Lecture 1: Actions of Finite Groups on C*-Algebras and Introduction to Crossed Products

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26 July 2014

2014 Summer School for Operator Algebras

East China Normal University, Shanghai

26–31 July 2014

- Lecture 1 (26 July 2014): Actions of Finite Groups on C*-Algebras and Introduction to Crossed Products.
- Lecture 2 (27 July 2014): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 3 (28 July 2014): Crossed Products by Actions with the Rokhlin Property.
- Lecture 4 (29 July 2014): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 5 (30 July 2014): Examples and Applications.

A rough outline of all five lectures

- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.

General motivation

The material to be described is part of the structure and classification theory for simple nuclear C*-algebras (the Elliott program). More specifically, it is about proving that C*-algebras which appear in other parts of the theory (in these lectures, certain kinds of crossed product C*-algebras) satisfy the hypotheses of known classification theorems.

To keep things from being too complicated, we will consider crossed products by actions of finite groups. Nevertheless, even in this case, one can see some of the techniques which are important in more general cases.

Crossed product C*-algebras have long been important in operator algebras, for reasons having nothing to do with the Elliott program. It has generally been difficult to prove that crossed products are classifiable, and there are really only three cases in which there is a somewhat satisfactory theory: actions of finite groups on simple C*-algebras, free minimal actions of groups which are not too complicated (not too far from $\mathbb{Z}^d$) on compact metric spaces, and "strongly outer" actions of such groups on simple C*-algebras.

Background

These lectures assume some familiarity with the basic theory of C*-algebras, as found, for example, in Murphy's book. K-theory will be occasionally used, but not in an essential way. A few other concepts will be important, such as tracial rank zero. They will be defined as needed, and some basic properties mentioned, usually without proof. Various side comments will assume more background, but these can be skipped.

Group actions on C*-algebras

Definition

Let G be a group and let A be a C*-algebra. An *action of G on A* is a homomorphism $g \mapsto \alpha_g$ from G to $\mathrm{Aut}(A)$.

That is, for each $g \in G$, we have an automorphism $\alpha_g \colon A \to A$, and $\alpha_1 = \mathrm{id}_A$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$.

In these lectures, almost all groups will be discrete (usually finite). If the group has a topology, one requires that the function $g \mapsto \alpha_g(a)$, from G to A, be continuous for all $a \in A$.

We give examples of actions of groups (mainly finite groups), considering first actions on commutative C*-algebras. These come from actions on locally compact spaces, as described next.

Group actions on spaces

Definition

Let G be a group and let X be a set. Then an *action of G on X* is a map $(g, x) \mapsto gx$ from $G \times X$ to X such that:

- $1 \cdot x = x$ for all $x \in X$.
- $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

If G and X have topologies, then $(g, x) \mapsto gx$ is required to be (jointly) continuous.

When G is discrete, continuity means that $x \mapsto gx$ is continuous for all $g \in G$. Since the action of g^{-1} is also continuous, this map is in fact a homeomorphism.

Group actions on commutative C*-algebras

An action of G on X is a continuous map $(g, x) \mapsto gx$ from $G \times X$ to X such that:

- $1 \cdot x = x$ for all $x \in X$.
- $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

Lemma

Let G be a topological group and let X be a locally compact Hausdorff space. Suppose G acts continuously on X. Then there is an action $\alpha \colon G \to \mathrm{Aut}(C_0(X))$ such that $\alpha_g(f)(x) = f(g^{-1}x)$ for $g \in G$, $f \in C_0(X)$, and $x \in X$.

Every action of G on $C_0(X)$ comes this way from an action of G on X.

If G is discrete, this is obvious from the correspondence between maps of locally compact spaces and homomorphisms of commutative C*-algebras. (In the general case, one needs to check that the two continuity conditions correspond properly.)

Examples of group actions on spaces

An action of G on X is a continuous map $(g, x) \mapsto gx$ from $G \times X$ to X such that:

- $1 \cdot x = x$ for all $x \in X$.
- $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

Every action on this list of a group G on a compact space X gives an action of G on $C(X)$.

