Topics in the Theory of Algebraic Groups

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1 Introduction

This article is a collection of notes from a series of talks given at the Bernoulli center. The attendees ranged from people who have never studied algebraic groups to experts. Consequently the series began with two introductory talks on the structure of algebraic groups, supplemented by two lectures of Steve Donkin on representation theory. The notes from his lectures appear in this volume and we encourage readers new to the theory of algebraic groups to consult his notes as this material will be used throughout this article.

The first section of this article contains a brief introduction to the theory of algebraic groups starting from the classification of simple Lie algebras over $\mathbb{C}$. The next section covers morphisms of simple algebraic groups and this leads to a discussion of the finite groups of Lie type and Lang's theorem. Sections three and four concern maximal subgroups of algebraic groups. Section 5 concerns a problem on the finiteness of double coset spaces. The last two sections cover some current work on unipotent classes in simple algebraic groups. None of the sections represents anything near a complete account of the subject at hand so the interested reader should consult the references for further information.

2 Algebraic groups: Introduction

In this section and the next one we give an informal introduction to the theory of simple algebraic groups and their finite analogs, the finite groups of Lie type. These sections, together with the material on representation theory presented by Steve Donkin, will form the foundation for later sections on various topics in the theory. We assume prior knowledge of the theory of complex Lie algebras in what follows, but familiar examples will be present at all times. The articles [39] and [22] from the 1994 Istanbul NATO conference provide an expanded account of the topics of these lectures and include references for the main results. A full development of the basic theory can be found in books such as Borel [5] and Humphreys [17].

Let K be an algebraically closed field and consider the algebra $M_n(K)$, of all $n \times n$ matrices over K. Let $sl_n(K)$ denote the subspace of trace 0 matrices. This subspace is not closed under matrix product, but it does have additional structure, a bracket operation $[x, y] = xy - yx$, which turns $sl_n(K)$ into a Lie algebra. Another important object for us is $G = SL_n(K)$, the group of matrices of determinant 1. There is extra structure here as well which will be discussed below. When working in $sl_n(K)$, the notion of vector space dimension is an important tool that is used repeatedly. It allows us to talk about large or small subalgebras. We want a similar notion for $SL_n(K)$, but this requires some background material.

2.1 Affine algebraic varieties and algebraic groups

We illustrate the ideas in $SL_n(K)$ and then generalize the theory. A vector space basis of $M_n(K)$ is given by the elementary matrix units x_{ij}. We can regard these basis elements as functions on matrices. The functions generate a polynomial ring $K[x_{ij}]$, an algebra of functions on $M_n(K)$. Then $SL_n(K)$ is the set of matrices annihiliated by the principal ideal $I = (det - 1)$ and $K[x_{ij}]/I$ is an algebra of functions on $SL_n(K)$.

More generally, consider K^m and the polynomial ring $K[x_1, \dots, x_m]$ viewed as a ring of functions. There is a map $I \to V(I)$ sending an ideal to the subset of K^m annihilated by I and another map sending a subset $S \subset K^m$ to the ideal $I(S)$ of all polynomials annihilating the subset. Note that the first map is not 1-1 since a polynomial and its powers have the same zero set, Hence, $V(I) = V(\sqrt{I})$. The Hilbert Nullstellensatz implies that $I(V(I)) = \sqrt{I}$. Consequently, we focus on radical ideals so that the correspondence between radical ideals and zero sets of radical ideals is bijective.

An *affine algebraic variety* is defined to be a pair $(V, K[V])$, where for some m, $V \subset K^m$ is the zero set of a radical ideal $I \leq K[x_1, \dots, x_m]$ and $K[V] = K[x_1, \dots, x_m]/I$.

There is a notion of morphism from one algebraic variety $(V, K[V])$ to another $(W, K[W])$. Let $\phi : V \to W$ be a function. For $f \in K[W], \phi^*(f) = f \circ \phi$ is a K-valued function on V. We say that ϕ is a *morphism* provided $\phi^*(f) \in K[V]$ for each f. In this case $\phi^* : K[W] \to K[V]$ is an algebra homomorphism, called the *comorphism of* ϕ. Given algebraic varieties $V \subset K^n$ and $W \subset K^m$, the cartesian product $V \times W \subset K^{n+m}$ is also an algebraic variety, where $K[V \times W] = K[V] \otimes K[W]$, subject to suitable identifications.

An *algebraic group*, $(G, K[G])$, is an affine algebraic variety which also has a group structure such that multiplication, $m : G \times G \to G$, and inverse, $i : G \to G$, are morphisms. For example, one checks that for $G = SL_n(K), m^*(x_{ij}) = \sum_k x_{ik}x_{kj}$, where we are identifying x_{ij} with its restriction to $SL_n(K)$. There is a considerable amount of algebraic geometry lurking in the background, which we mostly ignore in this brief survey. However, the proofs of essentially all the major results are heavily dependent on the extra structure.

The *Zariski topology* on K^m is the topology where the closed sets have the form $V(I)$ with I an ideal of $K[x_1, \dots, x_m]$. An algebraic variety V is said to be *irreducible* if it cannot be written as the union of two proper closed subsets.. If V is reducible and $V = V_1 \cup V_2$ is a proper decomposition with I_1, I_2 the radical ideals annihilating V_1, V_2, respectively. Then $I_1 I_2 \subset I_1 \cap I_2$ is an ideal annihilating all of V, hence must be the 0 ideal. Here is a key result.

Proposition 2.1 *An algebraic variety V is irreducible if and only if $K[V]$ is an integral domain.*

We can now introduce the notion of dimension for irreducible algebraic varieties. Namely, if V is an irreducible variety we define $\dim(V)$ to be the transcendence degree over K of the quotient field of $K[V]$. The following result shows that dimension works nicely with respect to containments.

Proposition 2.2 *Let V_1, V_2 be irreducible varieties such that $V_1 \subseteq V_2$. Then $\dim(V_1) \leq \dim(V_2)$. Moreover, the dimensions coincide if and only if the varieties are equal.*

An arbitrary algebraic variety can be uniquely expressed as the union of finitely many irreducible varieties, called the *irreducible components* of the variety. One defines the dimension as the largest of the dimensions of the irreducible components.

Let G be an algebraic group. Let G^0 denote the irreducible component containing the identity. Then G^0 is a normal subgroup of G and the other irreducible components are the (finitely many) cosets of G^0. In this case the irreducible components are both open and closed, so G is irreducible if and only if it is connected.

The coordinate ring provides a link between the theory of algebraic groups and the theory of Lie algebras. If G is an algebraic group define $L(G) = T(G)_1$, the *tangent space at the identity*, to be the space of functions $\gamma : K(G) \to K$ satisfying the property $\gamma(fg) = \gamma(f)g(1) + f(1)\gamma(g)$. This space is closely connected to a certain derivation algebra of $K[G]$ and so inherits a Lie algebra structure. A homomorphism $\phi : G \to H$ between algebraic groups gives rise to a homomorphism $\partial\phi : L(G) \to L(H)$ via $\partial\phi(\gamma) = \gamma \circ \phi^*$. Taking $\phi = inn_g$ for $g \in G$ we obtain an isomorphism of $L(G)$ which is called $\mathrm{Ad}(g)$. Then $\mathrm{Ad} : G \to GL(L(G))$ and this is called the *adjoint representation* of G.

2.2 Lie algebras to algebraic groups

Here we discuss a method for passing from Lie algebras to algebraic groups. If e is a nilpotent element of $sl_n(K)$ (e.g. one of the e_{ij}), then $exp(e) = 1 + e + e^2/2 + \cdots$, a finite sum, is an element of $SL_n(K)$. For the time being assume that K has characteristic 0 so that there are no problems with denominators in this expression. It can be shown that SL_n is generated by such unipotent elements. Indeed, $SL_n(K)$ is generated by transvections and each transvection has the above form. This depends on the particular presentation of $sl_n(K)$ as acting on a vector space, but something similar holds more generally. Let L be a simple Lie algebra and suppose $e \in L$ has the property that the map $ad(e) : l \to [e, l]$ is a nilpotent endomorphism of L. Then $u = exp(ad(e)) = 1 + ad(e) + ad(e)^2/2 + \cdots$ is well-defined and it turns out to be an automorphism of L. The elements $u(c) = exp(ad(ce))$ for $c \in K$ is an abelian subgroup of $SL(L)$ with entries being polynomials in c. It is an algebraic group isomorphic to K^+. There is a result showing that the subgroup of $Aut(L)$

generated by these irreducible subgroups is itself a connected algebraic group. So in this way we can pass from L to an irreducible algebraic group $G(L)$.

If we take $L = sl_n(K)$ in the above situation, then the corresponding unipotent elements generate $G(L) = PSL_n(K) = SL_n(K)/Z(SL_n(K))$. We get $PSL_n(K)$ rather than $SL_n(K)$ since we are looking at the action on L, by conjugation, rather than the action on the original vector space. This shows that exponentiation gives slightly different groups, depending on the representation.

The exponentiation process described above is more complicated when the field involved has positive characteristic, since the denominators cause problems. However, one can proceed by choosing a basis defined over $\mathbb{Z}$ and then carefully passing to arbitrary fields. A very good reference for this can be found in the Steinberg Yale notes [46].

Let L be a simple, finite dimensional Lie algebra over $\mathbb{C}$, the complex field. The classification theorem shows that there exists an indecomposable root system Σ and base Π of this root system such that L has vector space basis consisting of elements $\{h_{\alpha_i} : \alpha_i \in \Pi\}$ and root vectors $\{e_\alpha : \alpha \in \Sigma\}$ subject to the following relations:

(i) If $\alpha, \beta, \alpha + \beta \in \Sigma$, then $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$, where $N_{\alpha,\beta}$ is an integer.
(ii) If $\alpha \in \Sigma^+$, then $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $\langle e_\alpha, h_\alpha, e_{-\alpha} \rangle \cong sl_2$.
(iii) $[h_{\alpha_i}, e_\beta] =< \beta, \alpha_i > e_\beta$ and $[h_{\alpha_i}, h_{\alpha_j}] = 0$ for all i, j.

Also recall that the simple roots give rise to a Dynkin diagram and the diagram determines the isomorphism type of L. The possible types are $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$. The infinite familes are called the *classical types* and the other are said to be of *exceptional type*.

We next discuss analogs of the above relations which hold within the group $G(L)$. We work with an algebraically closed field of arbitrary characteristic, ignoring the complications that occur in finite characteristic.

For each root α and $c \in K$, set $U_\alpha(c) = exp(ad(ce_\alpha)$. Then the root space $\langle e_\alpha \rangle$ exponentiates to yield a unipotent group $U_\alpha = \{U_\alpha(c) : c \in K\} \cong K^+$ (isomorphism of algebraic groups). These groups generate G. The group analog of (i) is as follows.

(a) If $\alpha, \beta, \alpha + \beta \in \Sigma$, then $[U_\alpha(x), U_\beta(y)] = \prod_{i,j>0} U_{i\alpha+j\beta}(c_{ij} x^i y^j)$. Also , $c_{11} = N_{\alpha,\beta}$.

If the Dynkin diagram is simply laced (no multiple bonds), then there is just one term in the product (i = j = 1), so this closely resembles (i). The analog of (ii) is

(b) If $\alpha \in \Sigma^+, \langle U_\alpha, U_{-\alpha} \rangle \cong SL_2(K)$. or $PSL_2(K)$.

Under this isomorphism we regard U_α (resp. $U_{-\alpha}$) as the group of lower (resp. upper) triangular matrices. Then for $c \in K$, we let $T_\alpha(c)$ denote the image of the diagonal matrix having entries c, c^{-1} and set $T_\alpha = \{T_\alpha(c) : c \in K^*\}$. Then $T_\alpha \cong K^*$ is a commutative algebraic group. We obtain the group analog of the maximal toral subalgebra $H = \langle h_{\alpha_1}, \ldots, h_{\alpha_n} \rangle$, by setting $T = \langle T_{\alpha_1}, \ldots, T_{\alpha_n} \rangle$. Then T is a connected abelian algebraic group isomorphic to $(K^*)^n$ and it is called a *maximal torus*. T has the property (analgous to maximal toral subalgebras) that it can be diagonalized in any linear representation of G. In the case $G = SL_n(K)$, the maximal tori are just the full diagonal groups with respect to a given basis. We have the relation

(c) $T_{\alpha_i}(c)^{-1}U_\beta(d)T_{\alpha_i}(c) = U_\beta(c^{<\beta,\alpha_i>}d)$.

2.3 (BN)-pair and Classification

To determine the isomorphism type of a Lie algebra one finds a convenient basis of the space and determine commutators among basis elements. It is not so clear how this carries over to the group setting. Let's continue to work with $G = G(L)$.

Let Σ^+ denote the set of positive roots. It follows from (a) above that $U = \prod_{\alpha\in\Sigma^+} U_\alpha$ is a subgroup of G. Indeed, if we order the roots in Σ^+ according to descending height, say $U_{\gamma_1}, \ldots, U_{\gamma_k}$, then the commutator relations in (a) imply that each of the products $U_{\gamma_1} \cdots U_{\gamma_i}$, for $i \leq k$, is a normal subgroup of U and hence U is a nilpotent group. Also (b) shows that T normalizes each U_α and so U is normalized by T. The group $B = UT$ is a solvable group which can be shown to be maximal among connected solvable subgroups of G. Such subgroups of algebraic groups are called *Borel subgroups* and they play a fundamental role in the theory.

A key result shows that if a connected solvable group acts on a special type of variety (a complete variety), then it fixes a point. We will not go into this here, but one important consequence is the fact that all Borel subgroups of G are conjugate. An easier, but still important, result shows that connected solvable groups leave invariant a 1-space in any linear representation of G. An induction shows that they fix a maximal flag. For $G = SL_n(K)$, this implies that the Borel subgroups are precisely the lower triangular groups with respect to a suitable basis.

At certain stages in the classification of simple Lie algebras one goes outside the Lie algebra. In particular, the Weyl group is introduced. It should be no surprise to see it appear here as well, although here it appears in a more intrinsic manner.

The simplest case is $G = SL_2(K)$. We describe matrices as follows.

$$U(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, V(c) = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A matrix calculation shows that $s = U(-1)V(1)U(-1)$. Also, notice that s interchanges the 1-spaces spanned by the basis vectors and normalizes the diagonal group.

Now consider $G = SL_n(K)$. For each consecutive pair of basis vectors, G contains a subgroup $SL_2(K)$ acting on the corresponding 2-space and fixing the remaining basis vectors. It is then easy to see that each of the matrices corresponding to s above normalizes T, the full diagonal group, and acts as a 2-cycle on the set of 1-spaces spanned by the given basis. Hence $N_G(T)$ induces the full symmetric group Sym_n on the set of 1-spaces spanned by basis elements.

We wish to record one more piece of information about $SL_n(K)$. For a matrix $g \in SL_n(K)$, consider what can be achieved by applying elementary row operations. Such an operation is just multiplication by a matrix of the form $U_{ij}(c)$. We perform row reduction by such matrices with $i > j$. That is we multiply row j by c and add it to row i, a lower row. Since g is invertible we can find a matrix u, a product of elementary matrices, so that ug has the property that for each i there is a row with first nonzero entry in column i and all entries lower in this column are 0. Now we can apply a suitable permutation matrix, to get a matrix in upper triangular form. That is $nug = b^-$. So $g \in BNB^-$. Now, $B^- = B^w$ for some permutation w. So in fact, we have proved $G = BNBw$, and multiplying by the inverse of w we have $G = BNB$.

By a *simple algebraic group* we mean a nonabelian irreducible algebraic group G having no closed normal subgroups of positive dimension. This definition allows for finite normal subgroups. However, G has no closed normal subgroups of finite index, so it must act trivially by conjugation on any finite normal subgroup. So G may have a finite center, but one shows that $G/Z(G)$ is abstractly simple. So our definition allows for $SL_n(K)$ as a simple algebraic group.

The theory of simple algebraic groups leads to many of the above concepts. In particular, if G is a simple algebraic group, then one shows that all Borel subgroups are conjugate and that given a maximal torus $T < G$, the group $N_G(T)/T \cong W$, the Weyl group, is a group generated by reflections. W is associated with a root system Σ and to each root $\alpha \in \Sigma$ there is a corresponding root subgroup $U_\alpha < G$. The reflections act as usual: $s_\alpha(\beta) = \beta - <\beta, \alpha> \alpha$. Moreover, T normalizes each root group, so elements of the Weyl group permute the root subgroups just as they do the corresponding roots. One shows that $W = \langle s_1, \ldots, s_n \rangle$, where the s_i are the fundamental reflections. The length of a Weyl group element is the minimal length of the element as a product of fundamental reflections. The following result is a generalization of the above example and shows how the fundamental reflections arise.

Proposition 2.3 *If $N = N_G(T)$, then $N/T \cong W$. Moreover, the isormorphism can be chosen such that for each root α a representative of s_α can be taken to be $n_\alpha = U_\alpha(-1)U_{-\alpha}(1)U_\alpha(-1)$*

So for each root α we can think of s_α as the coset $n_\alpha T$. More generally, for each $w \in W$, we can choose an element $n_w \in N$ and regard identify w with the coset $n_w T$.

At the level of Lie algebras we can express elements uniquely as a linear combination of basis elements. Things are more complicated at the level of algebraic groups, but we next describe a variation of this idea. We require one additional piece of notation. For $w \in W$ let U_w^- denote the product of all root groups U_β such that β is a positive root and $w(\beta)$ is a negative root. Then U_w^- is a subgroup and we have the following fundamental result

Theorem 2.4 *Let G be a simple algebraic group. Then*

i) $G = BNB = \bigcup_{w\in W} Bn_wB$.
ii) $G = \bigcup_{w\in W} UTn_wU_w^-$ *with uniqueness of expression of elements.*

The above theorem says two important things. (i) shows that there are only finitely many double cosets of B in G or, equivalently, B has only finitely many orbits in the action on G/B. As advertised, (ii) provides a unique expression for each element of G. In the context of Lie algebras, the unique expression of elements also tells us how to add elements. Of course group multiplication is more complicated than addition. So how do we multiply group elements? Well, from (i) this comes down to knowing something about multiplying elements from double cosets of B. The following result provides an indication of how this occurs.

