Large Subalgebras and the Structure of Crossed Products, Lecture 4: Large Subalgebras in Crossed Products by $\mathbb{Z}$

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- Lecture 1 (1 June 2015): Introduction, Motivation, and the Cuntz Semigroup.
- Lecture 2 (2 June 2015): Large Subalgebras and their Basic Properties.
- Lecture 3 (4 June 2015): Large Subalgebras and the Radius of Comparison.
- Lecture 4 (5 June 2015 [morning]): Large Subalgebras in Crossed Products by $\mathbb{Z}$.
- Lecture 5 (5 June 2015 [afternoon]): Application to the Radius of Comparison of Crossed Products by Minimal Homeomorphisms.

A rough outline of all five lectures

- Introduction: what large subalgebras are good for.
- Definition of a large subalgebra.
- Statements of some theorems on large subalgebras.
- A very brief survey of the Cuntz semigroup.
- Open problems.
- Basic properties of large subalgebras.
- A very brief survey of radius of comparison.
- Description of the proof that if B is a large subalgebra of A, then A and B have the same radius of comparison.
- A very brief survey of crossed products by $\mathbb{Z}$.
- Orbit breaking subalgebras of crossed products by minimal homeomorphisms.
- Sketch of the proof that suitable orbit breaking subalgebras are large.
- A very brief survey of mean dimension.
- Description of the proof that for minimal homeomorphisms with Cantor factors, the radius of comparison is at most half the mean dimension.

A brief reminder on crossed products

Let G be a (discrete) group, let A be a unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A.

The *skew group ring* $A[G]$ is the set of all formal sums

$$\sum_{g \in G} a_g u_g$$

with $a_g \in A$ for all $g \in G$ and $a_g = 0$ for all but finitely many $g \in G$. The product and adjoint are determined by requiring that:

- u_g is unitary for $g \in G$.
- $u_g u_h = u_{gh}$ for $g, h \in G$.
- $u_g a u_g^* = \alpha_g(a)$ for $g \in G$ and $a \in A$.

Thus,

$$(a \cdot u_g)(b \cdot u_h) = (a[u_g b u_{g^{-1}}]) \cdot u_{gh} = (a\alpha_g(b)) \cdot u_{gh}$$

$$(a \cdot u_g)^* = \alpha_g^{-1}(a^*) \cdot u_{g^{-1}}$$

for $a, b \in A$ and $g, h \in G$, extended linearly.

As above, G is a discrete group, A is a unital C*-algebra, and $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of G on A.

Fix a faithful representation $\pi\colon A \to L(H_0)$ of A on a Hilbert space H_0. Set $H = l^2(G, H_0)$, the set of all $\xi = (\xi_g)_{g\in G}$ in $\prod_{g\in G} H_0$ such that $\sum_{g\in G} \|\xi_g\|^2 < \infty$, with the scalar product

$$\big\langle (\xi_g)_{g\in G},\, (\eta_g)_{g\in G} \big\rangle = \sum_{g\in G} \langle \xi_g, \eta_g \rangle.$$

Then define $\sigma\colon A[G] \to L(H)$ as follows. For $a = \sum_{g\in G} a_g u_g$,

$$(\sigma(a)\xi)_h = \sum_{g\in G} \pi(\alpha_h^{-1}(a_g))(\xi_{g^{-1}h})$$

for all $h \in G$. In particular, $(\sigma(u_g)\xi)_h = \xi_{g^{-1}h}$ for $g \in G$ and $(\sigma(au_1)\xi)_h = \pi(\alpha_h^{-1}(a))(\xi_h)$ for $a \in A$.

We take $C^*_{\mathrm{r}}(G, A, \alpha)$, the *reduced crossed product* of the action $\alpha\colon G \to \mathrm{Aut}(A)$, to be the completion of $A[G]$ in the norm $\|a\| = \|\sigma(a)\|$. It is a theorem that this norm does not depend on π as long as π is injective.

Conditional expectation and coefficients

As above, G is a discrete group, A is a unital C*-algebra, and $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of G on A.

As above, $C^*_{\mathrm{r}}(G, A, \alpha)$ is the completion of $A[G]$ in the norm coming from a representation modelled on the regular representation of G.