- Any group G has a trivial action on any space X, given by $gx = x$ for all $g \in G$ and $x \in X$.
- Any group G acts on itself by (left) translation: gh is the usual product of g and h.
- The finite cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ acts on the circle S^1 by rotation: the standard generator acts as multiplication by $e^{2\pi i/n}$.
- $\mathbb{Z}_2$ acts on S^1 via the order two homeomorphism $\zeta \mapsto \overline{\zeta}$.
- $\mathbb{Z}_2$ acts on S^n via the order two homeomorphism $x \mapsto -x$.

More examples of group actions on spaces

- Fix $\theta \in \mathbb{R}$. Then there is an action of $\mathbb{Z}$ on S^1, given by $n \cdot \zeta = e^{2\pi i n\theta}\zeta$ for $n \in \mathbb{Z}$ and $\zeta \in S^1$. (This action is generated by the rotation homeomorphism $\zeta \mapsto e^{2\pi i\theta}\zeta$.)
- If G is a group and H is a (closed) subgroup (not necessarily normal), then G has a translation action on $X = G/H$, given by $g \cdot (kH) = (gk)H$ for $g, k \in G$.
- If G is a group and $\sigma\colon G \to H$ is a continuous homomorphism to another group H, then there is an action of G on $X = H$ given by $g \cdot h = \sigma(g)h$ for $g \in G$ and $h \in H$. For example, G might be a closed subgroup of H. (The action on the previous slide of $\mathbb{Z}_n$ on S^1 by rotation comes this way.) The previous example comes from the homomorphism $\mathbb{Z} \to S^1$ given by $n \mapsto e^{2\pi i\theta}$. If $\theta \notin \mathbb{Q}$, this homomorphism has dense range.
- Let Y be a compact space, and set $X = Y \times Y$. Then $G = \mathbb{Z}_2$ acts on X via the order two homeomorphism $(y_1, y_2) \mapsto (y_2, y_1)$. Similarly, the symmetric group S_n acts on Y^n.

Yet more examples of group actions on spaces

Every action on this list of a group G on a compact space X gives an action of G on $C(X)$.

- Let Z be a compact manifold, or a connected finite complex. (Much weaker conditions on Z suffice, but Z must be path connected.) Let $X = \widetilde{Z}$ be the universal cover of Z, and let $G = \pi_1(Z)$ be the fundamental group of Z. Then there is a standard action of G on X. Spaces with finite fundamental groups include real projective spaces (in which case this example was already on the first slide of examples) and lens spaces.
- The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ via the usual matrix multiplication. This action preserves $\mathbb{Z}^2$, and so is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$. $\mathrm{SL}_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by
$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$
Restriction gives actions of these on $S^1 \times S^1$.

Group actions on noncommutative C*-algebras

Some elementary examples:

- For every group G and every C*-algebra A, there is a trivial action $\iota\colon G \to \mathrm{Aut}(A)$, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in A$.
- Suppose $g \mapsto z_g$ is a (continuous) homomorphism from G to the unitary group $U(A)$ of a unital C*-algebra A. Then $\alpha_g(a) = z_g a z_g^*$ defines an action of G on A. (We write $\alpha_g = \mathrm{Ad}(z_g)$.) This is an *inner* action. (If A is not unital, use the multiplier algebra $M(A)$, and the strict topology on its unitary group.)
- As a special case, let G be a finite group, and let $g \mapsto z_g$ be a unitary representation of G on $\mathbb{C}^n$. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on M_n.

Pointwise inner does not imply inner

Let A be a unital C*-algebra. An automorphism $\varphi \in \mathrm{Aut}(A)$ is *inner* if there is a unitary $z \in A$ such that $\varphi = \mathrm{Ad}(z)$. Recall also that $\alpha \colon G \to \mathrm{Aut}(A)$ is *inner* if there is a homomorphism $g \mapsto z_g$ from G to $U(A)$ such that $\alpha_g = \mathrm{Ad}(z_g)$ for all $g \in G$.

Let $A = M_2$, let $G = (\mathbb{Z}_2)^2$ with generators g_1 and g_2, and set

$$\alpha_1 = \mathrm{id}_A, \qquad \alpha_{g_1} = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\alpha_{g_2} = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \alpha_{g_1 g_2} = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These define an action $\alpha \colon G \to \mathrm{Aut}(A)$ such that α_g is inner for all $g \in G$, but for which there is no homomorphism $g \mapsto z_g \in U(A)$ for which $\alpha_g = \mathrm{Ad}(z_g)$ for all $g \in G$. The point is that the implementing unitaries for α_{g_1} and α_{g_2} commute up to a scalar, but can't be appropriately modified to commute exactly. Exercise: Prove this.