To ease notation write BwB instead of Bn_wB. This is an abuse of notation, but since B contains T the expression is independent of the choice of coset representative.

Proposition 2.5 *If s_i is a fundamental reflection and $w \in W$, then $(Bs_iB)(BwB) \subset BwB \cup Bs_iwB$. Moreover, if $l(s_iw) > l(w)$, then $(Bs_iB)(BwB) = Bs_iwB$.*

This might appear confusing, but in principle one can use these results to work out the multiplication table of G. Indeed, the commutator relations tell us how to find the structure of U and we have seen how to conjugate elements of U by elements of T. This determines the structure of B and the additional information follows from the above result.

We are almost ready to state the classification theorem, but we first have to face the issue of how to differentiate between two groups which differ only by a finite center such as $SL_n(K)$ and $PSL_n(K)$. We need one more ingredient. Let T be a maximal torus of G and set $X(T) = Hom(T, K^*)$. Then $X(T)$ is a $\mathbb{Z}$-lattice such that $\mathbb{Z}\Sigma \leq X(T) \leq \Lambda$, where $\mathbb{Z}\Sigma$ is the root lattice and Λ is the weight lattice. The two extremes of this containment are both free abelian groups of rank n and $\mathbb{Z}\Sigma/\Lambda$ is a finite group, so that there are only finitely possibilities for $X(T)$.

Theorem 2.6 *Let G be a simple algebraic group over an algebraically closed field. If T is a maximal torus of G, then G is determined up to isomorphism by the pair $(\Sigma, X(T))$.*

So the classification theorem shows that the simple algebraic groups are very closely related to simple Lie algebras over the complex numbers. They are more or less determined by a root system. The above result can be stated more generally, so as to classify semisimple and reductive groups. A connected group is called *semisimple* if it does not not contain a connected solvable normal subgroup and *reductive* if it does not contain a connected normal unipotent subgroup. Semisimple groups turn out to be commuting products of simple groups and reductive groups can be expressed as a commuting product of a semisimple group and a torus.

If we fix Σ and take the simple group corresponding to Λ, then we obtain what is called the *simply connected* group of type Σ, say $\hat{G}$, which is universal in the sense that there is a natural surjection $\pi : \hat{G} \to G$ for any simple algebraic group of the same type. We note that SL_n is simply connected.

2.4 Parabolic subgroups and subsystem subgroups

In this section we discuss two types of subgroups of a simple algebraic group G which arise naturally from the root system.

Parabolic subgroups. A *parabolic subgroup* is defined to be a subgroup containing a Borel subgroup. In the case of SL_n these are just the stabilizers of a flag of subspaces of the usual module. If P is such a subgroup, then $P = QL$, where Q is a unipotent group which can be defined by inducing the identity on quotients of successive terms in the filtration and L is block diagonal. L is called a *Levi subgroup* and it roughly speaking a product of smaller $SL's$.

Parabolic subgroups occur quite naturally in simple algebraic groups. Namely, let $B = UT$ be a Borel subgroup. So T determines a set of root groups and U is the product of root groups for positive roots. Given any subset $J \subseteq \Pi$, set $P = P_J = \langle B, s_i : \alpha_i \in J \rangle$. Then P is a parabolic subgroup and all parabolic subgroups are conjugate to precisely one group of this type. Thus the conjugacy classes of parabolic subgroups is in bijective correspondence with subsets of Π. We can think of parabolic subgroups as being obtained by removing some subset of nodes from the Dynkin diagram. The Levi subgroup of a parabolic contains the commuting product of simple groups one for each connected component of the diagram after the nodes have been removed. Maximal parabolic subgroups occur by removing a single node.

We highlight one special feature of parabolic subgroups and illustrate with the case of maximal parabolics. Assume $J = \Pi - \{\alpha_i\}$ and write $P = P_J = QL$. If α is a positve root we can write $\alpha = \sum_j c_j \alpha_j$. Then $Q = \prod_\alpha U_\alpha$, such that $c_i \geq 1$. Moreover, for each $j > 0$, let $Q(j)$ denote the product of all root groups U_α such that $c_i \geq j$. It follows from the commutator relations that $Q(j)$ is a normal subgroup of P_J for each j. To state a uniform result we rule out cases where the Dynkin diagram of G has a double bond and the characteristic is 2 (also 3 for G_2). Then we can state the following result which combines results of [4] and [36].

Proposition 2.7 *For each positive integer j the quotient $V(j) = Q(j)/Q(j+1)$ has the structure of an irreducible module for the Levi subgroup L of P and L has finitely many orbits on this module.*

Subsystem subgroups. Let Σ be a root system. A subset Δ is said to be *closed* if it satisfies two conditions: (i) $\alpha \in \Delta$ if and only if $-\alpha \in \Delta$; (2) If $\alpha, \beta \in \Delta$ and $\alpha + \beta \in \Sigma$, then $\alpha + \beta \in \Delta$. If Δ is closed, then using the above information we can show that $G(\Delta) = \langle U_\alpha : \alpha \in \Delta \rangle$, is a T-invariant semisimple group. Except for some special situations in characteristic 2 and 3 (the latter only for G_2), these are the only semisimple groups normalized by T. The group $G(\Delta)$ is called a *subsystem subgroup* of G.

Let's consider a couple of examples. Let $G = SL_n(K)$ and consider a subgroup of the form $SL_k(K) \times SL_{n-k}(K)$. It is a subsystem group, in fact, it is the semisimple part of a Levi subgroup. Now consider the symplectic group. Certainly $Sp_{2n}(K) > Sp_{2k}(K) \times Sp_{2n-2k}(K)$, corresponding to an orthogonal decomposition of the underlying space. However, this is not part of a Levi subgroup. Similarly for the orthogonal groups. However, these groups are subsystem groups.

There is a lovely algorithm of Borel and de Siebenthal that determines all subsystem groups. Start from the Dynkin diagram and form the extended diagram by adding the negative of the root of highest height. Remove any collection of nodes and repeat the process with each of the connected components of the resulting graph. All diagrams obtained in this way are Dynkin diagrams of subsystem groups and there is one conjugacy class of subsystem groups for each such subsystem.

We illustrate with some examples. First consider $Sp_{2n}(K)$. The extended diagram adds a long root at the end of the diagram adjacent to the short root α_1. If we remove any short node, we will get a subsystem of type $C_k \times C_{n-k}$. So this accounts for the subgroups described above. Let's do the same thing for E_8. In this case the negative of the root of highest height can be joined to α_8. We can remove any of the 8 other nodes to get subsystems. Here is a list of some of the subsystems we obtain in this manner: $D_8, A_8, A_4A_4, A_2E_6, A_1E_7$.

The five subsystem subgroups of E_8 listed above turn out to be maximal among closed connected subgroups, but they are not necessarily maximal among closed subgroups. This is because three of the subgroups are proper in their normalizers. We record these normalizers for future reference

$$N_G(A_8) = (A_8)Z_2$$
$$N_G(A_2E_6) = (A_2E_6)Z_2$$
$$N_G(A_4A_4) = (A_4A_4)Z_4$$

3 Morphisms of algebraic groups

In this section our focus will be on morphisms of algebraic groups. The finite groups of Lie type are introduced as fixed points of certain morphisms of algebraic groups.

We begin with a simple algebraic group G defined over an algebraically closed field of characteristic p. Let σ be a nontrivial endomorphism of G. Since G is simple modulo a finite center and $G = G'$, it follows that σ is an isomorphism of G viewed as an abstract group. However, the inverse map need not be a morphism of algebraic groups. How could this happen? Consider the example of $G = K^+$, where K has positive characteristic $p > 0$. Of course, this not a simple group, but it will serve as an example. We consider morphisms, σ_1 and σ_2 of K^+ as follows. For $0 \neq s \in K$ fixed, let $\sigma_1 : c \to sc$. Now let $\sigma_2 : c \to c^p$. It is easy to see that σ_1 is an isomorphism (the inverse sends $c \to s^{-1}c$). On the other hand, σ_2 is an isomorphism of K^+ as an abstract group, but there is no algebraic inverse. Indeed, the comorphsm, σ_2^*, sends x to x^p and hence is not surjective. The following fundamental result of Steinberg clarifies the situation.

Theorem 3.1 *Assume σ is an endomorphism of the simple algebraic group G. Then one of the following holds:*
i) σ is an automorphism of G.
ii) G_σ is finite.

Endomorphisms of type (ii) are called *Frobenius morphisms*. The difference between the types of endomorphisms can be detected at the level of Lie algebras. Let σ be an endomorphism. Then $\partial\sigma$ is either an isomorphism of $L(G)$ or nilpotent, according to whether σ is an automorphism or a Frobenius morphism.

The following general result of Steinberg is another key ingredient in understanding endomorphisms of simple algebraic groups.

Proposition 3.2 *Let σ be an endomorphism of the simple algebraic group G. Then σ stabilizes a Borel subgroup of G.*

3.1 Automorphisms

So what are the automorphisms of a simple algebraic group G? First, there are the inner automorphisms. Since G is simple modulo a finite center, analyzing the different types of inner automorphisms is roughly equivalent to analyzing elements of G and conjugacy classes.

There is a Jordan decomposition for elements of G just like for simple Lie algebras. If $g \in G$, then we can write $g = su = us$, where s has the property that it can be diagonalized in all representations of G, whereas u is represented by a unipotent matrix in all representations of G. Moreover, the expression is unique, so the study of conjugacy classes can be reduced to studying semisimple and unipotent classes. If $G = SL_n(K)$, then the semisimple elements are just matrices of determinant 1 which are conjugate to diagonal matrices, whereas the unipotent matrices are conjugate to lower triangular matrices with $1's$ on the main diagonal.

What about conjugacy classes? A semisimple element in $SL_n(K)$ is described up to conjugacy by the multiplicity of its eigenvalues. A unipotent element, can be put into Jordan form. That is, the matrix is a product of blocks and each block is lower triangular with $1's$ on the main diagonal and subdiagonal and $0's$ elsewhere. Both types of elements yield partitions of n, but there are infinitely many semisimple classes because one can assign different eigenvalues to each block (provided there are no coincidences), whereas there are only finitely many classes of unipotent elements.

We will want to know about the fixed points of automorphisms, in particular centralizers of elements of G. If $G = SL_n(K)$ and s is semisimple, then $C_G(s)$ is just the full group of block diagonal matrices corresponding to the partition. This is a Levi subgroup. Understanding $C_G(u)$ is more complicated we will come back to this in a later section.

Something similar occurs for other types of groups. If s is a semisimple element of a simple algebraic group G, then it can be shown that there is a maximal torus T of G with $s \in T$. Hence, $T \le C_G(s)$. A general lemma shows that any T-invariant subgroup has its connected component generated by root groups. So $C_G(s)^0$ is generated by the root groups centralized by s. As s has inverse action on a root group and its negative, these come in opposite pairs and one checks that $C_G(s)^0$ is a subsystem group, though perhaps not a Levi subgroup. For example, in E_8 we have observed the existence of a subystem group of type A_4A_4. This group is the central product of two copies of SL_5, so if the characteristic is not 5, then there is a center of order 5 and we find that $A_4A_4 = C_G(s)$ for such a central element.

Here is a general result regarding the semisimple classes of simple algebraic groups. Note that it shows that a simple algebraic group has infinitely many conjugacy classes of semisimple elements.

Proposition 3.3 *Let G be a simple algebraic group and T a maximal torus of G.*
i) Every semisimple conjugacy class meets T.
ii) Two elements of T conjugate in G are conjugate in $N(T)$ (i.e. by the Weyl group).

Proof We offer a proof of (ii) as an illustration of the use of the (BN)-pair structure of G. Suppose $t_1, t_2 \in T$ and $g \in G$ with $t_1^g = t_2$. Write $g = u_1 t n_w u_2$, where $u_1 \in U$, $u_2 \in U_w^-$, and $t \in T$. The equation $t_1^g = t_2$ can be rewritten to yield $t_1 u_1 t n_w u_2 = u_1 t n_w u_2 t_2$. Rearrange each side to get $u_1^{t_1^{-1}} t_1 t n_w u_2 = u_1 t n_w t_2 u_2^{t_2} = u_1 t t_2^{n_w^{-1}} n_w u_2^{t_2}$. Now compare the two extreme terms of this expression using uniqueness. We find that u_i centralizes t_i for $i = 1, 2$. So $t_1^g = t_2$ reduces to $t_2 = t_1^{t n_w u_2} = t_1^{n_w u_2}$ and conjugating by u_2^{-1} we have $t_1^{n_w} = t_2$, as required.

What about outer automorphisms of a simple algebraic group? Start with an automorphism σ. Proposition 3.2 shows σ normalizes a Borel subgroup, say B, and adjusting by an inner automorphism we can assume that σ normalizes a maximal torus T of B. But then σ permutes the minimal T-invariant unipotent subgroups, namely the root subgroups of G. On the other hand, σ normalizes B, which is built from the positive root groups. Further, if $Q = R_u(B)$, then the root groups in $Q - Q'$ are the root groups for simple roots, so these are permuted by σ. It follows that σ gives rise to a permutation of the Dynkin diagram of G.

This sort of analysis ultimately shows that $Aut(G)$ is generated by inner automorphisms together with certain graph automorphisms, where the possibilities are visible from the Dynkin diagram. Namely, the types A_n, D_n, E_6 all have involutory automorphisms called *graph automorphisms*. In type D_4, there is a group of graph isomorphisms of type Sym_3. (Note also that for types B_2, G_2, F_4, there is a symmetry of the diagram provided one ignores the arrow. Such symmetries do play a role and we will discuss them later.)

For each symmetry of the Dynkin diagram there is a corresponding *graph automorphism* of G. Indeed, starting with a maximal torus T we can find an automorphism τ of G such that for each fundamental root $\alpha, U_\alpha(c)^\tau = U_\beta(c)$, where β is the image of α under the given symmetry of the Dynkin diagram. For the most part the above graph automorphisms of simple groups are involutory, although for D_4 graph automorphisms of order 3 also exist, called *triality* automorphisms.

It is not difficult to determine the fixed points of a graph automorphism. Suppose for example that τ is an involutory graph automorphism. If α is fixed by the symmetry, then $U_\alpha < G_\tau$. Otherwise, one of $U_\alpha(c)U_\beta(c), U_\alpha(c)U_\beta(c)U_{\alpha+\beta}(-c^2/2)$, or $U_{\alpha+\beta}$ is in G_τ. The latter two cases occur only if $\alpha+\beta$ is a root, in which case the second case occurs if $p \neq 2$ and the third case if $p = 2$. It turns out that G_τ^0 is a simple algebraic group where the Dynkin diagram has nodes corresponding to the orbits on simple roots, and the root groups for fundamental roots are given above. The same holds for triality graph automorphisms of D_4, although orbits may have length 3. The fixed point groups of graph automorphisms are given below, where we indicate a group simply by giving its Dynkin diagram.

$(A_n)_\tau = C_{(n+1)/2}$ or $B_{n/2}$ ($C_{n/2}$ if $p = 2$), according to whether n is odd or even.
$(D_n)_\tau = B_{n-1}$
$(E_6)_\tau = F_4$
$(D_4)_\tau = G_2$, if τ is a triality graph automorphism.

If τ is a graph automorphism, there may be other types of involutions (or elements of order 3) in the coset $G\tau$. Consider the case of D_n with $\mathrm{char}(K) \neq 2$, viewed as the orthogonal group SO_{2n}. We can choose an orthonormal basis and consider the diagonal matrix $-1, 1^{2n-1}$. This is an orthogonal matrix, but it is not in SO_n. It is not difficult to see that it's connected centralizer in SO_{2n} is $SO_{2n-1} = B_{n-1}$. Indeed, this matrix induces an involutory graph automorphism, τ. However, adjusting this matrix by a diagonal matrix in SO_{2n} we get the matrix $(-1)^{2k+1}, 1^{2n-2k-1}$ which has connected centralizer B_kB_{n-k-1}. So in this case, $G\tau$ contains several classes of involutions. It turns out each involution in this coset is conjugate to one of the ones we just described.

The following result provides a complete analysis of the situation.

Theorem 3.4 *Let G be a simple simply connected algebraic group. Suppose that τ is a graph automorphism of G of order 2 or 3 (only for D_4) and that $\mathrm{char}(K) \neq |\tau|$. Listed below are the types of fixed points for elements of $G\tau$ of the same order as τ. Each type of fixed points corresponds to precisely one G-class in $G\tau$.*
A_n(n even) : $B_{n/2}$
A_n(n odd): $C_{(n+1)/2}, D_{(n+1/2)}$
$E_6 : F_4, C_4$
$D_n : B_kB_{n-k-1}$ for $n-1 \geq k \geq n/2$
$D_4 : G_2, A_2$

We have discussed several types of involutory outer automorphisms. Of course there are also involutions within G. These involutions share a remarkable feature which is described in the following result, first established in [34] and [51]. We will present an alternate proof in a later section.

Proposition 3.5 *Assume τ is an involutory automorphism of the simple algebraic group G and $\mathrm{char}(K) \neq 2$. Then $|G_\tau \backslash G/B| < \infty$, where G_τ denotes the fixed points of τ on G. That is, there are finitely many orbits of G_τ on Borel subgroups.*

3.2 Frobenius morphisms

We next discuss the other case of Theorem 1.1. Namely, consider an endomorphism, say σ, of the simple algebraic group G for which G_σ is finite.

We begin with an example. Consider $G = SL_n(K)$, where K is an algebraically closed field of characteristic p. View this group as a group of matrices and consider the morphism $\sigma : (a_{ij}) \to (a_{ij}^q)$, where $q = p^a$. It is clear that $G_\sigma = SL_n(q)$, a finite group. Notice also, that σ does have an inverse, but it is not a polynomial map and hence is not a morphism of G as an algebraic group. Also, with σ as above, $\partial\sigma$ is the 0 map.