The full crossed product $C^*(G, A, \alpha)$ is in principle bigger, but for amenable groups, including $\mathbb{Z}$, it is known to be the same. In this case, we just write $C^*(G, A, \alpha)$.

If 1 is the identity of the group, then $a \mapsto au_1$ is an injective unital homomorphism from A to $C^*_{\mathrm{r}}(G, A, \alpha)$. Identify A with its image.

The map $\sum_{g\in G} a_g u_g \mapsto a_1$, from $A[G]$ to $A \subset A[G]$, extends to a faithful conditional expectation $E\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A \subset C^*_{\mathrm{r}}(G, A, \alpha)$. We can recover the coefficient a_h of u_h in $a = \sum_{g\in G} a_g u_g$ as $E(au_h^*)$, and thus get a formal series $\sum_{g\in G} a_g u_g$ for any $a \in C^*_{\mathrm{r}}(G, A, \alpha)$. Unfortunately, this series need not converge in any standard topology. (However, under suitable conditions, its Cesàro means converge to a in norm.)

Crossed products by $\mathbb{Z}$

In this lecture, $G = \mathbb{Z}$. An action $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(A)$ is determined by its generator α_1, which is an arbitrary automorphism of A; we usually just call it α, and for $\alpha \in \mathrm{Aut}(A)$ we write $C^*(\mathbb{Z}, A, \alpha)$ for the crossed product by the action generated by α. We conventionally take $u = u_1$ the standard unitary in the crossed product corresponding to the generator $1 \in \mathbb{Z}$. (Note the change of notation: 1 is not the identity of $\mathbb{Z}$.) Thus $u_n = u^n$. We also let $E\colon C^*(\mathbb{Z}, A, \alpha) \to A$ be the standard conditional expectation (picking out the coefficient of the group identity). We will usually have $A = C(X)$; see below.

See the incomplete draft lecture notes on my website, *Crossed Product C*-Algebras and Minimal Dynamics*, especially Sections 8 and 9, for a lot more on crossed products. Full and reduced crossed products exist in much greater generality: A need not be unital, and G can be any locally compact group.

Crossed products by homeomorphisms

The irrational rotation algebra A_θ, for $\theta \in \mathbb{R} \setminus \mathbb{Q}$, is a famous example. By definition, it is the universal C*-algebra generated by unitaries u and v satisfying $vu = e^{2\pi i\theta} uv$. We can rewrite the relation as $uvu^* = e^{-2\pi i\theta} v$. Take $A = C^*(v) \cong C(S^1)$, using the isomorphism sending v to the function $v(\zeta) = \zeta$ for $\zeta \in S^1$. Then u corresponds to the standard unitary in $C^*(\mathbb{Z}, C(S^1), \alpha)$ corresponding to generator $1 \in \mathbb{Z}$ for $\alpha \in \mathrm{Aut}(C(S^1))$ determined by $\alpha(v) = e^{-2\pi i\theta} v$. For general $f \in C(S^1)$, this is $\alpha(f)(\zeta) = f(e^{-2\pi i\theta}\zeta)$ for $\zeta \in S^1$.

For a compact Hausdorff space X and a homeomorphism $h\colon X \to X$, we use the automorphism $\alpha(f)(x) = f(h^{-1}(x))$ for $f \in C(X)$ and $x \in X$ to define an action of $\mathbb{Z}$ on X, and write $C^*(\mathbb{Z}, X, h)$ for $C^*(\mathbb{Z}, C(X), \alpha)$. For the irrational rotation algebra A_θ, $h(\zeta) = e^{2\pi i\theta}\zeta$ for $\zeta \in S^1$.

Orbit breaking subalgebras

Standing hypothesis for the rest of this lecture: X is a compact Hausdorff space and $h \colon X \to X$ is a homeomorphism. Very soon, X will be required to be infinite and h will be required to be minimal.

Recall from Lecture 1:

Definition

Let $Y \subset X$ be a nonempty closed subset, and define

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, C_0(X \setminus Y)u\big) \subset C^*(\mathbb{Z}, X, h).$$

We call it the Y-*orbit breaking subalgebra* of $C^*(\mathbb{Z}, X, h)$.