Product type actions

We describe a particular "product type action". Let $A_n = (M_2)^{\otimes n}$, the tensor product of n copies of the algebra M_2 of 2×2 matrices. Thus $A_n \cong M_{2^n}$. Define

$$\varphi_n \colon A_n \to A_{n+1} = A_n \otimes M_2$$

by $\varphi_n(a) = a \otimes 1$. Let A be the (completed) direct limit $\varinjlim_n A_n$. (This is just the 2^∞ UHF algebra.) Define a unitary $v \in M_2$ by

$$v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define $z_n \in A_n$ by $z_n = v^{\otimes n}$. Define $\alpha_n \in \mathrm{Aut}(A_n)$ by $\alpha_n = \mathrm{Ad}(z_n)$. Then α_n is an inner automorphism of order 2. Using $z_{n+1} = z_n \otimes v$, one can easily check that $\varphi_n \circ \alpha_n = \alpha_{n+1} \circ \varphi_n$ for all n, and it follows that the α_n determine an order 2 automorphism α of A. Thus, we have an action of $\mathbb{Z}_2$ on A. This action is not inner, although it is "approximately inner".

General product type actions

We had $A = \varinjlim_n (M_2)^{\otimes n}$ with the action of $\mathbb{Z}_2$ generated by the direct limit automorphism

$$\varinjlim_n \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes n}$$

We write this automorphism as

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

In general, one can use an arbitrary group, one need not choose the same unitary representation in each tensor factor, and the tensor factors need not all be the same size.

More examples of product type actions

We will later use the following two additional examples:

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_3,$$

and

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}\big(\mathrm{diag}(-1, 1, 1, \ldots, 1)\big) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_{2^n+1}.$$

In the second one, there are supposed to be 2^n ones on the diagonal, giving a $(2^n + 1) \times (2^n + 1)$ matrix.

The tensor product of copies of conjugation by the regular representation

Let G be a finite group. Set $m = \text{card}(G)$. Let G act on the Hilbert space $l^2(G) \cong \mathbb{C}^m$ via the left regular representation. That is, if $g \in G$, then g acts on $l^2(G)$ by the unitary operator $(z_g \xi)(h) = \xi(g^{-1}h)$ for $\xi \in l^2(G)$ and $h \in G$. Now let G act on $M_m \cong L(l^2(G))$ by conjugation by the left regular representation: $g \mapsto \text{Ad}(z_g)$. Then take $A = \varinjlim_n (M_m)^{\otimes n}$ (which is the m^∞ UHF algebra), with the action of G given by

$$g \mapsto \bigotimes_{n=1}^{\infty} \text{Ad}(z_g).$$

The first example we gave of a product type action is the case $G = \mathbb{Z}_2$. The left regular representation of $\mathbb{Z}_2$ is generated by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{rather than} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

but these two matrices are conjugate. Using this, one can show (see below) that the two product type actions are "essentially the same".

The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra A_θ, the universal C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i\theta}uv$. If $\theta_1 - \theta_2 \in \mathbb{Z}$, then $A_{\theta_1} = A_{\theta_2}$. (The commutation relation is the same.) Some standard facts, presented without proof.

- If $\theta \notin \mathbb{Q}$, then A_θ is simple. In particular, *any* two unitaries u and v in any C*-algebra satisfying $vu = e^{2\pi i\theta}uv$ generate a copy of A_θ.
- If $\theta \in \mathbb{Q}$, then A_θ is Type I. In fact, if $\theta = \frac{m}{n}$ in lowest terms, with $n > 0$, then A_θ is isomorphic to the section algebra of a locally trivial continuous field over $S^1 \times S^1$ with fiber M_n.
- In particular, if $\theta = 0$, or if $\theta \in \mathbb{Z}$, then $A_\theta \cong C(S^1 \times S^1)$.
- A_θ is the crossed product of the action of $\mathbb{Z}$ on S^1 generated by rotation by $e^{2\pi i\theta}$.
- There is a "natural" continuous field over S^1 whose fiber over $e^{2\pi i\theta}$ is A_θ. (Obviously it isn't locally trivial.)