As mentioned earlier, a morphism σ is a Frobenius morphism if and only if $\partial\sigma : L(G) \to L(G)$ is nilpotent. It follows that if τ is any endomorphism of G, then the morphisms $\tau\sigma$ and $\sigma\tau$ must also be Frobenius morphisms. Indeed, it is easy to see that the differential of each of these morphisms fails to be surjective, so Theorem 3.1 implies they are both Frobenius morphisms.

In the the special case where τ is a graph automorphism of $G = SL_n(K)$, then $G_{\sigma\tau} = SU_n(q)$, the unitary group over the base field $\mathbb{F}_{q^2}$. Thus the algebraic group $SL_n(K)$ gives rise to two types of finite groups, $SL_n(q)$ and $SU_n(q)$, by taking fixed points of different Frobenius morphisms. Of course, q can be taken as any power of the characteristic.

Let G be a simple algebraic group and σ a Frobenius morphism. Then G_σ is a finite group. If G is taken to be simply connected, then excluding a small handful of cases, G_σ is a simple group except for a finite center. We call G_σ a *finite group of Lie type* and $G_\sigma/Z(G_\sigma)$ a *finite simple group of Lie type.*

The classication theorem of finite simple groups states

Theorem 3.6 *Let G be a finite simple group. Then one of the following holds*
i) G is a simple group of Lie type.
ii) $G \cong Alt_n$, an alternating group for $n \geq 5$.
iii) G is isomorphic to one of 26 sporadic simple groups.

We next discuss the various types of groups G_σ that occur. For each type of simple algebraic group there is a finite group of the same type, such as. $SL_n(q) = A_{n-1}(q)$. If there is a symmetry of the Dynkin diagram, then there is also a twisted group, such as $SU_n(q) = {}^2A_{n-1}(q)$. For example, we obtain groups of type $E_6(q)$ and ${}^2E_6(q)$.

There is one other special situation to discuss. Consider the cases $G = B_2, F_4, G_2$. If one ignores the arrow on the diagram, then there is a symmetry of the graph. When $char(K) = p = 2, 2, 3$, respectively, there is an exceptional morphism τ of G with the following property. If α_i corresponds to α_j under the symmetry, then $U_{\alpha_i}(c) \to U_{\alpha_j}(c^{n_i})$, where $n_i = 1$ or p, according to whether α_i is a long or short root. Here τ^2 is a usual field automorphism. Multiplying τ by various powers of this field morphism, we get additional finite groups, ${}^2B_2(q)$, ${}^2F_4(q)$, ${}^2G_2(q)$where q is an odd power of p.

In the following we list the possible types of groups G_σ. Throughout q is a power of the characteristic of K.

Chevalley groups: $A_n(q), B_n(q), C_n(q), D_n(q), E_6(q), E_7(q), E_8(q), F_4(q), G_2(q)$
Twisted types: ${}^2A_n(q)$, ${}^2D_n(q)$, ${}^3D_4(q)$, ${}^2E_6(q)$
Special types: ${}^2B_2(2^{2k+1})$, ${}^2F_4(2^{2k+1})$, ${}^2G_2(3^{2k+1})$

The finite groups of Lie type play a central role in the theory of finite groups. One of the best ways to study finite simple groups of Lie type is to first obtain results for simple algebraic groups and then use the σ-setup to pass from the algebraic group to the finite group. Many considerations are easier at the level of algebraic groups, due to the presence of extra structure, and it is easier to obtain insight. For example, we have seen that both $SL_n(q)$ and $SU_n(q)$ arise from the same algebraic group $SL_n(K)$. It follows that there are connections between these groups that might not be apparent if one studies the groups just as finite groups. The similarity is already apparent in the group order. For example, for n even

$$|SL_n(q)| = q^{n(n-1)/2}(q^n - 1)(q^{n-1} - 1)(q^{n-2} - 1) \cdots (q^3 - 1)(q^2 - 1)$$
$$|SU_n(q)| = q^{n(n-1)/2}(q^n - 1)(q^{n-1} + 1)(q^{n-2} - 1) \cdots (q^3 + 1)(q^2 - 1)$$

There are many instances where a feature of $SL_n(q)$ can be translated to one of $SU_n(q)$ by changing certain signs.

The finite groups of Lie type also have a (BN)-pair which is inherited from that of the corresponding algebraic group. For example, $SO^+_{2n}(q)$ has Dynkin diagram of type D_n, with all parts of the structure clearly related to the corresponding objects in the algebraic group. If α is a root, then $(U_\alpha)_\sigma = \{U_\alpha(c) : c \in \mathbb{F}_q\}$, a group isomorphic to $\mathbb{F}_q^+$.

But now consider $\tau\sigma$. If $\alpha = \alpha_i$ for $1 \leq i \leq n-2$, then $(U_\alpha)_{\tau\sigma} = \{U_\alpha(c) : c \in \mathbb{F}_q\} \cong \mathbb{F}_q^+$, as before. On the other hand $\tau\sigma$ interchanges $U_{\alpha_{n-1}}$ and U_{α_n} and we get fixed points $(U_{\alpha_{n-1}} U_{\alpha_n})_{\tau\sigma} = \{U_{\alpha_{n-1}}(c) U_{\alpha_n}(c^q) : c \in \mathbb{F}_{q^2}\} \cong (F_{q^2})^+$. What we find here is that there is again a BN-structure, but the Dynkin diagram is of type B_{n-1}. Moreover, the root groups have different orders. Even though these two orthogonal groups have different BN-structures, they still have closely related order formulas just as in the case of $SL_n(q)$ and $SU_n(q)$.

We next discuss a fundamental result concerning Frobenius morphisms which is often referred to as Lang's theorem.

Theorem 3.7 *Assume G is a connected algebraic group and σ is a morphism of G with G_σ finite. Then the map $g \to g^{-1}\sigma(g)$ is surjective. So, in the semi-direct product $G\langle\sigma\rangle$ the coset $G\sigma = \sigma^G$.*

An important application occurs when one considers orbits of a simple group G on an algebraic variety V. Suppose that σ is a Frobenius morphism of G having compatible action on V (i.e $\sigma(gv) = \sigma(g)\sigma(v)$). For $v \in V, \sigma(v) = g(v)$ for some $g \in G$, so that $g^{-1}\sigma$ fixes v. By Lang's theorem $g^{-1}\sigma$ is a conjugate of σ, so it follows that V_σ is nonempty. It is easy to see that G_σ acts on V_σ. But how many orbits are there in this action?

Suppose $\sigma(v) = v$ and $\sigma(w) = w$. There exists $g \in G$ such that $g(v) = w$. Then $\sigma g(v) = g(v)$ and so $g^{-1}\sigma g$ fixes v. Hence, $g^{-1}\sigma g \in G\sigma \cap stab(v) = G_v\sigma$. Suppose we know that G_v is irreducible. Then Lang's theorem tell us that $g^{-1}\sigma g = h\sigma h^{-1}$ for some $h \in G_v$. Hence, $gh \in G_\sigma$. Then $w = g(v) = gh(v)$ which implies that G_σ is transitive on V_σ. That is, under the irreducibility assumption, an orbit of G gives rise to an orbit of the finite group G_σ.

If we relax the assumption that G_v is connected, we can still apply a version of Lang's theorem, but we have to be more careful since it only applies to connected groups. There is a completely general result in this direction which we now state.

Proposition 3.8 *Let G be an irreducible algebraic group with Frobenius morphism σ and suppose that G acts morphically and transitively on a variety V with compatible action of σ. Then $V_\sigma \neq \emptyset$. Further, the number of orbits of G_σ on V_σ is in bijective correspondence with the H classes in the coset $H\sigma$, where $H = G_v/G_v^0$ and v is an element of V_σ.*

The above result is of great importance for understanding the structure of the finite groups of Lie type. We illustrate with a few examples. Consider a simple algebraic group, G, with Frobenius morphism σ.

Maximal tori. Assume σ is a field morphism. Thus $G_\sigma = G(q)$ is a finite group of Lie type and has the same Dynkin diagram as G. Let T be a maximal torus of G. If α is a root, then σ normalizes U_α and induces a field morphism. It follows that σ stabilizes the coset Ts_α. As this occurs for each root, we conclude that σ centralizes $W = N_G(T)/T$. Now $N_G(T)$ is the stabilizer of a point in the action of G on T^G. What does the above theorem say in this case? Well, consider the orbits of G_σ on the fixed points of σ in this transitive action. The last result shows that these orbits are in bijective correspondence with W classes in $W\sigma$. But since σ centralizes W, in fact the orbits are in bijective correspondence with conjugacy classes in W. The fixed points of σ stable maximal tori of G are called *maximal tori of* $G(q)$. What do the maximal tori of $G(q)$ look like?

We next describe a method for answering that question. We will work within the present context of maximal tori, but the method is quite general. Let $n \in N_G(T)$. Notice that $n\sigma$ normalizes T. On the other hand, Lang's theorem shows that $n\sigma$ is conjugate of σ. Hence, taking conjugates, there exists a σ-stable conjugate of T for which σ has fixed points isomorphic to those of $n\sigma$ on T.

Lets try this in $G = GL_m(K)$, regarded as matrices, with σ the q-power morphism. We use GL_m rather than SL_m in order to simplify matters. Take T to be the invertible diagonal matrices. Taking $n = 1, T_\sigma$ consists of the diagonal matrices over $\mathbb{F}_q$ and thus is isomorphic to $(\mathbb{Z}_{q-1})^m$. On the other hand, if $n = n_w$ corresponds to an m-cycle in the symmetric group, then one checks that $T_{n\sigma} \cong (F_{q^m})^* \cong Z_{q^m-1}$. This is often called a Singer cycle. Of course, the general case is where $n = n_w$ corresponds to a product of cycles corresponding to a partition $m = m_1 + \cdots + m_k$. Here $T_{n\sigma} \cong Z_{q^{m_1}-1} \times \cdots \times Z_{q^{m_k}-1}$. So clearly the maximal tori can have very different structures. The maximal tori in the finite classsical groups can also be understand from linear algebra considerations. But for exceptional groups the connection with algebraic groups provides the best approach for understanding these important subgroups.

What is important here is that we are learning a great deal about important subgroups of $G(q)$ by considering a much easier structure in the algebraic group.

Subsystem subgroups. Exactly the same thing occurs if one consider other sorts of orbits of G. Here is an interesting example. Let $G = E_8$. We can see from the extended Dynkin diagram that $E_8 > A_4A_4 > T$. There is just one class of subgroups of this type. Both factors are of type SL_5, but they have the same center. We have already mentioned that $N_G(A_4A_4) = A_4A_4Z_4$.

The group Z_4 is generated by an element of the Weyl group. Taking σ to be a field morphism we see that it centralizes a representative of each fundamental reflection, hence centralizes $N_G(T)/T \cong W$, and hence we can take σ to centralize the Z_4 factor. So we conclude that G_σ has four orbits on fixed points of σ on $(A_4A_4)^G$. We want to know the structure of the fixed points of σ on representatives of the σ-stable orbits. We again use the above method. If $Z_4 = \langle s \rangle$, then we need to consider fixed points of $\sigma, s^2\sigma, s\sigma, s^{-1}\sigma$. It turns out that s interchanges the A_4 factors and s^2 induces a graph automorphism on each factor. Starting from the direct product of two copies of SL_5 and replacing s by s^{-1}, if necessary, we obtain

$(A_4A_4)_\sigma \cong SL_5(q)SL_5(q)$
$(A_4A_4)_{s^2\sigma} \cong SU_5(q)SU_5(q)$
$(A_4A_4)_{s\sigma} \cong SU_5(q^2)$
$(A_4A_4)_{s^3\sigma} \cong PGU_5(q^2)$.

If one was working entirely within the finite group $E_8(q)$, some of the above subgroups would be very difficult to find. Yet, from the perspective of the algebraic group they are easy to understand and one can predict what might happen.

4 Maximal subgroups of classical algebraic groups

In this section we discuss the problem of determining the maximal connected subgroups of the classical groups $SL(V), Sp(V), SO(V)$. Let M be maximal among closed connected subgroups of $I(V)$, where $I(V)$ denotes one of the classical groups. Analysis of this problem began with work of Dynkin [14], in characteristic 0 and was redone and extended to arbitrary characteristic by Seitz [41] and Testerman [47]

We begin with a reduction.

Proposition 4.1 *Let M be maximal among closed connected subgroups of $G = I(V)$. Then one of the following holds:*

i). *$M = G_W$ where W is a singular subspace, a non-degenerate subspace, or a nonsingular 1-space (only if $p = 2$ with G orthogonal).*

ii). *$V = V_1 \otimes V_2$ and $M = I(V_1) \circ I(V_2)$.*

iii). *M is a simple group and $V \downarrow M$ is irreducible and restricted (if $p > 0$)..*

Proof We will sketch the proof and in the process describe the groups that arise in parts (i) and (ii). In the $SL(V)$ case we consider V to have the trivial bilinear form.

First assume M is a reducible subgroup and let W be a proper M-invariant subspace, which we can take to be minimal. If V is equipped with a non-degenerate form, then by minimality W is either non-degenerate or totally singular under the bilinear form. If W is singular (i.e. the bilinear form restricted to W is trivial), then, with one exception, G_W is a proper parabolic subgroup of G. The exception occurs when $p = 2, G = O(V) = D_n$ and $G_W = B_{n-1}$. On the other hand, if W is non-degenerate with respect to the bilinear form, then $V = W \perp W^{\perp}$, this is preserved by M, and hence $M = I(W) \circ I(W^{\perp})$.

So now assume that $V \downarrow M$ is irreducible. Notice that this forces M to be reductive, else M would act on the fixed points of the unipotent radical of M. Also, M is semisimple, since otherwise M would act on the various weight spaces of a central torus. If M is a commuting product of more than one simple factor, then M preserves a tensor product decomposition of V. Indeed, irreducible representations of commuting products are just the tensor products of irreducibles for the given factors. So we ask what is the full group preserving a tensor product, $V = V_1 \otimes V_2$. If $G = SL(V)$, then this is clearly $SL(V_1) \circ SL(V_2)$. Suppose V has a nondegenerate form. Then each simple factor acts homogeneously on the space and preserves the bilinear form. So the corresponding irreducible module for the simple factor is self dual and the factor preserves a form on the corresponding tensor factor. An irreducible group can preserve only one type of bilinear form up to scalar multiples, so the bilinear form on V must be a multiple of the tensor product form. Thus M must be a group such as $Sp(V_1) \circ SO(V_2) < Sp(V)$ or $Sp(V_1) \circ Sp(V_2) < SO(V)$. In this way we settle the case where M is not simple.

Finally, suppose that M is a simple group. If $p > 0$, then the tensor product theorem shows that $V \downarrow M$ is the tensor product of twists of restricted irreducibles. If there is more than one tensor factor then M is properly contained in a group of type (ii), against maximality. Hence, there is just one factor. So the embedding is the composition of a Frobenius twist of M followed by a restricted representation. The twist does not change the image group, so as a subgroup of $I(V)$, the representation is restricted.

One now wants to determine which of the groups appearing in (i), (ii), and (iii) are maximal in $I(V)$. This is not too difficult for the first two cases. The real mystery occurs in part (iii). Here the problem is the following. Let X be a simple algebraic group and consider a restricted irreducible representation of X. When is the image of X maximal in a corresponding classical group? If this is not the case, then there is an embedding $X < Y$ of irreducible subgroups and the goal is to explicitly determine such containments.

Before continuing we recall some notation from representation theory. If X is a semisimple algebraic group and λ is a dominant weight we let $L_X(\lambda)$ denote the irreducible module for X with high weight λ. There is an expression $\lambda = \sum_i a_i\omega_i$, where the ω_i are the fundamental dominant weights and the a_i are nonnegative integers. Then λ determines a *labelling* of the Dynkin diagram of X where the nodes are labelled by the integers a_i.

A first goal is to understand when an irreducible representation is self-dual and then determine what type of form is stabilized. The following lemmas largely settle these issues.

Lemma 4.2 *Suppose X is a simple algebraic group and $V = L_X(\lambda)$ is an irreducible representation.*

i). *$V^* \cong L_X(-w_0\lambda)$, where w_0 is the long word in the Weyl group.*

ii). *All irreducible modules of X are self-dual unless the Dynkin diagram of X has type A_n, D_n(n odd), or E_6.*

iii). *If the Dynkin diagram of X has type A_n, D_n(n odd), or E_6, then $L_X(\lambda)$ is self-dual if and only if $\lambda = \tau(\lambda)$, where τ is the involutory graph automorphism.*

Proof Here is a sketch of (i). Let $V = L_X(\lambda)$ and let B be a Borel subgroup of X with maximal torus T. The T-weights of V^* are the negatives of the weights of V and weights of V can be obtained by subtracting positive roots from λ. Hence the weights of V^* all have the form $-\lambda$ plus a sum of positive roots. Therefore, $-\lambda$ is the lowest weight of V^*. Consequently, if w_0 is the long word in the Weyl group, then B^{w_0}, the opposite of B, stabilizes a 1-space of V which affords the weight $-\lambda$ for T. Conjugating by w_0 we then see that B stabilizes a 1-space affording the weight $-w_0\lambda$.

For (ii) consider the action of $-w_0$ on roots. Aside from the special cases indicated, one sees that w_0 is -1 on a maximal torus, so sends each root to its negative. Hence $-w_0$ is the identity and we get (ii). In the exceptional cases, $-w_0$ affords the involutory graph automorphism and (iii) holds.

The following lemma of Steinberg describes which form is fixed once the module is self-dual.

Lemma 4.3 *Assume $V = L_X(\lambda)$ and $V \cong V^*$ and $p \neq 2$. Let $z = \Pi_{\alpha\in\Sigma^+} T_\alpha(-1)$. Then the representation is symplectic if $\lambda(z) = -1$ and orthogonal if $\lambda(z) = 1$.*

Remarks. 1. Consider what happens forf $X = SL_2$, where λ is restricted if $p > 0$. Then $\lambda = n\lambda_1$ and $n < p$, if $p > 0$. The corresponding irreducible module $L_X(n)$ has dimension $n+1$. The element z in Lemma 4.3 generates center of SL_2. Moreover, $\lambda_1(z) = -1$ and $n\lambda_1(z) = (-1)^n$, so that $L_X(n)$ is symplectic if n is odd and orthogonal if n is even.

