm $\|c\| = \|\sigma(c)\|$ is therefore the only norm in which $A[G]$ is a C*-algebra. In particular, it does not depend on the choice of π.

We return to the notation $C^*(G, A, \alpha)$ for the crossed product.

Things are more complicated if G is discrete but not finite. (In particular, there may be more than one reasonable norm—since $A[G]$ isn't complete, this is not ruled out.) The situation is even more complicated if G is merely locally compact.

Universal property of crossed products by finite groups

Crossed products are supposed to have the following property:

Theorem

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then the integrated form construction defines a bijection from the set of nondegenerate covariant representations of (G, A, α) on a Hilbert space H to the set of nondegenerate representations of $C^*(G, A, \alpha)$ on the same Hilbert space.

Exercise: When G is finite and A is unital, prove that $C^*(G, A, \alpha)$, as constructed above, has the universal property in this theorem. Hint: All the calculations are algebra; no analysis is needed. The key to the algebra is to compare the definition of the product in $A[G]$ (recall that $u_g a u_g^* = \alpha_g(a)$) with the condition $v_g \pi(a) v_g^* = \pi(\alpha_g(a))$ in the definition of a covariant representation. The integrated form sends u_g to v_g.

Exercise: Do the same without the requirement that A be unital. Hint: Now one needs one piece of analysis: an approximate identity for A.

Examples of crossed products by finite groups

Let G be a finite group, and let $\iota\colon G \to \mathrm{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of G. (So far, G could be any locally compact group.)

Since we are assuming that G is finite, this is a finite dimensional C*-algebra, with $\dim(C^*(G)) = \mathrm{card}(G)$. If G is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\mathrm{card}(G)}$.

If G is a general finite group, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand M_k for each unitary equivalence class of irreducible representations of G, with k being the dimension of the representation.

Now let A be any C*-algebra, and let $\iota_A\colon G \to \mathrm{Aut}(A)$ be the trivial action. It is not hard to see that $C^*(G, A, \iota_A) \cong C^*(G) \otimes A$. The elements of A "factor out", since $A[G]$ is just the ordinary group ring.

Exercise: prove this. (Since $C^*(G)$ is finite dimensional, $C^*(G) \otimes A$ is the algebraic tensor product.)

Examples of crossed products (continued)

Let G be a finite group, acting on $C(G)$ via the translation action on G. Set $n = \mathrm{card}(G)$. We describe how to prove that $C^*(G,\, C(G)) \cong M_n$.

Let $\alpha\colon G \to \mathrm{Aut}(C(G))$ denote the action. For $g \in G$, we let u_g be the standard unitary (as above), and we let $\delta_g \in C(G)$ be the function $\chi_{\{g\}}$. Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. (Exercise: Prove this.) For $g, h \in G$, set

$$v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G,\, C(G),\, \alpha).$$

These elements form a system of matrix units. We check:

$$\begin{aligned} v_{g_1,h_1} v_{g_2,h_2} &= \delta_{g_1} u_{g_1 h_1^{-1}} \delta_{g_2} u_{g_2 h_2^{-1}} \\ &= \delta_{g_1} \alpha_{g_1 h_1^{-1}}(\delta_{g_2}) u_{g_1 h_1^{-1}} u_{g_2 h_2^{-1}} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}. \end{aligned}$$

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is v_{g_1,h_2}. Similarly (do it as an exercise), $v_{g,h}^* = v_{h,g}$.

Since the elements δ_g span $C(G)$, the elements $v_{g,h}$ span $C^*(G,\, C(G),\, \alpha)$. So $C^*(G,\, C(G),\, \alpha) \cong M_n$ with $n = \mathrm{card}(G)$.

Examples of crossed products (continued)

Let G be a finite group, acting on $C(G)$ via the translation action on G. Set $n = \mathrm{card}(G)$. Then $C^*(G,\, C(G)) \cong M_n$.

Now consider G acting on $G \times X$, by translation on G and trivially on X. Exercise: Use the same method to prove that $C^*(G,\, C_0(G \times X)) \cong C_0(X, M_n)$.