The algebra A_θ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ (more accurately, a noncommutative analog of $C(S^1 \times S^1)$).

The gauge action on the rotation algebra

Recall: A_θ is the universal C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i\theta}uv$.

There is a unique action $\gamma \colon S^1 \times S^1 \to Aut(A_\theta)$ such that

$$\gamma_{(\lambda,\zeta)}(u) = \lambda u \quad \text{and} \quad \gamma_{(\lambda,\zeta)}(v) = \zeta v$$

for $\lambda, \zeta \in S^1$. This essentially follows from the fact that λu and ζv satisfy the same commutation relation that u and v do. One must also check that $(\lambda, \zeta) \mapsto \gamma_{(\lambda,\zeta)}$ is a group homomorphism. (A bit of work is required to show that $(\lambda, \zeta) \mapsto \gamma_{(\lambda,\zeta)}(a)$ is continuous for all $a \in A_\theta$. Exercise: Do it. Hint: Show that it is true for a in the linear span of all $u^m v^n$, and then use an $\frac{\varepsilon}{3}$ argument.)

In particular, there are actions of $\mathbb{Z}_n$ on A_θ generated by the automorphism

$$u \mapsto e^{2\pi i/n}u \quad \text{and} \quad v \mapsto v$$

and by the automorphism

$$u \mapsto u \quad \text{and} \quad v \mapsto e^{2\pi i/n}v.$$

The action of $\mathrm{SL}_2(\mathbb{Z})$ on the rotation algebra

Recall: A_θ is the universal C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i\theta}uv$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}$$

and

$$\alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

Exercise: Check that α_n is an automorphism, and that $n \mapsto \alpha_n$ is a group homomorphism.

This action is the analog of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. It reduces to that action when $\theta = 0$.

The action of $\mathrm{SL}_2(\mathbb{Z})$ on the rotation algebra (continued)

Recall: A_θ is the universal C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

Recall that $\mathrm{SL}_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Restriction gives actions of these groups on the irrational rotation algebras.

In terms of generators of A_θ, and omitting the scalar factors (which are not necessary when one restricts to these subgroups), the action of $\mathbb{Z}_2$ is generated by

$$u \mapsto u^* \quad \text{and} \quad v \mapsto v^*,$$

and the action of $\mathbb{Z}_4$ is generated by

$$u \mapsto v \quad \text{and} \quad v \mapsto u^*.$$

Cuntz algebras

We will be more concerned with stably finite simple C*-algebras here, but the basic examples of purely infinite simple C*-algebras should at least be mentioned.

Let $d \in \{2, 3, \ldots\}$. Recall that the *Cuntz algebra* $\mathcal{O}_d$ is the universal C*-algebra on generators $s_1, s_2, \ldots, s_d$ satisfying the relations

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1.$$

Thus, $s_1, s_2, \ldots, s_d$ are isometries with orthogonal ranges which add up to 1. The Cuntz algebra $\mathcal{O}_\infty$ is the universal C*-algebra generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges. Thus, $s_1^* s_1 = s_2^* s_2 = \cdots = 1$ and $s_j^* s_k = 0$ for $j \neq k$.

These algebras are purely infinite, simple, and nuclear. Details and other properties are on the next slide.

Cuntz algebras (continued)

Some standard facts, presented without proof.

- $\mathcal{O}_d$ is simple for $d \in \{2, 3, \ldots, \infty\}$. For $d \in \{2, 3, \ldots\}$, for example, this means that whenever elements $s_1, s_2, \ldots, s_d$ in any unital C*-algebra satisfy

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1,$$

  then they generate a copy of $\mathcal{O}_d$.
- $\mathcal{O}_d$ is purely infinite and nuclear.
- $K_1(\mathcal{O}_d) = 0$, $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, generated by $[1]$, and $K_0(\mathcal{O}_d) \cong \mathbb{Z}_{d-1}$, generated by $[1]$, for $d \in \{2, 3, \ldots\}$.
- If A is any simple separable unital nuclear C*-algebra, then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.
- If A is any simple separable purely infinite nuclear C*-algebra, then $\mathcal{O}_\infty \otimes A \cong A$.

The last two facts are Kirchberg's absorption theorems. They are much harder.