2. The situation for $p = 2$ is still open. In this case $SO(V) < Sp(V)$, so one always has an embedding in the symplectic group. But deciding when it is orthogonal is nontrivial.

The next lemma and its corollary set the stage for an inductive approach to the maximal subgroup problem.

Lemma 4.4 *(Smith) Let $V = L_X(\lambda)$ be an irreducible module for X and let $P = QL$ be a standard parabolic with unipotent radical Q and Levi subgroup L. Then V_Q (the fixed points of Q) affords an irreducible module for L of high weight λ.*

Proof We give an argument only for characteristic 0. First note that if v is a weight vector of weight λ, then $v \in V_Q$. Also, $U = QU_L$, where U_L is a maximal unipotent subgroup of L. Now, U has a one-dimensional fixed space on V, namely the span of v. So the same must hold for the action of U_L on V_Q. But this implies the action is irreducible of high weight λ,; otherwise the space would decompose into irreducibles and there would be other fixed points.

Corollary 4.5 *Suppose $X < Y$ are simple algebraic groups, both irreducible on V. Assume P_X, P_Y are parabolic subgroups of X and Y such that $P_X = Q_X L_X, P_Y = Q_Y L_Y$ and $Q_X \leq Q_Y$. Then $V_{Q_X} = V_{Q_Y}$ and both L_X and L_Y are irreducible on this space.*

Proof This is easy from the lemma. We have $Q_X \leq Q_Y$, so $0 < V_{Q_Y} \leq V_{Q_X}$. Also, $L_X < P_X \leq P_Y$, so L_X normalizes Q_Y and acts on V_{Q_Y}. However, the lemma shows that L_X is irreducible on V_{Q_X}, so this is only possible if equality holds.

The case $X = A_1$ is an important case. This case is important in its own right, but in addition it forms a base for the induction. Fortunately, there are some nice features about A_1 that can be applied.

So now assume $X = A_1$, $V = L_X(n)$, and $X < I(V)$, where $I(V) = Sp(V)$ or $SO(V)$. We are interested in whether or not X is maximal. So assume that $X < Y < I(V)$. How can we find the possibilities for Y?

One key point is that irreducible representations of SL_2 have all weight spaces of dimension 1. If $X < Y$ and we choose a containment $T_X < T_Y$ of maximal tori, then the same must hold for weight spaces of T_Y. This is a very strong condition and is of interest independent of this particular problem.

In certain cases we can get a quick contradiction and we illustrate with $Y = A_2$. First note that X cannot lie in a proper parabolic of Y, since parabolics of Y are reducible on all modules for Y. So if $Y = A_2$, then X would be irreducible on the natural 3-dimensional module. It follows that if α is the fundamental root for X, then $\alpha_1 \downarrow T_X = \alpha_2 \downarrow T_X = \alpha$ for each of the fundamental roots of Y. Suppose V has high weight $\lambda = a\lambda_1 + b\lambda_2$. If both $a, b > 0$, then $\lambda - \alpha_1$ and $\lambda - \alpha_2$ are distinct weights of V with the same restriction to T_X a contradiction. Say $a > 0$ and $b = 0$. If $a > 1$, then $\lambda - 2\alpha_1$ and $\lambda - \alpha_1 - \alpha_2$ are distinct weights restricting to $\lambda - 2\alpha$ on T_X, again a contradiction. So this shows that $\lambda = \lambda_1$ for A_2. But then $SL(V) = A_2$, so $Y = SL(V)$, a contradiction.

Eventually, we obtain the following result.

Proposition 4.6 *Suppose $X < Y < I(V)$, where $I(V) = Sp(V)$ or $SO(V)$ with X of type A_1 and $V \downarrow X$ and $V \downarrow Y$ are irreducible and restricted. Then $Y = G_2, \dim(V) = 7$, and $I(V) = SO_7$.*

Consider what this means for the maximal subgroup problem.

Corollary 4.7 *Let X be the image of SL_2 under an irreducible restricted representation. Then X is a maximal connected subgroup of either $Sp(V)$ or $SO(V)$ unless* $\dim(V) = 7$ *and $X < G_2 < SO(V)$.*

This is a stunning result because it shows that classical groups of arbitrarily large dimension contain maximal subgroups of dimension 3.

Now we discuss some of the issues involved in dealing with higher rank configurations. First we note that there is a major division in the analysis according to whether Y is a classical group (Seitz) or an exceptional group (Testerman). Each case has its advantages and disadvantages and the analysis is very long in either case. If Y is a classical group, then the rank is unbounded, so there are infinitely many possibilities. On the other hand, one can study the embedding $X < Y$ by analyzing the action of X on the natural module for Y. This is an important tool. By way of contrast, if Y is an exceptional group, then Y has rank at most 8, so the rank of X is also bounded. Even so the possible embeddings are difficult to understand due to the lack of a classical module.

We next discuss how to proceed with the induction. Recall the setup of Corollary 4.5. Take one of our embeddings, $X < Y$, and choose a corresponding embedding of parabolic subgroups $P_X < P_Y$. Then both L_X and L_Y are irreducible on the space $V_{Q_X} = V_{Q_Y}$. We do not necessarily have a containment of the Levi subgroups, but this does hold modulo Q_Y, so there is a containment of the acting groups and we proceed as if there were a containment.

One proceeds by taking a convenient choice for P_X, usually a maximal parabolic corresponding to an end node. Then L'_X is a simple group of rank one less than that of X. On the other hand, a priori we do not have much information on L'_Y. All we can say is that $L'_Y = L_1 \times \ldots \times L_r$, for some product of simple groups.

Now V_{Q_X} is irreducible for both L'_X and L'_Y and the former acts as a restricted module. These are nearly always tensor indecomposable so this implies that at most one simple factor of L'_Y can act nontrivially, say L_i. So this means that all the other simple factors act trivially, implying that there are a certain number of $0's$ in the labelling of the high weight for Y. Now consider $\pi_i(L'_X) \leq L_i$, the image of the projection map. These two groups both act irreducibly on the fixed point space. So we are set up for induction. But we must always allow for the possibility that this containment is an equality.

In order get more information we dig a little deeper.

Lemma 4.8 *Let X be simple and let $V = L_X(\lambda)$. If $P = QL$ is a standard parabolic subgroup then $V/[V,Q]$ is irreducible of high weight $w_0^L w_0^G(\lambda)$.*

Proof Here is the idea of the proof. Let V^* denote the dual module of high weight $\lambda^* = -w_0^G\lambda$. We already know that V_Q is irreducible for L of high weight λ. Now consider the annihilator in V^* of this fixed point space. An easy argument shows that this annihilator is $[V^*, Q]$. Thus V_Q and $V^*/[V^*,Q]$ are dual modules for L. Hence the high weight of the latter is $-w_0^L\lambda$. Now do the same for $V/[V,Q]$, which is dual to V^*_Q. We conclude that this module is irreducible of high weight $-w_0^L\lambda^* = w_0^L w_0^G\lambda$. This is the desired result.

For technical reasons it will be helpful to make one additional observation. Suppose that in the last result we replaced the standard parabolic subgroup $P = QL$ by the opposite to P. Namely let $P^- = Q^-L$ be the parabolic with the same Levi subgroup, but for which the unipotent radical is the product of root groups for the negatives of those that appear in Q. If we do this, then the irreducible module $V/[V,Q^-]$ affords $V_L(\lambda)$ for L and the fixed points affords $V_L(w_0^L w_0^G(\lambda))$.

We next lemma providing additional information obtained from the embedding of parabolic subgroups. Say $X < Y$ are both irreducible on V and $V \downarrow X$ is restricted. Suppose that $P_X < P_Y$ is an embedding of opposite standard parabolics such that the Levi factor, L'_X is simple. Write $L'_Y = L_1 \times \ldots \times L_r$ and recall that there is a unique factor L_i acting nontrivially and irreducibly on $V/[V,Q_X] = V/[V,Q_Y]$. In the Dynkin diagram for Y consider a node adjacent to L_i. Since we are taking P_Y as the opposite of the standard parabolic subgroup, the corresponding node is the negative of a fundamental root, say $-\delta$. Let $\gamma = \delta + \beta_1 + \ldots + \beta_s$, where the β_i are the fundamental roots up to and including the nonzero label of L_i closest to $-\delta$. We note that $\lambda - \gamma$ restricts to a dominant weight of L. We will present an example of all this following the statement of the lemma.

Lemma 4.9 *Let $X < Y$ be irreducible on V, suppose that $P_X < P_Y$ be an embedding of opposite standard parabolics, and assume that P_X is a maximal parabolic of X. Then*

$$\dim(V/[V,Q_X]) \cdot \dim(Q_X/Q'_X) \geq \dim[V,Q_X]/[V,Q_X,Q_X] \geq \dim V_{L_Y}(\lambda - \gamma).$$

If there happens to be more than one node adjacent to L_i then the argument gives a sum of terms and stronger information.

We illustrate how 4.9 can be applied in a very specific configuration. Say $X < Y$ are both irreducible on V with $X = A_2$ and $V \downarrow X = L_X(a\lambda_1 + b\lambda_2)$, with the weight restricted. Take P_X so that the action on $V/[V, Q_X]$ is irreducible for $L'_X = A_1$ of high weight a. Write $L_Y = L_1 L_2 \dots L_k$ with $k > 1$, ordered such that $L_2 \dots L_X$ acts nontrivially on $V/[V, Q_X] = V/[V, Q_Y]$. Assume there is just a node $-\delta$ adjacent to both L_1 and L_2.

One possibility arising from induction is that $L_1 = A_1$ and this factor acts the same way on $V/[V, Q_X]$ as L'_X. To be definite we assume L_2 has type A_k and the label on $-\delta$ is nonzero. Then $\lambda - \delta$ affords the module of high weight$(a+1)\lambda_1$ for L_1 and the high weight of a natural module for L_2. So 4.9 reads

$$(a+1) \cdot 2 \geq (a+2)(k+1),$$

unless $a = p - 1$, where we use a slightly different argument. This gives a contradiction.

There are several additional levels of analysis and we briefly mention one of these. Given an embedding of parabolics $P_X < P_Y$, one can obtain information on how a root group $U_\beta \leq Q_X$ sits in Q_Y. Indeed, one obtains precise information regarding how $U_\beta(c)$ is expressed in terms of root elements of Y. In turn this provides information on how certain roots in the root system of Y restrict to a maximal torus of X. In some sense this links the group theory to the representation theory. It is a very important tool.

The ultimate result determines all possibilities for $X < Y$, assuming $Y \neq SL(V), Sp(V)$ or $SO(V)$. There are a number of configurations. Nonetheless, the main point is that if one considers an irreducible restricted representation of X, then the image is usually maximal in the smallest classical group containing it.

To complete the discussion we provide a few examples that occur. We refer the reader to [41] for a complete statement. Let $X < Y$ both be irreducible on V, acting via restricted representations. Let λ denote the hight weight and let $T_X < T_Y$ be a containment of maximal tori of the respective groups.

$(X, Y) = (O_n, SL_n)$ (natural embedding). The wedge powers of the natural module for SL_n are irreducible for both groups, provided $p \neq 2$.

$(X, Y) = (Sp_n, SL_n)$ (natural embedding). The restricted symmetric powers of the natural module for SL_n are irreducible for both groups.

$(X, Y) = (A_n, A_{(n^2+n-2)/2})$, where the embedding is obtained from the wedge square of the natural module for X. The wedge square of the natural module for Y remains irreducible for X, provided $p \neq 2$.

$(X, Y) = (E_7, C_{28})$, with $\lambda \downarrow T_X = \omega_2 + \omega_4$ and $\lambda \downarrow T_Y = \omega_5$, and $p \neq 2, 3, 5$.

$(X, Y) = (F_4, E_6)$, with $\lambda \downarrow T_X = (p-2)\omega_3 + \omega_4$ and $\lambda \downarrow T_Y = (p-2)\omega_5 + \omega_6$, where $2 \neq p > 0$.

5 Maximal subgroups of exceptional algebraic groups

In this section we discuss the problem of finding the maximal, closed, connected, subgroups of simple algebraic groups of exceptional type. Here the large group has bounded rank (at most 8) and consequently the goal is a complete list of the maximal subgroups. The analysis of this problem began with Dynkin who settled the case where K has characteristic 0 [15]. Seitz extended the work to positive characteristic, but subject to some mild characteristic restrictions. Very recently, Liebeck and Seitz [26] completed the analysis. So there now exist results for exceptional groups in arbitrary characteristic. Each of the papers is very long.

Below is a simplified version of the main theorem which covers only connected maximal subgroups. The more general result in [26] also covers disconnected maximal subgroups of positive dimension and also allows for the presence of certain morphisms of G, so that the result can be applied to yield information about the finite groups of Lie type.

Theorem 1 *Let G be a simple algebraic group of exceptional type. Let $X < G$ be maximal among proper closed connected subgroups of G. Then X is a parabolic subgroup, a semisimple subgroup of maximal rank, or X is in the table below.*

Table

G	X simple	X not simple
G_2	$A_1\,(p \geq 7)$	
F_4	$A_1\,(p \geq 13)$, $G_2\,(p = 7)$,	$A_1G_2\,(p \neq 2)$
E_6	$A_2\,(p \neq 2, 3)$, $G_2\,(p \neq 7)$, $C_4\,(p \neq 2)$, F_4	A_2G_2
E_7	A_1 (2 classes, $p \geq 17, 19$ resp.), $A_2\,(p \geq 5)$	$A_1A_1\,(p \neq 2, 3)$, $A_1G_2\,(p \neq 2)$, A_1F_4, G_2C_3
E_8	A_1 (3 classes, $p \geq 23, 29, 31$ resp.), $B_2\,(p \geq 5)$	$A_1A_2\,(p \neq 2, 3)$, G_2F_4

The maximal parabolic subgroups are easy-they correspond to removing a single node from the Dynkin diagram. Similarly, the maximal rank subgroups are well-known. Except for a few special cases in characteristic 2 (3 for G_2) they are just subsystem subgroups and correspond to removing nodes from the extended Dynkin diagram. Beyond this, there are remarkably few types of maximal subgroups and most of these have rather small rank in comparison with the rank of G. In particular, notice that maximal subgroups of type A_1 occur. Of course this should not surprise us at this point, as we have seen this behavior for classical groups as well.

In the following we describe some of the main ideas involved in the proof of the main theorem. The complete proofs are very long, but it is not so difficult to understand the overall strategy. Assume G is of adjoint type.

From now on assume that X is not a proper parabolic subgroup or a group of maximal rank.

Lemma 5.1 *The following conditions hold.*
i). *X is semisimple.*
ii). $C_G(X) = 1$
iii). $C_{L(G)}(X) = 0$
iv). $X = N_G(L(X))^0$.
v). $C_G(L(X)) = 1$.

Proof This is relatively easy. For (i) first note that $R_u(X) = 1$ as otherwise X would lie in a proper parabolic subgroup of G. So X is reductive. If X is not semisimple, then it centralizes a torus, whose full centralizer is a Levi subgroup. Hence (i) holds. If $C_G(X) > 1$, then X centralizes either a unipotent or an semisimple element. In the former case, X is contained in a proper parabolic subgroup and in the latter X lies in a maximal rank subgroup (as the semisimple element lies in a maximal torus of G). This gives (ii) and essentially the same idea gives (iii). For (iv) we certainly have $X \leq N_G(L(X))^0$, so the equality follows from maximality. For (v), let $C = C_G(L(X))$. We argue that $X \cap C = 1$, so maximality implies that C is finite. But then X centralizes C and so (ii) implies that $C = 1$.

We next describe an important 1-dimensional torus of X. Fix a maximal torus T_X of X and a system of T_X-invariant root subgroups of X, one for each root in the root system $\Sigma(X)$ of X. Let $\Pi(X)$ be a system of fundamental roots. If $\gamma \in \Sigma(X)^+$ and if $U_\gamma, U_{-\gamma}$ are the corresponding T_X-root subgroups of X, then we let $T_\gamma(c)$ be the image of the matrix $\mathrm{diag}(c, c^{-1})$ under the usual surjection $SL_2 \to \langle U_\gamma, U_{-\gamma}\rangle$.

For $c \in K^*$ set

$$T(c) = \Pi_{\gamma \in \Sigma(X)^+} T_\gamma(c),$$

and

$$T = \langle T(c) : c \in K^* \rangle.$$

Lemma 5.2 (i) $T(c)e_\alpha = c^2 e_\alpha$ *for each* $\alpha \in \Pi(X)$.
(ii) $T(c)h = h$ *for all* $h \in L(T_X)$.

Proof (ii) is immediate since $T \leq T_X$ and T_X acts trivially on $L(T_X)$. For (i) fix $\alpha \in \Pi(X)$. Then $T(c)e_\alpha = c^r e_\alpha$, where $r = \sum_{\gamma \in \Sigma(X)^+} \langle \alpha, \gamma \rangle$. Let $\Sigma(X)^*$ denote the dual root system consisting of roots $\delta^* = \delta/(\delta, \delta)$, for $\delta \in \Sigma(X)$. Then $r = \Sigma_{\gamma \in \Sigma(X)^+} \langle \gamma^*, \alpha^* \rangle =$

$2\langle \rho^*, \alpha^* \rangle$, where ρ is the half-sum of positive roots in $\Sigma(X)$. But it is well known that ρ is the sum of all fundamental dominant weights of $\Sigma(X)^*$ and α^* is a fundamental root in $\Sigma(X)^*$. (i) follows.

Since each root in $\Sigma(X)$ is an integral combination of roots in $\Pi(X)$ the previous lemma determines all weights of T on $L(X)$, showing, in particular, that these weights are all even.