A harder exercise: Prove that for any action of G on X, and using the diagonal action on $G \times X$, we still have $C^*(G,\, C_0(G \times X)) \cong C_0(X, M_n)$. Hint: A trick reduces this to the previous exercise.

This result generalizes greatly: for any locally compact group G, one gets $C^*(G,\, C_0(G)) \cong K(L^2(G))$, etc.

Equivariant homomorphisms

We will describe several more examples, mostly without proof. To understand what to expect, the following is helpful.

For $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$, we say that a homomorphism $\varphi\colon A \to B$ is *equivariant* if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

An equivariant homomorphism $\varphi\colon A \to B$ induces a homomorphism

$$\overline{\varphi}\colon C^*(G, A, \alpha) \to C^*(G, B, \beta),$$

just by applying φ to the algebra elements. Thus, if G is discrete, the standard unitaries in $C^*(G, A, \alpha)$ are called u_g, and the standard unitaries in $C^*(G, B, \beta)$ are called v_g, then

$$\overline{\varphi}\left(\sum_{g\in G} c_g u_g\right) = \sum_{g\in G} \varphi(c_g) v_g.$$

Exercises: Assume that G is finite. Prove that $\overline{\varphi}$ is a *-homomorphism, that if φ is injective then so is $\overline{\varphi}$, and that if φ is surjective then so is $\overline{\varphi}$. (Warning: the surjectivity result is true for general G, but the injectivity result can fail if G is not amenable.)

Digression: Conjugacy

For $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$, we say that a homomorphism $\varphi\colon A \to B$ is *equivariant* if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

If φ is an isomorphism, we say it is a *conjugacy*. If there is such a map, the C* dynamical systems (G, A, α) and (G, B, β) are *conjugate*. This is the right version of isomorphism for C* dynamical systems.

Recall that equivariant homomorphisms induce homomorphisms of crossed products. It follows easily that if G is locally compact and φ is a conjugacy, then φ induces an isomorphism from $C^*(G, A, \alpha)$ to $C^*(G, B, \beta)$.

Recall from the discussion of product type actions on UHF algebras that we claimed that the actions of $\mathbb{Z}_2$ on $A = \bigotimes_{n=1}^{\infty} M_2$ generated by

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are "essentially the same". The correct statement is that these actions are conjugate. Exercise: prove this. Hint: Find a unitary $w \in M_2$ such that $w \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) w^* = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and take $\varphi = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(w)$.

Equivariant exact sequences

The homomorphism φ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

Recall that equivariant homomorphisms induce homomorphisms of crossed products.

Theorem

Let G be a locally compact group. Let $0 \to J \to A \to B \to 0$ be an exact sequence of C*-algebras with actions γ of G on J, α of G on A, and β of G on B, and equivariant maps. Then the sequence

$$0 \longrightarrow C^*(G, J, \gamma) \longrightarrow C^*(G, A, \alpha) \longrightarrow C^*(G, B, \beta) \longrightarrow 0$$

is exact.

When G is finite, the proof is easy:

$$0 \longrightarrow J[G] \longrightarrow A[G] \longrightarrow B[G] \longrightarrow 0$$

is clearly exact.

Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_n$ acts on the circle S^1 by rotation, with the standard generator acting by multiplication by $\omega = e^{2\pi i/n}$.

For any point $x \in S^1$, let

$$L_x = \{\omega^k x\colon k = 0, 1, \ldots, n-1\} \quad \text{and} \quad U_x = S^1 \setminus L_x.$$

Then L_x is equivariantly homeomorphic to $\mathbb{Z}_n$ with translation, and U_x is equivariantly homeomorphic to

$$\mathbb{Z}_n \times \left\{ e^{2\pi i t/n} x\colon 0 < t < 1 \right\} \cong \mathbb{Z}_n \times (0, 1).$$

The equivariant exact sequence

$$0 \longrightarrow C_0(U_x) \longrightarrow C(S^1) \longrightarrow C(L_x) \longrightarrow 0$$

gives the following exact sequence of crossed products:

$$0 \longrightarrow C_0((0,1),\, M_n) \longrightarrow C^*(\mathbb{Z}_n,\, C(S^1)) \longrightarrow M_n \longrightarrow 0.$$

With more work (details are in my crossed product notes), one can show that $C^*(\mathbb{Z}_n,\, C(S^1)) \cong C(S^1, M_n)$. The copy of S^1 on the right arises as the orbit space $S^1/\mathbb{Z}_n$.

Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, \, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle S^1 by rotation, we got

$$C^*(\mathbb{Z}_n, \, C(S^1)) \cong C(S^1/\mathbb{Z}_n, \, M_n) \cong C(S^1, M_n).$$

Recall the example from earlier: $\mathbb{Z}_2$ acts on S^n via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle S^1 by rotation, one might hope that $C^*(\mathbb{Z}_2, S^n)$ would be isomorphic to $C(S^n/\mathbb{Z}_2, \, M_2)$. This is almost right, but not quite. In fact, $C^*(\mathbb{Z}_2, S^n)$ turns out to be the section algebra of a bundle over $S^n/\mathbb{Z}_2$ with fiber M_2, and the bundle is locally trivial—but not trivial.

We still have the general principle: A closed orbit $Gx \cong G/H$ in X gives rise to a quotient in the crossed product isomorphic to $K(L^2(G/H)) \otimes C^*(H)$. What we have done illustrates this when G is finite (so that all orbits are closed) and H is either G or $\{1\}$.

Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_2$ acts on S^1 via the order two homeomorphism $\zeta \mapsto \overline{\zeta}$.

Set

$$L = \{-1, 1\} \subset S^1 \quad \text{and} \quad U = S^1 \setminus L.$$

Then the action on L is trivial, and U is equivariantly homeomorphic to

$$\mathbb{Z}_2 \times \{x \in U \colon \mathrm{Im}(x) > 0\} \cong \mathbb{Z}_2 \times (-1, 1).$$

The equivariant exact sequence

$$0 \longrightarrow C_0(U) \longrightarrow C(S^1) \longrightarrow C(L) \longrightarrow 0$$

gives the following exact sequence of crossed products:

$$0 \longrightarrow C_0((-1, 1), \, M_2) \longrightarrow C^*(\mathbb{Z}_2, \, C(S^1)) \longrightarrow C(L) \otimes C^*(\mathbb{Z}_2) \longrightarrow 0,$$

in which $C(L) \otimes C^*(\mathbb{Z}_2) \cong \mathbb{C}^4$. With more work (details are in my crossed product notes), one can show that $C^*(\mathbb{Z}_n, \, C(S^1))$ is isomorphic to

$$\{f \in C([-1, \, 1], \, M_2) \colon f(1) \text{ and } f(-1) \text{ are diagonal matrices}\}.$$

Crossed products by inner actions

Recall the inner action $\alpha_g = \mathrm{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from G to the unitary group of a C*-algebra A. The crossed product is the same as for the trivial action, in a canonical way.

Assume G is finite. Let $\iota \colon G \to \mathrm{Aut}(A)$ be the trivial action of G on A. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism φ sends $a \cdot u_g$ to $a z_g \cdot v_g$. This is clearly a linear bijection of the skew group rings.

We check the most important part of showing that φ is an algebra homomorphism. Recall that $u_g b = \alpha_g(b) u_g$ (and $v_g b = \iota_g(b) v_g = b v_g$). So we need $\varphi(u_g)\varphi(b) = \varphi(u_g b)$. We have

$$\varphi(u_g b) = \varphi(\alpha_g(b) u_g) = \alpha_g(b) z_g v_g$$

and, using $z_g b = \alpha_g(b) z_g$,

$$\varphi(u_g)\varphi(b) = z_g v_g b = z_g b v_g = \alpha_g(b) z_g v_g.$$

Exercise: When G is finite, give a detailed proof that φ is an isomorphism. (This is written out in my crossed product notes.)

Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the 2^∞ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

Write it as $\alpha = \varinjlim_n \mathrm{Ad}(z_n)$ on $A = \varinjlim_n M_{2^n}$.

It is not hard to show that crossed products commute with direct limits. (Exercise: Prove this for finite groups.) Since $\mathrm{Ad}(z_n)$ is inner, we get

$$C^*(\mathbb{Z}_2, M_{2^n}, \mathrm{Ad}(z_n)) \cong C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong M_{2^n} \oplus M_{2^n}.$$

Now we have to use the explicit form of these isomorphisms to compute the maps in the direct system of crossed products, and then find the direct limit. In this particular case, the maps turn out to be unitarily equivalent to

$$(a, b) \mapsto \big(\mathrm{diag}(a, b), \; \mathrm{diag}(a, b)\big),$$

and a computation with Bratteli diagrams shows that the direct limit is again the 2^∞ UHF algebra. (For general product type actions, the direct limit will be more complicated, and usually not a UHF algebra.)

Crossed products by product type actions (continued)

Recall the action of $\mathbb{Z}_2$ on the 2^∞ UHF algebra generated by $\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Write it as $\alpha = \varinjlim_n \mathrm{Ad}(z_n)$ on $A = \varinjlim_n M_{2^n}$, with maps $\varphi_n \colon M_{2^n} \to M_{2^{n+1}}$.

Exercise: Find isomorphisms $\sigma_n \colon C^*(\mathbb{Z}_2, M_{2^n}, \mathrm{Ad}(z_n)) \to M_{2^n} \oplus M_{2^n}$ and homomorphisms $\psi_n \colon M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$ such that, with $\overline{\varphi}_n$ being the map induced by φ_n on the crossed products, the following diagram commutes for all n:

$$\begin{tikzcd} C^*\left(\mathbb{Z}_2,\, M_{2^n},\, \mathrm{Ad}(z_n)\right) \arrow[r, "\sigma_n"] \arrow[d, "\overline{\varphi}_n"'] & M_{2^n} \oplus M_{2^n} \arrow[d, "\psi_n"] \\ C^*\left(\mathbb{Z}_2,\, M_{2^{n+1}},\, \mathrm{Ad}(z_{n+1})\right) \arrow[r, "\sigma_{n+1}"] & M_{2^{n+1}} \oplus M_{2^{n+1}}. \end{tikzcd}$$

(You will need to use the explicit computation of the crossed product by an inner action and an explicit isomorphism $C^*(\mathbb{Z}_2) \to \mathbb{C} \oplus \mathbb{C}$.) Then prove that, using the maps ψ_n, one gets $\varinjlim_n (M_{2^n} \oplus M_{2^n}) \cong A$. (This part doesn't have anything to do with crossed products.) Conclude that $C^*(\mathbb{Z}_2, A, \alpha) \cong A$.

Motivation for the Rokhlin property

Recall that an action $(g, x) \mapsto gx$ of a group G on a set X is *free* if every $g \in G \setminus \{1\}$ acts on X with no fixed points. Equivalently, whenever $g \in G$ and $x \in X$ satisfy $gx = x$, then $g = 1$. (Examples: G acting on G by translation, $\mathbb{Z}_n$ acting on S^1 by rotation by $e^{2\pi i/n}$, and $\mathbb{Z}$ acting on S^1 by an irrational rotation.)

Let X be the Cantor set, let G be a finite group, and let G act freely on X.

Fix $x_0 \in X$. Then the points gx_0, for $g \in G$, are all distinct, so by continuity and total disconnectedness of the space, there is a compact open set $K \subset X$ such that $x_0 \in K$ and the sets gK, for $g \in G$, are all disjoint.

By repeating this process, one can find a compact open set $L \subset X$ such that the sets $L_g = gL$, for $g \in G$, are all disjoint, and such that their union is X.

Exercise: Carry out the details. (It isn't quite trivial.)