Actions on Cuntz algebras

For d finite, $\mathcal{O}_d$ is generated by isometries $s_1, s_2, \ldots, s_d$ with orthogonal ranges which add up to 1, and $\mathcal{O}_\infty$ is generated by isometries $s_1, s_2, \ldots$ with orthogonal ranges.

We give the general quasifree action here. Two special cases on the next slide have much simpler formulas.

Let $\rho \colon G \to L(\mathbb{C}^d)$ be a unitary representation of G. Write

$$\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix}$$

for $g \in G$. Then there exists a unique action $\beta^\rho \colon G \to \mathrm{Aut}(\mathcal{O}_d)$ such that

$$\beta^\rho_g(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g) s_j$$

for $j = 1, 2, \ldots, d$. (This can be checked by a computation.) For $d = \infty$, a similar formula works for any unitary representation of G on $l^2(\mathbb{N})$.

Actions on Cuntz algebras (continued)

The Cuntz relations: $s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1$ and $s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1$. (For $d = \infty$, $s_1, s_2, \ldots$ are isometries with orthogonal ranges.)

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For $G = \mathbb{Z}_n$, choose n-th roots of unity $\zeta_1, \zeta_2, \ldots, \zeta_d$ and let a generator of the group multiply s_j by ζ_j.
- Let G be a finite group. Take $d = \mathrm{card}(G)$, and label the generators s_g for $g \in G$. Then define $\beta^G \colon G \to \mathrm{Aut}(\mathcal{O}_d)$ by $\beta_g^G(s_h) = s_{gh}$ for $g, h \in G$. (This is the quasifree action coming from regular representation of G.)
- Label the generators of $\mathcal{O}_\infty$ as $s_{g,j}$ for $g \in G$ and $j \in \mathbb{N}$. Define $\iota \colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ by $\iota_g(s_{h,j}) = s_{gh,j}$ for $g \in G$ and $j \in \mathbb{N}$. (This is the quasifree action coming from the direct sum of infinitely many copies of the regular representation of G.)

The tensor flip

Assume (for convenience) that A is nuclear and unital. Then there is an action of $\mathbb{Z}_2$ on $A \otimes A$ generated by the "tensor flip" $a \otimes b \mapsto b \otimes a$.

Similarly, the symmetric group S_n acts on $A^{\otimes n}$.

The tensor flip on the 2^∞ UHF algebra $A = \bigotimes_{n=1}^\infty M_2$ turns out to be essentially the product type action generated by

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{on} \quad \bigotimes_{n=1}^{\infty} M_4.$$

Another interesting example is gotten by taking A to be the Jiang-Su algebra Z. It is simple, separable, unital, and nuclear. It has no nontrivial projections, its Elliott invariant is the same as for $\mathbb{C}$, and $Z \otimes Z \cong Z$.

Since $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ and $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \cong \mathcal{O}_\infty$, one gets actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$ and $\mathcal{O}_\infty$ the same way.

The free flip

Let A be a C*-algebra, and let $A \star A$ be the free product of two copies of A. Then there is an automorphism $\alpha \in \mathrm{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of a in the first free factor to the copy of the same element in the second free factor, and similarly the copy of a in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the "free flip". It generates an action of $\mathbb{Z}_2$ on $A \star A$.

There are many generalizations. One can take the amalgamated free product $A \star_B A$ over a subalgebra $B \subset A$ (using the same inclusion in both copies of A), or the reduced free product $A \star_{\mathrm{r}} A$ (using the same state on both copies of A). There is a permutation action of S_n on the free product of n copies of A. And one can make any combination of these generalizations.

Crossed products

Let G be a locally compact group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on a C*-algebra A. There is a *crossed product C*-algebra* $C^*(G, A, \alpha)$, which is a kind of generalization of the group C*-algebra $C^*(G)$. Crossed products are quite important in the theory of C*-algebras.

One motivation: Suppose G is a semidirect product $N \rtimes H$. The action of H on N gives an action $\alpha \colon H \to \mathrm{Aut}(C^*(N))$, and one has $C^*(G) \cong C^*(H, C^*(N), \alpha)$. Thus, crossed products appear even if one is only interested in group C*-algebras and unitary representations of groups.