We next pass to weights of T on L. Consider $T \leq T_X < T_G$, with T_G a maximal torus of G. Then for $\beta \in \Sigma(G)$, e_β is a weight vector of T and we write

$$T(c)e_\beta = c^{t_\beta} e_\beta,$$

where t_β is an integer.

Lemma 5.3 (i) *The T_X-weights on L are each integral combinations of elements of $\Sigma(X)$.*

(ii) *There exists a system of fundamental roots $\Pi(G)$ of $\Sigma(G)$ such that $t_\beta = 0$ or 2 for each $\beta \in \Pi(G)$.*

Proof Here is a sketch of the proof. Consider the action of X on $L(G)$ Using representation theory we can write $L(G) = I \oplus J$, where $0 \neq I$ is the sum of all weight spaces for T_X where the corresponding weights are integral combinations of roots and J is the sum of the remaining weight spaces. One argues that the full stabilizer of I is a group of maximal rank. Indeed, I is stabilized by the group generated by T_G and all root groups corresponding to roots that restrict to T_X as an integral combinations of roots of X. This contradicts maximality of X, unless $I = L(G)$. So this gives (i).

We can choose a system of fundamental roots for G such that each of the root vectors affords a nonnegative (even) weight for T. We must show that the corresponding weights are only 0 or 2. The idea here is to take the Levi subgroup generated by all such fundamental roots and their negatives, then argue using Lemma 5.2 that $L(X)$ is contained in the Lie algebra of this Levi. If this is a proper Levi, then it is centralized by a torus, which contradicts Lemma 1.1(iv).

Let's now consider what the above lemma says. Given X we obtain a labelled Dynkin diagram for G, where the labels are the weights of T on fundamental roots. From such a labelling we can easily obtain all weights of T on $L(G)$. In particular, the largest T weight on $L(G)$ is bounded by twice the height of the highest root in the root system of G. We also know that each composition factor of X on $L(G)$ has all weights being combinations of roots of X. Restricting to T we get a collection of T-weights. Combining these over all composition factors we must get T weights which are precisely those from the given labelling. This, then, is a restriction on the possible composition factors that can appear. In particular, there are only finitely many possible composition factors that can occur!

At this point we have narrowed our search to some degree. To go further we separate the analysis according to whether or not X is simple.

Suppose first that X is not simple. Say $X = X_1 \cdots X_r$, with $r > 1$ and each X_i a simple group. One can see from the statement of the theorm, there are a number of possibilities that occur, so this cannot be a trivial step. How do we get some insight to this problem? Well at the outset we can use maximality to see that if we take any proper subproduct of the $X_i's$, then the remaining terms comprise the full connected centralizer of this subproduct. This simple observation gives us something to work with.

For example, fix i and write $X = X_i X^i$. Let T^i be a maximal torus of X^i. Then $C_G(T^i) = L$ is a Levi subgroup and $X_i \leq L'$. Maximality implies that $T^i = Z(L)^0$. On the other hand, $X^i = C_G(X_i)^0 \geq C_G(L')^0$. Now we know all about centralizers of Levi factors and hence obtain the following lemma.

Lemma 5.4 *Either X^i contains a root subgroup of G or X^i has rank at most 2.*

Actually more is available, since in the second case we have explicit candidates for L. If we know that X^i contains a root group of G we are in a very nice situation. Indeed, we can argue that X^i, actually some X_j, contains a fundamental SL_2 generated by opposite root groups of G. We know the precise composition factors of such an SL_2 on $L(G)$ and from this we can deduce detailed information about the action of X_j on $L(G)$, which provides information about the embedding of X_j and hence its centralizer.

In the other case, X^i has rank at most 2 and we obtain the possibilities for L'. Here we can study the embedding of X_i in L' to obtain further information. Also, we work with other factorizations $X = X_j X^j$. Ultimately this analysis comes down to cases where $r = 2$ with both factors of small rank, usually with one factor, say X_1, of type A_1. If T_1 is a maximal torus of this A_1 factor, then $C_G(T_1)$ is the Levi factor of a maximal parabolic subgroup. We can choose this parabolic so that a T_1-invariant unipotent subgroup of X_1 is contained in the unipotent radical. Using the fact that X_2 centralizes this group we eventually obtain precise information on both simple factors of X.

Now suppose that X is a simple group. The labelled diagram is particularly effective here. There is a computer program that does the following. For each possible labelling and each possible X, it determines the possible composition factors on $L(G)$, allowing for different characteristics.

We have progressed from knowing very little about our maximal subgroup to having a finite number of possibilities for the composition factors on $L(G)$. There are a few situations where the information is not very useful. For example, when $X = A_1$ with $p = 2$ there are hundreds of possibilities and they provide little insight. But for most cases the list is manageable.

The first thing to look for are trivial composition factors. For if we have one, then there is a chance of getting a fixed point, a contradiction. The problem is that in positive characteristic, a trivial composition factor need not give rise to a fixed point because of the presence of certain indecomposable modules. We therefore require precise information on the structure of Weyl modules for certain dominant weights.

In the first pass we reduce to cases where X either has no trivial composition factors or where they exist but there exist other composition factors that could prevent the existence of fixed points. At this point the role of the characteristic is clearly in evidence. In characteristic 0 one automatically has fixed points whenever there is a trivial composition factor, since the relevant modules are all completely reducible. If one assumes mild characteristic restrictions (e.g. $p \neq 2,3$), then many of the most serious problems are avoided. This may help explain the evolution of the results from Dynkin, to Seitz, to Liebeck-Seitz.

We now consider what one can do when faced with the possibility of trivial compositon factors but no fixed points. When this occurs we are typically in small characteristic and it is common to find composition factors of the form V^q, a module twisted by a nontrivial Frobenius morphism. These modules are centralized by $L(X)$, so any such submodule is contained in $A = C_{L(G)}(L(X))$. It is very fortunate that the most troublesome cases for finding fixed points often force $A \neq 0$ and this provides a wedge into handling these cases.

Lemma 5.5 *Let $A = C_{L(G)}(L(X))$.*
i). *A is a subalgebra of $L(G)$.*
ii). *Excluding the cases $X = A_1, B_2, C_3$ with $p = 2$, we have $A \leq L(D)$, where $D = \langle T_G, U_\beta : e_\beta \downarrow T$ is a multiple of $2p\rangle$.*

We can find D explicitly from the labelling of the Dynkin diagram of G determined by T. Moreover, the action of T determines a labelling of the diagram of D. Information on submodules of the form V^q implies the existence of certain weight vectors of A lying in $L(D)$. Such a weight vector centralizes $L(X)$. Now, in certain situations we can choose such a nonzero nilpotent weight vector $n \in A$ and "exponentiate" to get a unipotent element $g \in G$ having exactly the same fixed points as n. Then $1 \neq g \in C_G(L(X))$. However, this contradicts Lemma 1.1(v).

A simple example might be helpful. Say $p = 5$ and $D = A_4$ with all labels $10 = 2p$. If we happened to know that V^q had 40 as T-weight, then we could see that this weight vector must be a root vector of $L(G)$. Indeed, it corresponds to the root of highest height in the system for D. Hence a corresponding root element in G has precisely the same fixed points on $L(G)$.

This turns out to be a powerful method, although the examples that actually occur are considerably more elaborate than the one given and require more care in the exponentiation process.

Employing all the techniques mentioned so far we greatly narrow our seach for maximal subgroups. There are a handful of possibilities that either turn out to be maximal subgroups of G or require a great deal of effort to eventually place them in a larger group. For these cases we usually work hard to determine the Lie algebra of X up to conjugacy in G and then determine the full stabilizer of this algebra. Most of the simple groups listed in the statement of the theorem fall into this category and were constructed by various authors (Testerman, Seitz, and Liebeck-Seitz).

There are a few cases that we have yet to address, the most notable of these is $X = A_1$ with $p = 2$. This is a case that required a somewhat different approach. Eventually it was shown that such a group cannot be maximal, but this was not so easy.

We indicate some of the special considerations that can be used here to again highlight the important role of Lie algebra considerations in small characteristic.

First, consider the structure of $L(X)$. We have $X = A_1$. If $X = SL_2$, then as $p = 2$, we see that $L(X)$ has a nontrivial center and X has a fixed point on $L(G)$, a contradiction. Hence $X = PSL_2$. Here, $I = L(X)' = \langle e, f\rangle$, where $[ef] = 0$.

Lemma 1.1(iv) can be extended slightly to see that $C_G(I) = 1$. We then proceed by first getting a labelled diagram as before and get $e \in L(Q)$, where Q is the Levi subgroup of the corresponding parabolic subgroup (the one with Levi subgroup $C_G(T)$).

Let δ be the root in $\Sigma(G)$ of highest height. Then $[ee_\delta] = 0$. We consider $[fe_\delta]$. If this is also 0, then e_δ centralizes I and we argue that a corresponding root element also centralizes I, a contradiction. So suppose it is nonzero. Then we have

$$[e[fe_\delta]] = [[ef]e_\delta] + [f[ee_\delta]] = 0.$$

We then try to argue that $[f[fe_\delta]] = 0$. A careful analysis shows that $[fe_\delta]$ is usually either a multiple of a root vector or a linear combination of two orthogonal root vectors G and that the commutator is 0. At this point we argue that it is again possible to exponentiate to get a group element centralizing I, a contradiction.

Similar analysis must be carried out in a couple of other situations with $p = 2$, particularly $X = B_2$.

Altogether, it is a long, fascinating, story involving an intricate blend of group theory and representation theory.

6 On the finiteness of double coset spaces

In this section we discuss a problem concerning double cosets in algebraic groups. Throughout G will be a simple algebraic group over an algebraically closed field K of characteristic p.

Main Problem. Determine pairs, X, Y, of closed subgroups of G such that $|X\backslash G/Y|$ is finite.

What makes this problem fascinating is the range of examples coming from group theory and representation theory. We begin the discussion with several types of examples.

Parabolic subgroups. Let B denote a fixed Borel subgroup of G and write $B = UT$ where $U = R_u(B)$ is the unipotent radical of B and T is a maximal torus. The root system of G will be denoted by Σ with Π a fixed base of Σ. We take U to be the product of T-root subgroups corresponding to positive roots.

The fundamental result

$$G = \bigcup_{w \in W} BwB.$$

immediately implies that

$$|P_J \backslash G / P_K| < \infty$$

for all pairs of subsets $J, K \subset \Sigma$. Taking conjugates of P_J, P_K we obtain our first family of configurations.

Finite orbit modules. Another type of example occurs as follows. Given a semisimple group G we consider an irreducible representation of G on a module V and obtain an embedding $G < SL(V)$. Suppose that G has only finitely many orbits on subspaces of dimension k. Then

$$|G \backslash SL(V)/P_k| < \infty$$

where P_k is the stabilizer in $SL(V)$ of a k-space of V, a parabolic subgroup of $SL(V)$.

Proposition 2.7 shows that parabolic subgroups lead to modules for the Levi subgroup on which there are only finitely many orbits on vectors. These are called *internal modules.* We note that here the semisimple part of the Levi subgroup then has finitely many orbits on 1-spaces of these modules. So the internal modules provide a good supply of configurations where we have an irreducible subgroup $X < SL(V)$ such that $|X \backslash SL(V)/P_1| < \infty$.

In the special situation where the unipotent radical is abelian, there is a very nice connection between the number of orbits of L on Q and the number of double cosets of the given parabolic. The following result appears in a paper of Richardson-Röhrle-Steinberg.

Theorem 6.1 [38]*. Assume $P = QL$ is a parabolic subgroup and $Q = R_u(P)$ is abelian. If $p = 2$, assume that the Dynkin diagram is simply laced. Then the number of orbits of L on Q is precisely $|P \backslash G/P|$.*

For example, if $G = E_7$ and P is the parabolic subgroup with Levi factor of type E_6, then Q is abelian and affords a 27 dimensional irreducible module for E_6. Let P_1 be the stabilizer of a 1-space in $SL_{27}(K)$ and regard $E_6(K) < SL_{27}(K)$. We then have $|E_6 \backslash SL_{27}/P_1| = |P \backslash E_7/P| = 4$.

Involution centralizers. In an earlier section we stated the following result of Matsuki [34] and Springer [51]

Theorem 6.2 *Let τ be an involutory automorphism of the simple algebraic group G and assume that the underlying field does not have characteristic 2. Then $|C_G(\tau) \backslash G/B| < \infty$. That is, there are finitely many orbits of K on Borel subgroups.*

Of course Theorem 6.2 implies $|C_G(\tau) \backslash G/P| < \infty$ for all parabolic subgroups P of G. We will prove this result in the next section, but let's first consider a couple of examples. If

$G = SL(V)$ one can choose τ such that $C_G(\tau) = SO(V)$ or $Sp(V)$, the latter when $\dim V$ is even. If the parabolic subgroup of $SL(V)$ is taken as P_k, the stabilizer of a k-space of V, then we are looking at the action of $SO(V)$ or $Sp(V)$ on k-spaces of V. From Witt's theorem it follows that two k-spaces are in the same orbit if and only if, under the bilinear form, their radicals have the same dimension.

Consequently, Theorem 6.2 can be regarded as an extension of Witt's theorem. It provides many interesting examples, even for exceptional groups. For example, consider the case $G = E_8$. Here (assuming $\mathrm{char} K \neq 2$) there is an involution $\tau \in G$ such that $C_G(\tau) = D_8$. We conclude that $|C_G(\tau)\backslash G/B| < \infty$, where B is a Borel subgroup of E_8. We note that there is barely room for a dense orbit of D_8 on G/B, as both have dimension 120.

6.1 Proof of Theorem 6.2

In this section we sketch a proof of the result of Matsuki-Springer result, due to the author.

Form the semidirect product $\hat{G} = G\langle\tau\rangle$ and define an action of $\hat{G}$ on $\mathcal{B} \times \mathcal{B}$, where $\mathcal{B}$ denotes the set of Borel subgroups of G. The action is as follows. For $g \in G, g : (B_1, B_2) \to (B_1^g, B_2^g)$, while the action of τ is given by $\tau : (B_1, B_2) \to (B_2^\tau, B_1^\tau)$.

Fix the Borel subgroup $B < G$. Then a typical element of $\mathcal{B}\times\mathcal{B}$ has the form $(B_1, B_2) = (B^{g_1}, B^{g_2})$, where g_1, g_2 are arbitrary elements of G. Letting $\sim$ denote G-equivalence, we have

$$(B_1, B_2) = (B^{g_1}, B^{g_2}) \sim (B, B^{g_2 g_1^{-1}}) \sim (B, B^{b_1 n_w b_2}),$$

where b_1, b_2 are elements of B and $n_w \in N_G(T)$ corresponds to $w \in W$. We then have

$$(B_1, B_2) \sim (B, B^{n_w b_2}) \sim (B, B^{n_w}) = (B, B^w).$$

Hence we have shown that G has only finitely many orbits on $\mathcal{B} \times \mathcal{B}$.

Consider the fixed points of τ in this action. Notice that τ fixes the pair $(B^g, B^{g\tau})$ for each $g \in G$. Also, τ permutes the G-orbits on $\mathcal{B} \times \mathcal{B}$, fixing any orbit which contains a fixed point.

We will apply the following lemma.

Lemma 6.3 *Let H be an algebraic group over an algebraically closed field of characteristic other than 2. Then H has only finitely many conjugacy classes of involutions.*

Proof If Q is a unipotent group in H normalized by the involution t, then it is well-known that all involutions in Qt are conjugate to t (e.g. Claim 5 p.64 of [49]). Consequently we may assume H^0 is reductive. By 5.16 of [45] each involution of H normalizes a maximal torus so it will suffice to show that $N_H(T)$ contains just finitely many conjugacy classes of involutions. And to see this it suffices to show that $T\langle t\rangle$ has finitely many classes of involutions for all involutions $t \in N_H(T)$. Thus we may now assume $H = T\langle t\rangle$.

Now T is isomorphic to the direct sum of finitely many copies of the multiplicative group of K, so contains only finitely many involutions. Hence we are left with the case where $t \notin T$ and we must show that the coset Tt contains just finitely any classes of involutions.

Let $I = \{x \in T : x^t = x^{-1}\}$. Then I is a closed subgroup of T and I^0 is a torus. Then It is the set of involutions in Tt. Since I^0 has finite index in the abelian group I, it suffices to show that I^0t has finitely many classes of involutions, under the action of I^0. However, if $x \in I^0$, then since I is a torus, there is an element $y \in I^0$ such that $y^2 = x$. As y is also inverted by t we have $t^{y^{-1}} = xt$, completing the argument.

We are now in position to complete the proof. Let $g \in G$. Then τ fixes $(B^g, B^{g\tau})$, so $\tau^{g^{-1}}$ fixes $(B, B^{g\tau g^{-1}})$. The second term of this pair can be rewritten as B^x for some $x \in G$ and writing $x = b'wb$, using the Bruhat decomposition, we find that $\tau^{g^{-1}b^{-1}}$ fixes (B, B^w). Now $D = stab_{\hat{G}}(B, B^w) = (B \cap B^w)\langle\gamma_w\rangle$, where $\gamma_w \in G\tau$.

The lemma shows D, and hence $B \cap B^w$, have only finitely orbits on $\tau^G \cap D$. Say $\tau^{g_{1,w}}, ..., \tau^{g_{k_w,w}}$ are representatives of the latter orbits. Consequently, for some $1 \leq i \leq k_w$ and some $x \in (B \cap B^w) \subseteq B$ we have $\tau^{g^{-1}b^{-1}} = \tau^{g_{i,w}x}$. So there is an element $c \in C_G(\tau)$ such that $g^{-1}b^{-1} = cg_{i,w}x$. It follows that $g \in Bg_{i,w}^{-1}C_G(\tau)$. Since $g \in G$ was arbitrary, this completes the proof.

6.2 The Reductive Case.

In this section we consider configurations of the main problem where the subgroups X and Y are both reductive. We call this the *Reductive Case.* We begin with a result of Brundan whose work was motivated by the following result of Luna for algebraic groups over fields of characteristic 0.