The Rokhlin property

Definition

Let A be a unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

For C*-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras (Ph.D. thesis of Vaughan Jones). The Rokhlin property for actions of $\mathbb{Z}$ goes back further.

The original use of the Rokhlin property was for understanding the structure of group actions. Application to the structure of crossed products is much more recent.

The Rokhlin property (continued)

The conditions in the definition of the Rokhlin prperty:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

The projections e_g are the analogs of the characteristic functions of the compact open sets gL from the Cantor set example.

Condition (1) is an approximate version of $gL_h = L_{gh}$. (Recall that $L_g = gL$.)

Condition (3) is the requirement that X be the disjoint union of the sets L_g.

Condition (2) is vacuous for a commutative C*-algebra. In the noncommutative case, one needs something more than (1) and (3). Without (2) the inner action $\alpha \colon \mathbb{Z}_2 \to \mathrm{Aut}(M_2)$ generated by $\mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ would have the Rokhlin property. We don't want this. For example, M_2 is simple but $C^*(\mathbb{Z}_2, M_2, \alpha)$ isn't. (There is more on outerness in Lecture 4.)

Examples

The conditions in the definition of the Rokhlin property:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Exercise: Let G be finite. Prove that the action of G on G by translation gives an action of G on $C(G)$ which has the Rokhlin property.

Exercise: Let G be finite. Let A be any unital C*-algebra. Prove that the action of G on $\bigoplus_{g \in G} A$ by translation of the summands has the Rokhlin property.

Exercise: Let G be finite, and let G act freely on the Cantor set X. Prove that the corresponding action of G on $C(X)$ has the Rokhlin property. (Use the earlier exercise on free actions on the Cantor set.)

In the exercises above, condition (2) is trivial. Can it be satisfied in a nontrivial way? In particular, are there any actions on simple C*-algebras with the Rokhlin property?

An example using a simple C*-algebra

The conditions in the definition of the Rokhlin property:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

We want an example in which A is simple. Thus, we won't be able to satisfy condition (2) by choosing e_g to be in the center of A.

From Lecture 1, recall the product type action of $\mathbb{Z}_2$ generated by

$$\beta = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

We will show that this action has the Rokhlin property.

In fact, we will use an action conjugate to this one: we will use $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in place of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Reasons for using $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ will appear in Lecture 4.

An example (continued)

We had

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The action α of $\mathbb{Z}_2$ is generated by

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}(w) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

Define projections $p_0, p_1 \in M_2$ by

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$w p_0 w^* = p_1, \quad w p_1 w^* = p_0, \quad \text{and} \quad p_0 + p_1 = 1.$$

The action $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $w p_0 w^* = p_1$, $w p_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_g such that:

1. $\|\beta_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense in A, we can assume $F \subset A_n$ for some n. (Exercise: Check this!)

For $g = 0, 1 \in \mathbb{Z}_2$, take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$

Clearly $e_0 + e_1 = 1$. Check that $\beta(e_0) = e_1$ and $\beta(e_1) = e_0$, and that e_0 and e_1 actually commute with everything in F. (Proofs: See the next slide.) This proves the Rokhlin property.

An example (continued)

The projections e_0 and e_1 actually commute with everything in F, essentially because the nontrival parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_{2^{n+1}}$, which we identify with $M_{2^n} \otimes M_2$. In this tensor factorization, elements of F have the form

$$a \otimes 1,$$

and

$$e_g = 1 \otimes p_g.$$

Clearly these commute.

For $\beta(e_0) = e_1$: we have $\beta|_{A_{n+1}} = \mathrm{Ad}\big(w^{\otimes n} \otimes w\big)$, so

$$\beta(e_0) = \big(w^{\otimes n} \otimes w\big)(1 \otimes p_0)\big(w^{\otimes n} \otimes w\big)^* = 1 \otimes w p_0 w^* = 1 \otimes p_1 = e_1.$$

The proof that $\beta(e_1) = e_0$ is the same.