This lecture is about the proof of the following theorem from Lecture 1:

Theorem

Let X be a compact Hausdorff space and let $h \colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^*(\mathbb{Z}, X, h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$.

A more general result

Under some technical conditions on α and D, similar methods can be used to prove the analogous result for $C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)_Y$. The following theorem is a consequence of joint work with Archey and Buck.

Theorem

Let X be an infinite compact metric space, let $h \colon X \to X$ be a minimal homeomorphism, let D be a simple unital C*-algebra which has a tracial state, and let $\alpha \in \mathrm{Aut}(C(X, D))$ lie over h. Assume that D has strict comparison of positive elements, or that the automorphisms $\alpha_x \in \mathrm{Aut}(D)$, determined by $\alpha(a)(x) = \alpha_x(a(h^{-1}(x)))$ for all $x \in X$ and $a \in C_0(X, D)$, are all approximately inner. Let $Y \subset X$ be a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)_Y$ is a centrally large subalgebra of $C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)$.

The ideas of the proof of the previous theorem are all used in the proof of the general result behind this theorem, but additional work is needed to deal with the presence of D.

Describing $C^*(\mathbb{Z}, X, h)_Y$

We can say exactly what is in $C^*(\mathbb{Z}, X, h)_Y$ (for any closed subset $Y \subset X$).

Proposition

Let $u \in C^*(\mathbb{Z}, X, h)$ and $E \colon C^*(\mathbb{Z}, X, h) \to C(X)$ be as above. Let $Y \subset X$ be a closed subset. For $n \in \mathbb{Z}$, set

$$Y_n = \begin{cases} \bigcup_{j=0}^{n-1} h^j(Y) & n > 0 \\ \varnothing & n = 0 \\ \bigcup_{j=1}^{-n} h^{-j}(Y) & n < 0. \end{cases}$$

Then

$$C^*(\mathbb{Z}, X, h)_Y = \big\{a \in C^*(\mathbb{Z}, X, h) \colon E(au^{-n}) \in C_0(X \setminus Y_n) \text{ for all } n \in \mathbb{Z}\big\}$$

and

$$\overline{C^*(\mathbb{Z}, X, h)_Y \cap C(X)[\mathbb{Z}]} = C^*(\mathbb{Z}, X, h)_Y.$$

This says that for $a \in C^*(\mathbb{Z}, X, h)$ with formal series $\sum_{n=-\infty}^{\infty} a_n u^n$, we have $a \in C^*(\mathbb{Z}, X, h)_Y$ if and only if for all n the coefficient of u^n is in $C_0(X \setminus Y_n)$.

Sketch of proof

Define

$$B = \big\{a \in C^*(\mathbb{Z}, X, h) \colon E(au^{-n}) \in C_0(X \setminus Y_n) \text{ for all } n \in \mathbb{Z}\big\}$$

and

$$B_0 = B \cap C(X)[\mathbb{Z}].$$

We are supposed to prove that

$$C^*(\mathbb{Z}, X, h)_Y = B \quad \text{and} \quad \overline{C^*(\mathbb{Z}, X, h)_Y \cap C(X)[\mathbb{Z}]} = C^*(\mathbb{Z}, X, h)_Y.$$

It is equivalent to prove that $C^*(\mathbb{Z}, X, h)_Y = B$ and $\overline{B_0} = B$.

We claim that $\overline{B_0} = B$. Let $b \in B$ and define $b_k = E(bu^{-k}) \in C_0(X \setminus Y_k)$. So the formal series for b is $\sum_{k=-\infty}^{\infty} b_k u^k$. The element

$$a_n = \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) b_k u^k.$$

is clearly in B_0. The Cesàro means of the formal series do converge in norm (this theorem is in Davidson's book), that is, $\lim_{n \to \infty} a_n = b$. The claim follows.