Theorem 6.4 [31] *Assume $charK = 0$ and X, Y are reductive subgroups of G. Then the union of closed (X, Y) double cosets in G contains an open dense subset of G.*

This result has a stunning corollary which highlights the importance of factorizations for the reductive case of the main problem

Corollary 6.5 *Assume $charK = 0$ and that X, Y are reductive subgroups of G with $|X\backslash G/Y| < \infty$. Then $G = XY$.*

Brundan tried to obtain a similar result for algebraic groups in characteristic $p > 0$. He established a variation of the above Corollary and the following is a special case of his result.

Theorem 6.6 *(*[6], [7]) Let X, Y be reductive subgroups of the simple algebraic group G, with each subgroup either maximal connected or a Levi subgroup of a parabolic. If $|X\backslash G/Y| < \infty$, then $G = XY$.

Brundan's work centered on closed double cosets. It follows from the Mumford conjecture that if there are at least two closed (X, Y) double cosets in G, then there is no dense double coset and hence infinitely many double cosets. Thus one wants to find methods of producing closed double cosets. The following lemma of Brundan illustrates where these might come from.

Lemma 6.7 *Let $T_X \leq X$ and $T_Y \leq Y$ be maximal tori and suppose that $T_X \leq T_Y$. Then the double coset XnY is closed for each $n \in N_G(T_Y)$.*

The following corollary settles the case when $X = Y$ is reductive.

Corollary 6.8 *Let X be a proper connected reductive subgroup of G. There does not exist a dense (X, X)-double coset. In particular, $|X\backslash G/X| = \infty$.*

Proof. Lemma 6.7 shows that the double coset XnX is closed for each $n \in N_G(T_X)$ where T_X is a maximal torus of X. So it suffices to show that $N_G(T_X) \not\leq X$. Let T_G be a maximal torus of G containing T_X. If $T_X < T_G$, then there is nothing to prove. If equality holds, then X is a reductive maximal rank subgroup of G and $N_G(T_X)/X$ is the Weyl group of G. Since X is proper and connected in G it is now easy to check that its Weyl group must also be proper.

The results of Brundan show that to obtain complete information on the reductive case one only has to study factorizations. A reasonably complete analysis of factorizations was obtained in [32]. A first lemma, based on the connectedness of G, implies that we may take X, Y to be connected and we shall assume this in the following.

For exceptional groups, factorizations are rare, and the following result from [32] gives all factorizations iinvolving connected subgroups.

Theorem 6.9 [32] *Let $G = XY$ with G of exceptional type and X, Y connected. Then one of the following holds:*

(i). $p = 3$ and $G = G_2 = A_2\tilde{A}_2$.

(ii). $p = 2$ and $G = F_4 = B_4C_4 = D_4C_4 = B_4\tilde{D}_4 = D_4\tilde{D}_4$.

Remark. The groups $A_2 < G_2$ and $D_4 < F_4$ are those subgroups corresponding to the subsystem given by all long roots. In the latter case there is a containment $D_4 < B_4 < F_4$. When $p = 3, 2$, respectively, there is also a subgroup corresponding to the subsystem of short roots and these are the subgroups in the theorem indicated by a tilde. These occur as images of the previous subgroups under a special isogeny of G, which only exists for these characteristics. In the F_4 case there is a containment $\tilde{D}_4 < C_4 < F_4$.

The situation with classical groups is considerably more complicated as there are many more factorizations. The following result provides a complete list of factorizations when the subgroups involved are maximal.

Theorem 6.10 [32] *Suppose G is of classical type and let V be the (irreducible) natural module. Assume $G = XY$ where X, Y are maximal among closed connnected subgroups of G. Then one of the following occurs.*

(1). Parabolic factorizations.

$SL_{2m} = Sp_{2m}P_1$.
$SO_{2m} = N_1P_m = N_1P_{m-1}$.
$SO_8 = B_3P_1 = B_3P_i$, *with* $i = 3$ *or* $i = 4$, $\quad (V \downarrow B_3 = L_{B_3}(\lambda_3))$.
$SO_7 = G_2P_1$.
$Sp_6 = G_2P_1(p = 2)$.

(2). Reductive factorizations, p arbitrary.

$SO_{4m} = (Sp_{2m} \otimes Sp_2)N_1$.
$SO_{16} = B_4N_1, \quad (V \downarrow B_4 = L_{B_4}(\lambda_4))$.
$PSO_8 = B_3B_3^\tau = B_3(Sp_4 \otimes Sp_2), \quad$ *(* $V \downarrow B_3 = L_{B_3}(\lambda_3)$ *and* τ *a triality).*
$SO_7 = G_2N_1$.

(3). Reductive factorizations, small characteristic.

$p = 3, SO_{25} = F_4N_1, \quad (V \downarrow F_4 = L_{F_4}(\lambda_4))$.
$p = 3, SO_{13} = C_3N_1, \quad (V \downarrow C_3 = L_{C_3}(\lambda_2))$.
$p = 2, Sp_{2m} = SO_{2m}N_{2k},\ 1 \leq k \leq m-1$.
$p = 2, SO_{56} = E_7N_1, \quad (V \downarrow E_7 = L_{E_7}(\lambda_7))$.
$p = 2, SO_{32} = D_6N_1, \quad (V \downarrow D_6 = L_{D_6}(\lambda_i), i = 5$ *or* $6.)$
$p = 2, SO_{20} = A_5N_1, \quad (V \downarrow A_5 = L_{A_5}(\lambda_3))$.
$p = 2, Sp_6 = G_2N_2 = G_2SO_6$.

To prove Theorem 6.10 one must first reduce to the configurations listed and then show that these all occur. The first part makes use of work on the maximal subgroups of simple algebraic groups. The existence of the factorizations is in some cases far from obvious.

6.3 The Parabolic Case.

In view of the results of the previous section, for purposes of resolving the maximal configurations of the main problem we may now assume that one of the groups X, Y is a maximal parabolic. Say $Y = P$ is a maximal parabolic subgroup of G. If X is a maximal reductive group, we call this the *Parabolic Case.*

We begin with the case $G = SL(V)$, where V is a finite dimensional vector space over K. If X is maximal and reductive, then X is irreducible on V and the parabolic case occurs if $|X \backslash SL(V)/P_k| < \infty$ for some k. That is, X has finitely many orbits on k-dimensional subspaces of V.

Note that if $\dim V = n$ and this holds for k, then it also holds for $n-k$. For we can identify the k-spaces of V with the $(n-k)$-spaces of V^* and V^* can be obtained from V by an automorphism of X. Hence it suffices to consider $k \leq n/2$.

A complete analysis of such situations, assuming only that X is irreducible rather than maximal, is presented in [16]. The case $k = 1$ is of particular interest. Note that X has finitely many orbits on 1-spaces of V if and only if XK^*, the group obtained by adjoining scalars, has finitely many orbits on vectors. In this situation we call V a *finite orbit module*.

The finite orbit modules for char$K = 0$ were determined by Kac [18] in connection with a study of nilpotent orbits. Of course, if there are finitely many orbits, then there is also a dense orbit, so the problem is closely related to the theory of *prehomogeneous spaces*, studied by Sato and Kimura [50] for fields of characteristic 0 and by Chen [10], [11] in positive characteristic.

Theorem 1 of [16] gives a complete analysis of finite orbit modules. It is shown that such a module is either an internal module or one of several explicit exceptions. The following result which, for simple groups, connects the notions of finite orbit modules with prehomogeneous spaces.

Theorem 6.11 *(*[16], Cor. 1) *Assume X is a simple algebraic group and V is a rational irreducible module for X. Then X has finitely many orbits on $P_1(V)$ if and only if it has a dense orbit on $P_1(V)$.*

We note that there are examples showing that the above corollary fails to hold when X is semisimple, but not simple,

The next result together with the reductive case completes the analysis of the main problem when $G = SL(V)$ and $X < SL(V)$ is reductive and maximal. The results of [16] go well beyond just the maximal configurations, but the statements are a little more complicated.

Theorem 6.12 *(*[16], Thm 3) *Let $dimV = n$ and $G = SL(V)$. Suppose X is maximal reductive and $|X \backslash SL(V)/P_k| < \infty$ for some $1 \le k \le n/2$. Then one of the following holds:*

(i). X is a classical group and V is the natural module for X.

(ii). $X^0 = (GL_{n/r})^r \cap G$, with $r|n$, and either $r \le 3, k$ arbitrary; or $r \ge 4, k = 1$.

(iii). $X^0 = SL_r \otimes SL_s$ with one of $k = 1$; or $k = 2, r \le 3, s$ arbitrary; or $k = 3, r = 2, s \ge 3$ arbitrary.

(iv). X^0, V, k are as follows (up to duals):

$X^0 = A_n, V = L(\lambda_2), k = 1.$

$X^0 = A_n(p \ne 2), V = L(2\lambda_1), k = 1.$

$X^0 = A_n(n = 6, 7), V = L(\lambda_3), k = 1.$

$X^0 = A_3(p = 3), V = L(\lambda_1 + \lambda_2), k = 1.$

$X^0 = D_n(n = 5, 7), V = L(\lambda_n), k = 1.$

$X^0 = A_2(p \ne 2), V = L(2\lambda_1), k = 2.$

$X^0 = A_n(n \le 6), V = L(\lambda_2), k = 2.$

$X^0 = A_4, V = L(\lambda_2), k = 3, 4.$

$X^0 = D_5, V = L(\lambda_5), k = 2, 3.$

$X^0 = E_6, V = L(\lambda_1), k = 1, 2.$

What about other classical groups? One would like analogs of the above results, replacing $SL(V)$ by $Sp(V)$ and $SO(V)$, but such results have yet to be achieved. There are some interesting examples. Consider $G = G_2$ in its action on V, the 14-dimensional adjoint module (assume char$K \neq 3$), regarded as $L(G)$. Let T be a maximal torus of G_2 and consider $L(T)$ a 2-dimensional subspace of V. An argument like that of Proposition 3.3(ii) shows that two 1-spaces in $L(T)$ in the same G-orbit are actually in the same $N_G(T)$-orbit. However, T is trivial on $L(T)$ and $N_G(T)/T = W$ is finite. Consequently, $|G_2\backslash SL(V)/P_1|$ cannot be finite. On the other hand, using the fact that there are only finitely many nilpotent orbits, one can argue that there are only finitely many orbits on singular 1-spaces.

Spherical subgroups and maximal rank subgroups

If X is a reductive subgroup of G with $|X\backslash G/B| < \infty$, then X is said to be *spherical*. The possibilities for X are known in characteristic 0 but the situation is open in positive characteristic. One interesting result here is that sphericality and the existence of a dense double coset with B are equivalent.

Spherical subgroups must have large dimension, at least the codimension of B, so these subgroups are often maximal rank subgroups. A complete classification of spherical subgroups of maximal rank subgroups has been obtained by Duckworth [13].

We conclude this section with the a result of Duckworth [13] which gives considerable insight into the parabolic case when the reductive subgroup involved is a maximal rank subgroup. In this theorem Duckworth assumes that the corresponding subgroup occurs for all groups of the same type, independent of characteristic. So he excludes certain situations where there is a multiple bond in the Dynkin diagram and the characteristic is 2 (also 3 for G_2).

Theorem 6.13 *Let G be a simple algebraic group, X a maximal rank subgroup of G, and P a parabolic subgroup of G. If $G = F_4$, then assume P is not an end node parabolic. If $|X\backslash G/P| < \infty$, then either X is spherical or a Levi subgroup of P is spherical.*

This result has the following nice corollary. Again assume X is a maximal rank subgroup of G which occurs in all characteristics.

Corollary 6.14 *If $|X\backslash G/P| < \infty$ with $P < G$ a nonmaximal parabolic subgroup, then X is spherical. That is, if X has finitely many orbits on the cosets of a nonmaximal parabolic of G, then it has finitely many orbits on the cosets of each parabolic subgroup of G.*

7 Unipotent elements in classical groups

In this section and the next we discuss some current work of Liebeck and the author [30] aimed at obtaining a better understanding of the unipotent classes of simple algebraic groups. In this section we discuss the case of classical groups. While the results here are not new, our approach is somewhat novel and is formulated so as to blend with a similar treatment of unipotent classes in exceptional groups.

Let $G = GL(V), Sp(V), O(V)$ and assume $\mathrm{char}(K) \neq 2$ when $G = Sp(V)$ or $O(V)$. The symplectic and orthogonal groups appear as fixed points of involutory autorphisms of $GL(V)$ and this will play a prominent role in the approach to follow.

Let $u \in GL_n$ be a unipotent element. That is, in some basis u has $1's$ on the main diagonal and $0's$ above the main diagonal. Our goal is to discuss the conjugacy classes of such elements in the groups GL_n, Sp_n, O_n and to determine their centralizers

First consider GL_n where the conjugacy classes correspond to the different types of Jordan forms. That is, if V denotes the natural module, then $V \downarrow u = \bigoplus J_i^{r_i}$, where J_i denotes the i by i matrix with $1's$ on the diagonal and subdiagonal, but $0's$ elsewhere. Up to the order of the terms, this determines the classes in GL_n and we see that the conjugacy classes of unipotent elements corresponds to partitions of n.

The Jordan blocks can be thought of as the building blocks for all unipotent classes in GL_n. Our first lemma concerns the existence of these blocks in the corresponding symplectic and orthogonal groups.

Lemma 7.1 *Let τ be the standard graph automorphism of $G = SL_n$. Let $u = J_n$.*

i) *$G_\tau = Sp_n$ or SO_n, according to whether n is even or odd.*

ii) *$G\langle\tau\rangle$ contains an involutory automorphism of G inducing the inverse transpose map and the fixed points of this map is the orthogonal group.*

iii) *G_τ contains conjugates of u.*

iv) *If n is even, SO_n does not contain conjugates of u.*

Proof The structure of the fixed point group is a standard result on algebraic groups. We will not have time to discuss (ii) other than to say that one can argue that when n is even, the inverse transpose map is conjugate to τ times a certain diagonal element. The graph automorphism sends root elements $U_{\alpha_1}(1)$ to $U_{\alpha_{n-1}}(1)$, etc. If n is even, then it fixes $U_{\alpha_{n/2}}(1)$. Taking a fixed point from each of these orbits and multiplying (in any order) it is readily seen that this gives an element which is conjugate to J_n (try it for $n = 4$). Suppose n is odd. Here the same thing works except it is slightly more complicated as you reach the middle of the Dynkin diagram. So look at the smallest case: $n = 3$. Here one checks that τ fixes $U_{\alpha_1}(1)U_{\alpha_2}(1)U_{\alpha_1+\alpha_2}(1/2)$. The other orbits are as before, so this gives (iii).

(iv) There are a number of ways to establish this. The argument to follow emphasizes the role of the graph automorphism. Assume n is even and let $u \in G_\tau \cong Sp_n$ be a conjugate of J_n as guaranteed in (iii). One calculates that $C_G(u) = U \times Z(G)$, where U is the unipotent group with equal entries on the various subdiagonals. Work in $\bar{G} = G/Z(G)$ and view τ as acting on this quotient. Then $C_{\bar{G}\langle\tau\rangle}(u) = \bar{U}\langle\tau\rangle$. One then argues, sort of a Sylow theorem argument, that there is just one class of complements to $\bar{U}$ and hence only one type of involution can centralize u, namely the conjugates of τ. So this gives (iv).

7.1 Conjugacy classes

In this section we describe the conjugacy classes of unipotent elements, illustrating their connection with Jordan blocks. We begin with an easy reduction of the problem, which works both for classical and exceptional groups. It is the basis of a well-known result of Bala-Carter. We require the following definition. A unipotent element of a simple (or semisimple) group H is *distinguished* if $C_H(u)^0$ is a unipotent group.

Lemma 7.2 *Let G be a simple algebraic group. There exists a bijection between the set of unipotent classes of G and the G-classes of pairs (L, C_L) where L is a Levi subgroup of G and C_L is a distinguished unipotent class in L'.*

Proof Choose u unipotent and let T_0 denote a maximal torus of $C_G(u)$. Then $C_G(T_0)$ is a Levi subgroup of G and of course $u \in L'$, as $L = L'T_0$. Also, it is clear from the setup that u must be distinguished in L'. The correspondence is

$$u^G \to (L, u^{L'})^G = (L, u)^G.$$

The above lemma reduces the problem to finding the distinguished classes of simple (hence semisimple) groups. This is easy for SL (same as for GL). Indeed, if $V \downarrow u = J_i \oplus J_j \oplus \cdots$, then there is a torus of SL in the centralizer. So SL_n has a unique distinguished class, namely J_n.

Our next lemma determines the distinguished unipotent classes in $Sp(V)$ or $O(V)$.

Lemma 7.3 i) *If $u \in Sp(V)$ is distinguished, then $V \downarrow u = J_{n_1} \perp \ldots \perp J_{n_r}$, where the n_i are distinct even integers and the sum is a perpendicular sum. So $u \in Sp_{n_1} \times \cdots \times Sp_{n_r}$.*

(ii) *If $u \in O(V)$ is distinguished, then $V \downarrow u = J_{n_1} \perp \ldots \perp J_{n_r}$, where the n_i are distinct odd integers and the sum is a perpendicular sum. So $u \in O_{n_1} \times \cdots \times O_{n_r}$.*

Proof Write $V \downarrow u = \bigoplus J_i^{r_i} = \bigoplus V_i$, where for each $i, V_i = J_i^{r_i}$. We first claim that i$r_i = 1$ for each i. For suppose $r_i > 1$. We can write $V_i = W_i \otimes X_i$, where dim$W_i = i$ and dim$X_i = r_i$. Also, $SL(V_i) \geq SL(W_i) \cdot SL(X_i)$ and $u \in SL(W_i)$. Hence $C_{SL_n}(u) \geq SL_{r_i}$. Consequently, $C_{SL_n}(u)$ is nonsolvable. Now let τ be an involutory autormorphism of $G = SL_n$ such that G_τ is the desired classical group. Under the assumption that some $r_i > 1$ one argues that $C_G(u)_\tau$ contains a torus. Indeed, to prove this one reduces consideration to reductive groups and here the assertion follows from our discussion of automorphisms of simple groups. Thus u is not distinguished in G_τ, establishing the claim. So write $V \downarrow u = J_{n_1} \oplus \ldots \oplus J_{n_r}$

To complete the proof we must verify the conditions on the n_i and show that the sum is an orthogonal decomposition. For this we start by viewing $u \in GL(V)$ and write $G = GL(V)_\tau$ for an appropriate involutory automorphism. Then $u \in L = (SL_{n_1} \times \ldots \times SL_{n_r})T_0$, a Levi subgroup of SL_n, where T_0 is a maximal torus of $C_{GL(V)}(u)$ and u is distinguished in L'. We can assume T_0 is τ invariant, so that τ leaves invariant the group $C_{GL(V)}(T_0)' = SL_{n_1} \times \cdots \times SL_{n_r}$. Then τ acts on each factor (as they have different dimensions). Consider the action on the natural module. Each factor acts on a single space of the given dimension and the fixed points of τ has the form Sp_{n_i} or SO_{n_i}. This is only possible if this gives an orthogonal decomposition with the fixed points of τ being a sum of symplectic groups or orthogonal groups, according to whether $SL(V)_\tau = Sp(V)$ or $SO(V)$. So all n_i must be even in the symplectic case since u projects to a single Jordan block in each summand, Lemma 7.1(iv) shows that all n_i must be odd in the orthogonal case.