Another motivation (not applicable to finite groups acting on spaces): The noncommutative version of X/G is the fixed point algebra A^G. In particular, for compact G, one can check that $C(X/G) \cong C(X)^G$. For noncompact groups, often X/G is very far from Hausdorff and A^G is far too small. The crossed product provides a much more generally useful algebra, which is the "right" substitute for the fixed point algebra when the action is free.

Reminder: The group C*-algebra

Let G be a locally compact group. We recall that nondegenerate representations of the group C*-algebra $C^*(G)$ on a Hilbert space H are in one to one correspondence with the unitary representations of G on H.

To construct $C^*(G)$, one starts with $L^1(G)$ (using left Haar measure μ) with convolution multiplication:

$$(a * b)(g) = \int_G a(h)b(h^{-1}g)\, d\mu(h).$$

(We omit the formula for the adjoint.) If G is discrete and $\delta_g \in l^1(G)$ is the standard basis vector corresponding to $g \in G$, this amounts to declaring that $\delta_g * \delta_h = \delta_{gh}$ and $\delta_g^* = \delta_{g^{-1}}$. A unitary representation $g \mapsto v_g$ of G on a Hilbert space H gives a nondegenerate *-representation σ of $L^1(G)$ on H via the formula

$$\sigma(a)\xi = \int_G a(g)v_g\xi\, d\mu(g).$$

(One must check many things about this formula.) If G is discrete, this is just $\sigma(a) = \sum_{g\in G} a(g)v_g$, and in particular $\sigma(\delta_g) = v_g$.

The group C*-algebra (continued)

For a locally compact group G and a unitary representation v of G on H, we set

$$\sigma(a)\xi = \int_G a(g)v_g\xi\, d\mu(g)$$

for $a \in L^1(G)$ and $\xi \in H$. If G is discrete, this is just $\sigma(a) = \sum_{g\in G} a(g)v_g$, and in particular $\sigma(\delta_g) = v_g$.

Getting v from σ is easy if G is discrete, since $v_g = \sigma(\delta_g)$. In general, one must do some work with multiplier algebras; we omit the details.

We must still get a C*-algebra. To do this, define a C* norm on $L^1(G)$ by taking $\|a\|$ to be the supremum of $\|\sigma(a)\|$ over all nondegeratate *-representations σ of $L^1(G)$ on Hilbert spaces. Then complete in this norm.

If G is finite, this simplifies greatly. The sums $\sigma(a) = \sum_{g\in G} a(g)v_g$ are finite sums and no completion is necessary (because $L^1(G)$ is finite dimensional). One only needs to find the C* norm. (It is equivalent to the L^1 norm, but not equal to it.)

The universal property of the crossed product

The crossed product $C^*(G, A, \alpha)$ (for G locally compact) is defined in such a way as to have a universal property which generalizes the universal property of the group C*-algebra $C^*(G)$. We give the statements for the general case, and (below) go through some details in the much easier case that G is finite.

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, π) consisting of a unitary representation $v\colon G \to U(H)$ (the unitary group of H) and a representation $\pi\colon A \to L(H)$ (the algebra of all bounded operators on H), satisfying the *covariance condition*

$$v_g\pi(a)v_g^* = \pi(\alpha_g(a))$$

for all $g \in G$ and $a \in A$. It is called *nondegenerate* if π is nondegenerate.

The universal property of the crossed product (continued)

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let (v, π) be a covariant representation of (G, A, α) on a Hilbert space H. Then the *integrated form* of (v, π) is the representation $\sigma\colon C_c(G, A, \alpha) \to L(H)$ given by

$$\sigma(a)\xi = \int_G \pi(a(g))v_g\xi\, d\mu(g).$$

$C^*(G, A, \alpha)$ is then a completion of $C_c(G, A, \alpha)$, chosen to give:

Theorem

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then the integrated form construction defines a bijection from the set of nondegenerate covariant representations of (G, A, α) on a Hilbert space H to the set of nondegenerate representations of $C^*(G, A, \alpha)$ on the same Hilbert space.

Crossed products by finite groups

As for group C*-algebras, crossed products simplify greatly if the group is finite. We give details. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all finite formal linear combinations of elements in G with coefficients in A. The multiplication and adjoint are given by:

$$(a \cdot g)(b \cdot h) = (a[gbg^{-1}]) \cdot (gh) = (a\alpha_g(b)) \cdot (gh)$$
$$(a \cdot g)^* = \alpha_g^{-1}(a^*) \cdot g^{-1}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. There is a unique norm which makes this a C*-algebra. (See below.)