At this point we can describe the unipotent classes of $Sp(V)$ and $O(V)$. We take u unipotent and apply Lemma 7.2. So $u \in L'$, where $L = C_G(T_0)$, a Levi subgroup. For convenience take the symplectic case where $L' = SL_{a_1} \times \cdots \times SL_{a_t} \times Sp_d$. What we now have is that

$$V \downarrow u = (J_{a_1} \oplus J_{a_1}) \perp \cdots \perp (J_{a_t} \oplus J_{a_t}) \perp (J_{d_1} \perp \cdots \perp J_{d_r}),$$

where the d_i are distinct even numbers adding to $2d$. Note that all the odd blocks occur an even number of times. For orthogonal groups, we get just the opposite: the even blocks occur an even number of times.

Theorem 7.4 *Let $G = Sp(V)$ or $O(V)$.*

i) *Two unipotent elements of G are conjugate if and only if they are conjugate in $SL(V)$ (or $GL(V)$).*

ii) $u = \bigoplus J_i^{r_i}$ *is in a symplectic group if and only if r_i is even for all odd i.*

iii) $u = \bigoplus J_i^{r_i}$ *is in an orthogonal group if and only if r_i is even for all even i.*

7.2 Centralizers

In this section we discuss centralizers of unipotent elements. Our first goal will be to determine the dimensions of the centralizers. We first note that if u is a unipotent element of $GL(V)$, then $\dim C_{GL(V)}(u) = \dim C_{M_n(K)}(u)$. This follows from the fact that $GL(V)$ is open dense in $M_n(K)$ and so the same holds by intersecting with centralizers. So to find the dimension of the centralizer we need only count fixed points of u in its action on the linear space $M_n(K)$.

Now $M_n(K)$ is isomorphic to the Lie algebra of $GL(V)$ and it is a general result that if G is a simple algebraic group where the defining field has good characteristic, then $\dim C_G(u) = \dim C_{L(G)}(u)$. With this in mind we state a standard lemma describing the Lie algebras of the classical groups.

Lemma 7.5 i). $L(GL(V)) \cong V \otimes V^*$.

ii) $L(Sp(V)) \cong S^2V$

iii) $L(O(V)) \cong \wedge^2 V$

The following result gives the dimensions of centralizers of unipotent elements in classical groups.

Proposition 7.6 *Let $u = \bigoplus J_i^{r_i}$ be in the classical group $G = GL(V), Sp(V), O(V)$.*

i) $\dim C_{GL_n(K)}(u) = \sum_i ir_i^2 + 2\sum_{i<j} ir_ir_j$,

ii) $\dim C_{Sp_n(K)}(u) = \frac{1}{2}\sum_i ir_i^2 + \sum_{i<j} ir_ir_j + \frac{1}{2}\sum_{i\ odd} r_i$,

iii) $\dim C_{O_n(K)}(u) = \frac{1}{2}\sum_i ir_i^2 + \sum_{i<j} ir_ir_j - \frac{1}{2}\sum_{i\ odd} r_i$.

Proof Assume $G = GL_(V)$. Then $L(G) \cong V \otimes V^*$. Then as above, $\dim C_G(u) = \dim C_{L(G)}(u) = \dim C_{V\otimes V^*}(u)$ So we must count the fixed points of u on $V \otimes V^*$. But

$$V \otimes V^* \downarrow u = V \otimes V \downarrow u = (\bigoplus_i (J_i \otimes J_i)^{r_i^2}) \oplus (\bigoplus_{i\neq j} (J_i \otimes J_j)^{r_ir_j}).$$

So we only need to find the fixed points of u on spaces such as $J_i \otimes J_j$. But

$$\dim Hom_u(K, J_i \otimes J_j) = \dim Hom_u(J_i, J_j) = min\{i, j\}$$

To see the last equality we can assume $i < j$. Any homomorphism must send J_i to a Jordan subblock of J_j of size at most i. A moments thought shows that there are precisely i possibilities. (i) follows.

A similar analysis leads to (ii) and (iii). In view of the above lemma one must work with $S^2(V)$ or $\wedge^2(V)$ according to whether $G = Sp(V)$ or $O(V)$. To carry out a calculation as for $GL(V)$ only ultimately requires information such as $\dim Hom_u(K, S^2(V))$ and $\dim Hom_u(K, \wedge^2(V))$. This can be accomplished by first computing the centralizer of a single Jordan block in the corresponding linear group and then counting fixed points under graph automorphisms.

We next discuss the structure of the centralizers. To do this exploit a connection between the unipotent elements of a classical group G and the nilpotent elements of its Lie algebra, $L(G)$. Namely, consider the following correspondences.

$G = GL: \quad u \to u - 1$

$G = Sp, O: \quad u \to (1 - u)/(1 + u)$

Lemma 7.7 i) *The above maps give G–equivariant maps between the set of unipotent elements of G and the set of nilpotent elements of $L(G)$.*

ii) *If u and e correspond under the above maps, then* $\dim(C_G(u)) = \dim(C_{L(G)}(u)) = \dim(C_G(e)) = \dim(C_{L(G)}(e))$.

We next introduce a certain torus. Let's begin with a single Jordan block, $u = J_r$, set $e = u - 1$ and let T be the torus where $T(c) = \mathrm{diag}(c^{r-1}, c^{r-3}, \ldots, c^{-(r-3)}, c^{-(r-1)})$. Then $T(c)(e) = T(c^{-1})eT(c) = c^2e$. More generally, we have this whenever u is distinguished in a Levi subgroup, by working in each of the simple factors and then taking the diagonal torus. Using this it follows that for each unipotent element in a classical group, u and its nilpotent correspondent $e \in L(G)$, we have a 1-dimensional torus with $T(c)(e) = c^2 e$ for each $c \in K^*$. This is because T can be taken to Lie in the relevant classical group. Note that T normalizes $C_G(e)$, since all nonzero multiples of e have the same centralizer in G.

We remark that such a torus can be obtained for all simple algebraic groups defined over fields of good characteristic. We then obtain the following key result.

Proposition 7.8 *Given u and e as above. T can be chosen such that $C_G(u) = C_G(e) = V(C_G(T) \cap C_G(e))$, where $V = R_u(C_G(e))$.*

The proof of the above proposition is not too difficult, but we will not discuss it here. The idea is to observe that T acts on $C_G(e)$ and with proper choice T it centralizes $C_G(e)$ modulo its unipotent radical. Then a Frattini argument gives the result.

In fact one can show that the product in the Proposition is actually a semidirect product, so that $C_G(T) \cap C_G(e)$ is a complement to V and thus is the reductive part of the centralizer. We will see a sample of this in the following.

Consider $G = GL(V)$. Let $u \in G$ be given such that $V \downarrow u = \bigoplus J_i^{r_i} = \bigoplus V_i$. For each i, write $V_i = W_i \otimes X_i$, where $u \downarrow V_i = J_i \otimes 1$. Set

$$J = \Pi_i SL(W_i) = \Pi_i SL_i \qquad R = \Pi_i GL(X_i) = \Pi_i GL_{r_i}.$$

Then $u \in J, T < J, e \in L(J)$, and $R < C_G(J) < C_G(u)$.

Let's calculate the dimension of $C_G(e) \cap C_G(T)$. One checks that

$$\dim(Hom_{T,e}(K, J_i \otimes J_j)) = \dim(Hom_{T,e}(J_i, J_j)) = \delta_{ij}$$

and hence

$$\dim(C_G(T) \cap C_G(e)) = \Sigma_i r_i^2.$$

On the other hand, we have $R = \Pi_i GL_{r_i}$ centralizes J and hence is contained in $C_G(e) \cap C_G(T)$. So this shows that $(C_G(T) \cap C_G(e))^0 = \Pi_i GL_{r_i}$. An additional argument shows that the connected component sign is not needed here and this gives the precise structure of $C_G(u)$.

To complete the discussion we indicate the necessary changes required for the symplectic and orthogonal groups. Take $u \in G = Sp(V), O(V)$. Write $V \downarrow u = \bigoplus J_i^{r_i} = \bigoplus V_i$, a perpendicular sum, where $V_i = J_i^{r_i}$. Now we observe the following facts:

$G = Sp(V)$:

For i even, $Sp(V_i) \geq Sp_i O_{r_i}$

For i odd, $Sp(V_i) \geq O_i Sp_{r_i}$

G = O(V):

For i even, $O(V_i) \geq Sp_i Sp_{r_i}$
For i odd, $O(V_i) \geq O_i O_{r_i}$.

The embeddings are obtained using a product form. Using the above techinques we obtain the following theorem. A key idea of the proof is to obtain information about Sp_n and O_n by starting from GL_n and then taking fixed points under appropriate graph automorphisms.

Theorem 2 *Let $G = GL_n(K)$, $Sp_n(K)$ or $O_n(K)$, where K is an algebraically closed field. Assume* $\mathrm{char}(K) \neq 2$ *when G is symplectic or orthogonal. Let $u = \bigoplus_i J_i^{r_i} \in G$ be a unipotent element.*

(i) *Two unipotent elements of G are G-conjugate if and only if they are $GL_n(K)$-conjugate (i.e. they have the same Jordan form).*

(ii) *If $G = Sp_n(K)$, then r_i is even for each odd i; and if $G = O_n(K)$, then r_i is even for each even i.*

(iii) *We have*

$$\begin{aligned}
&\dim C_{GL_n(K)}(u) = \textstyle\sum_i i r_i^2 + 2\sum_{i<j} i r_i r_j, \\
&\dim C_{Sp_n(K)}(u) = \tfrac{1}{2}\textstyle\sum_i i r_i^2 + \sum_{i<j} i r_i r_j + \tfrac{1}{2}\sum_{i\ odd} r_i, \\
&\dim C_{O_n(K)}(u) = \tfrac{1}{2}\textstyle\sum_i i r_i^2 + \sum_{i<j} i r_i r_j - \tfrac{1}{2}\sum_{i\ odd} r_i.
\end{aligned}$$

(iv) *We have $C_G(u) = VR$, where $V = R_u(C_G(u))$ and*

$$\begin{aligned}
&R = \textstyle\prod GL_{r_i}, \text{ if } G = GL_n(K), \\
&R = \textstyle\prod_{i\ odd} Sp_{r_i} \times \prod_{i\ even} O_{r_i}, \text{ if } G = Sp_n(K), \\
&R = \textstyle\prod_{i\ odd} O_{r_i} \times \prod_{i\ even} Sp_{r_i}, \text{ if } G = O_n(K).
\end{aligned}$$

(v) *Write $C = C_G(u)$. Then $C/C^0 = (Z_2)^k$, where*

$$\begin{aligned}
&k = 0, \text{ if } G = GL_n(K), \\
&k = |\{i : i \text{ even}, r_i > 0\}|, \text{ if } G = Sp_n(K), \\
&k = |\{i : i \text{ odd}, r_i > 0\}|, \text{ if } G = O_n(K).
\end{aligned}$$

8 Unipotent classes in exceptional groups

In this section we are concerned with the unipotent elements in simple algebraic groups of exceptional type. The goal is to describe the conjugacy classes and obtain precise information on centralizers of unipotent elements. There do exist papers in the literature providing complete results, but the results are spread over several papers, using different notation and methods. Moreover, some of the papers, particular the ones dealing with the groups E_6, E_7, E_8 are not in satisfactory shape. For these reasons a number of authors (Liebeck-Seitz, Lawther-Testerman) have undertaken revisions of the material. Here we report on the approach of Liebeck-Seitz. To date the analysis has been carried out for the exceptional groups in characteristic $p \neq 2, 3$. We hope to work through the cases $p = 2, 3$ using a similar approach.

Let X be a simple algebraic group defined over an algebraically closed field of characteristic p. We say p is **good** for X provided $p \neq 2$ if X has type $B_n, C_n, D_n, p \neq 2, 3$ if X is of exceptional type, and $p \neq 5$ if $X = E_8$.

In this section we describe the approach for G an exceptional group over a closed field of characteristic p, a good prime for G.

8.1 Conjugacy classes

As in the case of classical groups we describe conjugacy classes using a method of Bala-Carter which is based on a result of Richardson. Recall that a unipotent element of a simple (or semisimple) group H is *distinguished* if $C_H(u)^0$ is a unipotent group. Here again is the result from last time.

Lemma 8.1 *Let G be a simple algebraic group. There exists a bijection between the set of unipotent classes of G and the G-classes of pairs (L, C_L) where L is a Levi subgroup of G and C_L is a distinguished unipotent class in L'.*

The next step is to analyze the distinguished classes. We need a definition. A parabolic subgroup $P = QL$ of a semisimple group is *distinguished* if $\dim L = \dim(Q/Q')$. For example, a Borel subgroup is distinguished and in some cases (e.g. A_n) these are the only distinguished parabolic subgroups.

Proposition 8.2 *Let u be a distinguished unipotent element in a semisimple group H defined over a field of good characteristic. Then there is a distinguished parabolic subgroup, $P = QL$ of H such that $u \in Q$ and u^P is dense in Q. Moreover, in this case, $C_H(u)^0 = C_Q(u)^0$ and has dimension equal to* $\dim(L)$.

Proof Here is a sketch of a proof in characteristic 0. Using exponentiation techniques we can embed u in a subgroup $X = A_1$ of G. Then $L(G) \downarrow X$ is a direct sum of irreducible restricted modules of type $V(n)$. Now u is contained in a unique Borel subgroup of X with maximal torus T. Let e be a generator of the Lie algebra of the corresponding unipotent group, so that $T(c)e = c^2 e$ for all nonzero scalars c.

Now $C = C_{L(G)}(X)$ is the sum of all trivial modules. One can argue that this module is nondegenerate under the Killing form and from here conclude that $C_G(X)$ is a reductive group of dimension equal to that of C. However, u is distinguished, so $C = 0$. So each of the irreducible summands has the form $V(n)$ with $n > 0$. It then follows that $ad(e) : L(G)_0 \to L(G)_2$ is a bijective map, where the subscripts refer to weights of T.

Consider the parabolic subgroup $P = QL$ determined by T. That is, $L = C_G(T)$ and T acts on Q acting by positive weights on the quotients of the derived series. In particular $L(L) = L(G)_0 = L(P)_0$. Recall Richardson's result showing that P has a dense orbit on Q.

It follows that L has a dense orbit on Q/Q'. At the Lie algebra level, there is an element $l \in L(Q)$ such that $[L(P), l] = L(Q)$.

We use an argument of Jantzen to show that all weights of T are even. Write $l = \Sigma_{i>0} l_i$ for $l_i \in L(Q)_i$. Now intersect the equality of the last paragraph with $L(P)_1 + L(P)_2$. We have $[L(P)_0, l_1 + l_2] + [L(P)_1, l_1] = L(P)_1 + L(P)_2$. Suppose $L(P)_1 \neq 0$. Then as $l_1 \in L(P)_1$ we have $\dim([L(P)_1, l_1] < \dim(L(P)_1)$.. But then $\dim([L(P)_0, l_1 + l_2]) + \dim([L(P)_1, l_1]) < \dim(L(P)_0) + \dim(L(P)_1 = \dim(L(P)_2) + \dim(L(P)_1)$, a contradiction

It follows that all weights of T are even, so that $X \cong PSL_2$. At this point we have shown that P is a distinguished parabolic and from the above bijection we see that e is in the dense orbit of L on $L(Q/Q'))$. A little extra work shows that x is in the Richardson orbit on Q.

Corollary 8.3 *The unipotent classes in G are in bijective correspondence with the classes of distinguished parabolic subgroups of Levi factors.*

It is easy to find the possible Levi factors and it is not difficult to find the distinguished parabolic subgroups of Levi factors. The Borel subgroup is always distinguished and this corresponds to the regular class of the Levi. For this class the centralizer (in the Levi) has dimension precisely equal to the semisimple rank. Now we already have lots of information on unipotents in classical groups, so this provides information on the exceptional groups as well. But ultimately we need to know the distinguished classes and parabolics of exceptional groups. There are usually several of these. For example, in type E the parabolic with labelling all $2's$ except for a 0 over the triality node is a parabolic.

So far we have a description of the conjugacy classes of unipotent elements. But how do we use this to find the centralizers? It turns out that we need some additional information of the sort we talked about when discussing classical groups.

The next lemma is an extension of the correspondence between unipotent and nilpotent classes that we used for classical groups.

Lemma 8.4 i) *There exists a G–equivariant map (called a Springer map) between the set of unipotent elements of G and the set of nilpotent elements of $L(G)$.*

ii) *If u and e correspond under the above maps, then* $\dim(C_G(u)) = \dim(C_{L(G)}(u)) = \dim(C_G(e)) = \dim((C_{L(G)}(e))$.

The advantage of working with e is that we can associate a certain 1-dimensional torus which stabilizes the 1-space e generates and hence acts on $C_G(e)$.

Lemma 8.5 *If u is a distinguished unipotent element of the Levi subgroup L of G, then there is a nilpotent element e which corresponds to u under a Springer map and a 1-dimensional torus $T < L'$, such that $T(c)e = c^2 e$, for each nonzero c in the base field. Inparticular, T normalizes $C_G(u) = C_G(e)$.*

Here is the result we stated for classical groups, which holds for any simple algebraic group.

Proposition 8.6 *Given u, e, T as in the above lemma . Then $C_G(u) = C_G(e) = V(C_G(T) \cap C_G(e))$, where $V = R_u(C_G(e))$.*

Proof We will not go into much detail of the proof. However, in view of the importance of this result we provide a sketch. We know that T acts on $C = C_G(e)$ and so T acts on C/V where $V = R_u(C)$. Now C/V is a reductive group and the outer automorphism group of a reductive group is finite. Hence, there is 1-dimensional torus, say $Z < CT$, such that Z centralizes C/V. Using the fact that T lies in L', we argue that T centralizes C/V. Then VT is a normal subgroup of CT. A Fratttini argument shows that This shows that $C = V(N_C(T))$. However, T acts on C and is not contained in C, so $C = V(C_C(T)) = V(C_G(T) \cap C_G(e))$, as required.

8.2 Dual pairs of reductive subgrops

The above proposition shows us how to find the reductive part of the centralizer. Namely, we have to find T and then compute $C_G(T) \cap C_G(e) = C_G(T) \cap C_G(u))$. The trouble is, that this intersection is not particularly convenient. If we knew T precisely, then it would not be difficult to compute $C_G(T)$. Indeed, we could just find the labelled diagram corresponding to T and go from there. However, T and u do not commute, so u does not act on $C_G(T)$. So this is not the approach we will use.

We will work with certain reductive subgroups of G. The idea is to somehow determine a pair of reductive groups J and R with the following properties:

$$R = C_G(J)$$

$$J = C_G(R)$$

$$u \in J$$

$$T \leq J$$

$$C_G(T) \cap C_G(u) = R.$$

That is, we look for a particularly nice pair of reductive groups which are dual with respect to taking centralizers. If L is a Levi in which u is distinguished, then we could just set $J = L'$ and $R = C_G(L) = Z(L)$. In some cases this pair will satisfy the first four conditions, but it rarely satisfies the last condition, which is the key one. Indeed, the above proposition shows that R is a good candidate for the reductive part of $C_G(e) = C_G(u)$. So we have to be very careful in our choice.

So how do we go about finding a good pair of subgroups? Starting with the Levi L is a good idea. Then look at $N = N_G(L')$. This group acts on L'. The induced group involves inner automorphisms (from L') and finite group or outer automorphisms. We argue that the class of u in L' is stablilized under this action. So $C_N(u)$ covers the finite group. What we now do is choose a semisimple subgroup $J \le L'$ such that $u \in J$ and $C_N(J)$ also covers the finite group.

Example: The best way to understand this procedure is to work through an example. Say for example, that we are in $G = E_8$ and $L = D_7T_1$. We know how to find the distinguished classes in D_7. If we think in terms of the classical group O_{14}, then we simply choose an element with Jordan blocks which are of distinct odd size. So, let's take the class of u to correspond to J_{13}. This is the regular class and the unipotent element lies in a subgroup $B_6 < D_7$. This will be our group J. Notice, also that in the full orthogonal group J centralizes an involution acting as $1^{13}(-1^1)$. The full outer automorphism group of D_7 has order 2 and the long word in the Weyl group of E_8 does act on our Levi. So we conclude that the centralizer of J does cover the group of outer automorphisms.

The next issue to locate R. Of course, $Z(L) = T_1$ contained in the centralizer. We can do better, but before we do so, let's make one observation which will work in general. We claim that $C_G(J)$ has rank equal to $\dim Z(L)$. Certainly $Z(L)$ is a torus contained in $C_G(J)$. If it was properly contained in a maximal torus of $C_G(J)$, then this maximal torus would lie in $C_G(Z(L)) = L$ and centralize u. However, u is distinguished in L', a contradiction. So know the rank of $C_G(J)$.

Now consider the subgroup $D_8 < E_8$ (recall the extended diagram). This group certainly contains a subgroup B_6B_1. It is easy to argue that the first factor is conjugate to J, so we see that $C_G(J) \ge B_1$. Note also, that this group does contain an involution which normalizes T_1 and acts on $C_G(T_1) = L = D_7T_1$. So this accounts for the extra automorphism of D_7.

Set $R = B_1$. So far, so good. How do we choose T? Well, just as we did for classical groups. It is a diagonal group in $B_6 < D_7$ with weights $12, 10, \ldots, , -10, -12$ on the natural module. We now want to verify some of the above points. The approach will be to study the action of $JR = B_6B_1$ on $L(G)$. We start with D_8.

$$L(G) \downarrow D_8 = L(D_8) \oplus V_{D_8}(\lambda_8)$$

The last summand is one of the spin modules for D_8. Now we want to restrict to B_6B_1. The first summand is no problem, once we identify the Lie algebra of D_8 with the wedge square of the natural module. We get

$$L(D_8) \downarrow B_6B_1 = L(B_6) \oplus L(B_1) \oplus (V_{B_6}(\lambda_1) \otimes V_{B_1}(2\lambda_1)$$

Also, it is known that the spin module restricts to our subgroup as the tensor product of the spin modules for the factors.

$$V_{D_8}(\lambda_8) \downarrow B_6B_1 = V_{B_6}(\lambda_6) \otimes V_{B_1}(\lambda_1).$$

What do we see from here? Well, first we see that $C_{L(G)}(B_6) = L(B_1)$ and that $C_{L(G)}(B_1) = L(B_6)$. So this shows that $C_G(J)^0 = R$ and $C_G(R)^0 = J$.

Now let's try to compute $C_{L(G)}(T) \cap C_{L(G)}(e)$. One can prove a lemma showing that there is a decomposition of the Lie algebra into Jordan blocks for e which is compatible with the weights of T. That this, there is basis for each Jordan block can be chosen so that T has weights $s, s-2, \ldots, -(s-2), -s$ on the block and e sends the vector of weight r to the weight $r+2$, In particular, e fixes the vector of weight s. Given such a basis, then the intersection we are after is precisely the set of blocks of size 1.

How does this play out in the above example? Well, start with the spin module for B_6. Expressing λ_6 in terms of fundamental roots and noticing that T affords weight 2 on all fundamental roots of B_6, one checks that high weight vector affords weight 21. All other weights are obtained by subtracting roots. It follows that all weights of T on the spin module are odd. Hence T has no fixed points on this spin module and hence none on $V_{D_8}(\lambda_8)$. This was lucky - we do not even have to work out the Jordan blocks of e in this case. However, we will eventually want to have this, so I will just say that e has 5 Jordan blocks and the corresponding fixed points have high weights $21, 15, 11, 9, 3$. Note that there are total of ten fixed points of T, e on $V_{E_8}(\lambda_8)$ because of the other tensor factor. What about the other summands of $L(E_8)$? Well, $T < B_6$ and $e \in L(B_6)$, so both centralize $L(B_1)$. Also, $V_{B_6}(\lambda_1)$ is a single Jordan block of length 13, so we get no fixed points from this summand. So we need only consider $L(B_6)$. But e (also u) is regular in B_6, so it's centralizer (modulo center) is unipotent of rank 6. On the other, hand T is diagonal with distinct eigenvalues on the natural module, so $C_{B_6}(T)$ is a maximal torus of B_6. So $C_{B_6}(T) \cap C_{B_6}(e) = Z(B_6)$. Putting this altogether we see that

$$C_{L(G)}(T) \cap C_{L(G)}(e) = L(B_1).$$

It follows that $(C_G(T) \cap C_G(e))^0 = B_1$. Now we can argue that $N_G(B_1) = B_6 B_1$. This implies that we have found the reductive part of $C_G(e) = C_G(u)$.

Noice that we can also count the total number of fixed points of e on $L(G)$. There are 6 from $L(B_6), 3$ from $L(B_1), 3$ from the tensor product of the two orthogonal module, and 10 from trom $V_{D_8}(\lambda_8)$. Hence,

$$\dim(C_G(u)) = 22$$

and

$$C_G(u) = U_{19} B_1$$

This approach works in all cases, although there are some tricky points. For example, in some of the distinguished cases we have to work to find the component group. In particlar, there is a case where $R = Sym_5$ that takes some effort.

We remark that once we carry out the analysis for E_8, we get the results for other groups as well. This is because the other groups occur as centralizers of certain nice groups in E_8. For example $E_7 = C_{E_8}(A_1)$ and $E_6 = C_{E_8}(A_2)$.

Additional information is available from the above approach. In particular, one can get the precise action of R on successive factors of a certain filtration of V (the filtration is defined by the weights of T).

Centers of centralizers. If u is unipotent element then we know that $C_G(u) = C_G(e)$ for a suitable nilpotent element of $L(G)$. Now, e and ce have precisely the same centralizer in G for all nonzero scalars c. And each of these multiplies of e correspond to unipotent elements in G. It follows that $Z = Z(C_G(u))$ has positive dimension. The question is, what is this dimension? The above approach does yield the following

Lemma 8.7 *With notation as above, $Z \leq Z(C_J(u))$.*

Proof This is easy. Indeed, $Z = Z(C_G(u))$ and R is the reductive part of $C_G(e)$. Hence, $Z \leq C_G(R) = J$. Also, $C_J(u) \leq C_G(u)$, so Z is central in this group.

The lemma suggests that one can obtain information on Z by shifting attention to J, a smaller group. However, the containment in the lemma is not always an equality, so this is not the whole story. Additional information is available using the action of R as described above, but at this writing there remain some questions to resolve. We note that in unpublished work Lawther-Testerman have determined the dimension of the center of the connected center of $C_G(u)$.

Bad characteristic. Perhaps the most important issue to resolve is how to best deal with the bad primes. Several parts of the above analysis break down. For example, the Springer map is no longer available. Indeed, the number of nilpotent and unipotent classes can differ. Also, the Bala-Carter theorem breaks down. At the same time much is the same. Many of the same techniques can be applied, but one must allow for a few additional classes. The case of classical groups in characteristic 2 is already nontrivial.

References

[1] H.H. Andersen, J. Jorgensen and P. Landrock, "The projective indecomposable modules of $SL(2, p^n)$", *Proc. London Math. Soc.* **46** (1983), 495–528.

[2] M. Aschbacher, " On the maximal subgroups of the finite classical groups", Invent. Math., **76**, 469-514, (1984).

[3] M. Aschbacher and G.M. Seitz, "Involutions in Chevalley groups over fields of even order", *Nagoya Math. J.* **63** (1976), 1–91.

[4] H. Azad, M. Barry and G.M. Seitz, "On the structure of parabolic subgroups", *Comm. in Alg.* **18** (1990), 551-562.

[5] A. Borel, *Linear Algebraic Groups*, Springer-Verlag, 1991.

[6] J. Brundan, Double coset density in classical algebraic groups, (to appear Trans AMS).

[7] J. Brundan, Double coset density in exceptional algebraic groups, (to appear J. London Math. Soc.).

[8] J. Brundan, Dense orbits and double cosets, (preprint).

[9] R.W. Carter, *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*, John Wiley, London (1985).

[10] Z. Chen, "A classification of irreducible prehomogeneous vector spaces over an algebraically closed field of characteristic 2, I", Acta Math. Sinica **2** (1986), 168-177.

[11] Z. Chen, "A classification of irreducible prehomogeneous vector spaces over an algebraically closed field of characteristic p, II", Chinese Ann. Math. Ser. A **9** (1988), 10-22.

[12] A.M. Cohen, M.W. Liebeck, J. Saxl and G.M. Seitz, "The local maximal subgroups of exceptional groups of Lie type, finite and algebraic", *Proc. London Math. Soc.* **64** (1992), 21-48.

[13] W. E. Duckworth, "Infiniteness of double coset collections in algebraic groups", J. Alg., **273**, (2004), 718-733.

[14] E.B. Dynkin, "Maximal subgroups of the classical groups, Amer. Math. Soc. Translations, **6**, (1957), 245-378.

[15] E.B. Dynkin, "Semisimple subalgebras of semisimple Lie algebras", *Translations Amer. Math. Soc.* **6** (1957), 111-244.

[16] R. Guralnick, M. Liebeck, D. Macpherson, and G. Seitz, Modules for algebraic groups with finitely many orbits on subspaces, J. Algebra, **196**, (1997), 211-250.

[17] J.E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, New York, 1972.

[18] V. Kac, "Some remarks on nilpotent orbits", J. Alg., 64, (1980), 190-213.

[19] P.B. Kleidman and M.W. Liebeck, *The Subgroup Structure of Finite Classical Groups*, Cambridge University Press, 1990.

[20] R. Lawther and D. M. Testerman, "A_1 subgroups in exceptional algebraic groups", *Mem. Amer. Math. Soc.*, ???????

[21] R. Lawther and D. M. Testerman, (in preparation).

[22] M. W. Liebeck, "Subgroups of simple algebraic groups and of related finite and locally finite groups of Lie type", in *Finite and Locally Finite Groups*, (eds, B. Hartley, G. Seitz, A. Borovik, and R. Bryant) , Kluwer, NATO ASI Vol 471, (1995).

[23] M.W. Liebeck, J. Saxl and G.M. Seitz, "Subgroups of maximal rank in finite exceptional groups of Lie type", *Proc. London Math. Soc.* **65** (1992), 297-325.

[24] M.W. Liebeck and G.M. Seitz, "Maximal subgroups of exceptional groups of Lie type, finite and algebraic", *Geom. Dedicata* **36** (1990), 353-387.

[25] M.W. Liebeck and G.M. Seitz, "Reductive subgroups of exceptional algebraic groups", *Mem. Amer. Math. Soc.*, Vol. 121, No. 580, 1996.

[26] M.W. Liebeck and G.M. Seitz, "The maximal subgroups of positive dimension in exceptional algebraic groups", *Mem. Amer. Math. Soc.*, Vol. 169, No. 802, 2004.

[27] M.W. Liebeck and G.M. Seitz, "On the subgroup structure of exceptional groups of Lie type", *Trans. Amer. Math. Soc.* **350** (1998), 3409-3482.

[28] M.W. Liebeck and G.M. Seitz, "On the subgroup structure of classical groups", *Invent. Math.*, **134** (1998), 427-453.

[29] M.W. Liebeck and G.M. Seitz, "On finite subgroups of exceptional algebraic groups", *J. reine angew. Math.* **515** (1999), 25-72.

[30] M.W. Liebeck and G. M. Seitz. " Unipotent elements in simple algebraic groups", (in preparation).

[31] D. Luna, "Sur les orbites fermées des groupes algébrique réductifs", Invent. Math.,16, 1972, 1-5.

[32] M. Liebeck, J. Saxl, and G. Seitz, "Factorizations of algebraic groups", Trans. AMS, 348, (1996), 799-822.

[33] G. Lusztig, "On the finiteness of the number of unipotent classes", Invent. Math., 34, (1976), 201-213.

[34] T. Matsuki, "The orbits of affine symmetric spaces under the action of minimal parabolic subgroups", J. Math. Soc. Japan, 31, (1979), 331-357.

[35] K. Mizuno, "The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8", *Tokyo J. Math.* **3** (1980), 391–461.

[36] R. Richardson, R., "Finiteness theorems for orbits of algebraic groups", Indag. Math., 88, (1985), 337-344.

[37] R. Richardson, "On orbits of Lie groups and algebraic groups", Bull. Australian Math Soc., 25, (1982), 1-28.

[38] R. Richardson, G. Röhrle, and R. Steinberg, "Parabolic subgroups with abelian unipotent radical", Invent. Math., 110, (1992), 649-671.

[39] G. M. Seitz, "Algebraic groups", in *Finite and Locally Finite Groups*, (eds, B. Hartley, G. Seitz, A. Borovik, and R. Bryant) , Kluwer, NATO ASI Vol 471, (1995).

[40] G. M. Seitz, "Cross-Characteristic embeddings of finite groups of Lie type", Proc. London Math. Soc, (3) 50, (1990), 166-200.

[41] G.M. Seitz, "The maximal subgroups of classical algebraic groups", *Mem. Amer. Math. Soc.* Vol. 67, No. 365, 1987.

[42] G.M. Seitz, "Maximal subgroups of exceptional algebraic groups", *Mem. Amer. Math. Soc.* Vol. 90, No. 441, 1991.

[43] G.M. Seitz, "Unipotent elements, tilting modules, and saturation", *Invent. Math.* **141** (2000), 467-502.

[44] G. M. Seitz and A.E. Zalesskii, "On the minimal degrees of projective representations of the finite Chevalley groups, II, J. Alg., **158**, (1993), 233-243.

[45] T.A. Springer and R. Steinberg, "Conjugacy classes", in *Seminar on Algebraic Groups and Related Finite Groups* (eds. A. Borel et al.), Lecture Notes in Math. No. 131, Springer (1970), pp. 168–266.

[46] R. Steinberg, "Lectures on Chevalley Groups," Yale University, 1967

[47] D.M. Testerman, "Irreducible subgroups of exceptional algebraic groups", *Mem. Amer. Math. Soc.*, No. 390 (1988).

[48] D.M. Testerman, "A construction of certain maximal subgroups of the algebraic groups E_6 and F_4", *J. Algebra* **122** (1989), 299-322.

[49] D.M. Testerman, "A_1-type overgroups of elements of order p in semisimple algebraic groups and the associated finite groups", *J. Algebra*, **177** (1995), 34-76.

[50] M. Sato and T. Kimura, " A classification of irreducible prehomogeneous vector spaces and their relative invariants", Nagoya Math. J., 65 (1977), 1-155.

[51] T.A. Springer, "Some results on algebraic groups with involution, Advanced Studies in Pure Mathematics", 6, 1985, 525-543.

[52] D. Testerman, "A_1-type overgroups of elements of order p in semisimple algebraic groups and the associated finite groups", J. Algebra, 177, (1995), 34-76.