If A is unital, the group elements $g = 1 \cdot g$ are in $A[G]$, and are unitary. We conventionally write u_g instead of g for the element of $A[G]$. Thus, a general element of $A[G]$ has the form $c = \sum_{g \in G} c_g u_g$ with $c_g \in A$ for $g \in G$. (This actually works even if A is not unital.)

If G is discrete but not finite, $C^*(G, A, \alpha)$ is the completion of $A[G]$ in a suitable norm. (Before completion, we have the *skew group ring*.)

Crossed products by finite groups (continued)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. We construct a C* norm on the skew group ring $A[G]$.
Recall:

$$(a \cdot g)(b \cdot h) = (a\alpha_g(b)) \cdot (gh) \quad \text{and} \quad (a \cdot g)^* = \alpha_g^{-1}(a^*) \cdot g^{-1},$$

that is,

$$(au_g)(bu_h) = a\alpha_g(b)u_{gh} \quad \text{and} \quad (au_g)^* = \alpha_g^{-1}(a^*)u_{g^{-1}}.$$

Fix a faithful representation $\pi \colon A \to L(H_0)$ of A on a Hilbert space H_0. Set $H = l^2(G, H_0)$, the set of all $\xi = (\xi_g)_{g \in G}$ in $\bigoplus_{g \in G} H_0$, with the scalar product

$$\big\langle (\xi_g)_{g \in G}, \, (\eta_g)_{g \in G} \big\rangle = \sum_{g \in G} \langle \xi_g, \eta_g \rangle.$$

Then define $\sigma \colon A[G] \to L(H)$ as follows. For $c = \sum_{g \in G} c_g u_g$,

$$(\sigma(c)\xi)_h = \sum_{g \in G} \pi(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h})$$

for all $h \in G$.

Crossed products by finite groups (continued)

Recall:

$$(au_g)(bu_h) = a\alpha_g(b)u_{gh} \quad \text{and} \quad (au_g)^* = \alpha_g^{-1}(a^*)u_{g^{-1}}.$$

Also, for $c = \sum_{g \in G} c_g u_g$,

$$(\sigma(c)\xi)_h = \sum_{g \in G} \pi(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h}).$$

If A is unital, then for $a \in A$ and $g \in G$, identify a with au_1 and get

$$(\sigma(a)\xi)_h = \pi(\alpha_{h^{-1}}(a))(\xi_h) \quad \text{and} \quad (\sigma(u_g)\xi)_h = \xi_{g^{-1}h}.$$

One can check that σ is a *-homomorphism. We will just check the most important part, which is that $\sigma(u_g)\sigma(b) = \sigma(\alpha_g(b))\sigma(u_g)$. We have

$$\big[\sigma(\alpha_g(b))\sigma(u_g)\xi\big]_h = \pi\big(\alpha_{h^{-1}}(\alpha_g(b))\big)(\sigma(u_g)\xi)_h = \pi(\alpha_{h^{-1}g}(b))(\xi_{g^{-1}h})$$

and

$$\big(\sigma(u_g)\sigma(b)\xi\big)_h = \big(\sigma(b)\xi\big)_{g^{-1}h} = \pi\big(\alpha_{h^{-1}g}(b)\big)\xi\big)_{g^{-1}h}.$$

Crossed products by finite groups (continued)

Recall: for $c = \sum_{g \in G} c_g u_g$,

$$(\sigma(c)\xi)_h = \sum_{g \in G} \pi(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h}).$$

If A is unital, then for $a \in A$ and $g \in G$,

$$(\sigma(a)\xi)_h = \pi(\alpha_{h^{-1}}(a))(\xi_h) \quad \text{and} \quad (\sigma(u_g)\xi)_h = \xi_{g^{-1}h}.$$

For $c = \sum_{g \in G} c_g u_g$, it is easy to check that

$$\|\sigma(c)\| \leq \sum_{g \in G} \|c_g\|$$

and not much harder to check that

$$\|\sigma(c)\| \geq \max_{g \in G} \|c_g\|.$$

The norms on the right hand sides are equivalent, so $A[G]$ is complete in the norm $\|c\| = \|\sigma(c)\|$.