Sketch of proof (continued)

Recall our definitions:

$$B = \left\{ a \in C^*(\mathbb{Z}, X, h) \colon E(au^{-n}) \in C_0(X \setminus Y_n) \text{ for all } n \in \mathbb{Z} \right\}$$

and

$$B_0 = B \cap C(X)[\mathbb{Z}].$$

We proved $\overline{B_0} = B$, and we still must prove $C^*(\mathbb{Z}, X, h)_Y = B$.

The next step is to prove that B_0 is a *-algebra. It is enough to prove that if $f \in C_0(X \setminus Y_m)$ and $g \in C_0(X \setminus Y_n)$, then $(fu^m)(gu^n) \in B_0$ and $(fu^m)^* \in B_0$. The proof involves manipulations with h and the sets Y_n. The proof that $(fu^m)(gu^n) \in B_0$ must be broken into six cases: all combinations of signs of m, n, and $m+n$ which can actually occur.

Since $C(X) \subset B_0$ and $C_0(X \setminus Y)u \subset B_0$, it follows that $C^*(\mathbb{Z}, X, h)_Y \subset \overline{B_0} = B$.

Recall our definitions:

$$B = \left\{ a \in C^*(\mathbb{Z}, X, h) \colon E(au^{-n}) \in C_0(X \setminus Y_n) \text{ for all } n \in \mathbb{Z} \right\}$$

and

$$B_0 = B \cap C(X)[\mathbb{Z}].$$

Also, so far we have $C^*(\mathbb{Z}, X, h)_Y \subset \overline{B_0} = B$.

The last step is to show that for all $n \in \mathbb{Z}$ and $f \in C_0(X \setminus Y_n)$, we have $fu^n \in C^*(\mathbb{Z}, X, h)_Y$. For $n = 0$ this is trivial. Let $n > 0$, and let $f \in C_0(X \setminus Y_n)$. Define $f_0 = (\operatorname{sgn} \circ f)|f|^{1/n}$ and for $j = 1, 2, \ldots, n-1$ define $f_j = |f \circ h^j|^{1/n}$. The definition $Y_n = \bigcup_{j=0}^{n-1} h^j(Y)$ implies that $f_0, f_1, \ldots, f_{n-1} \in C_0(X \setminus Y)$. Therefore the element

$$a = (f_0 u)(f_1 u) \cdots (f_{n-1} u)$$

is in $C^*(\mathbb{Z}, X, h)_Y$. A computation shows that $a = fu^n$. The case $n < 0$ is reduced to the case $n > 0$ by taking adjoints; we omit the details.

It now follows that $B_0 \subset C^*(\mathbb{Z}, X, h)_Y$. Combining this result with $\overline{B_0} = B$ and $C^*(\mathbb{Z}, X, h)_Y \subset B$, we get $C^*(\mathbb{Z}, X, h)_Y = B$. This finishes the proof.

One can now deduce (details omitted) that the obvious isomorphism $\varphi \colon C^*(\mathbb{Z}, X, h^{-1}) \to C^*(\mathbb{Z}, X, h)$ gives

$$\varphi\left(C^*(\mathbb{Z}, X, h^{-1})_{h^{-1}(Y)} \right) = C^*(\mathbb{Z}, X, h)_Y.$$

From now on, X is an infinite compact metric space, $h \colon X \to X$ is a minimal homeomorphism, $Y \subset X$ is compact, and $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$.

Lemma

Let $U \subset X$ be a nonempty open subset. Then there exist $l \in \mathbb{Z}_{\geq 0}$, compact sets $Y_1, Y_2, \ldots, Y_l \subset X$, and $n_1, n_2, \ldots, n_l \in \mathbb{Z}_{>0}$, such that $Y \subset \bigcup_{j=1}^{l} Y_j$ and such that $h^{n_1}(Y_1)$, $h^{n_2}(Y_2)$, ..., $h^{n_l}(Y_l)$ are disjoint subsets of U.

Sketch of proof.

Choose a nonempty open subset $V \subset X$ such that $\overline{V}$ is compact and contained in U. Use minimality of h to cover Y with the images of V under finitely many negative powers of h, say $h^{-n_1}(V)$, ..., $h^{-n_l}(V)$. Set $Y_j = h^{-n_j}\left(\overline{V}\right) \cap Y$ for $j = 1, 2, \ldots, l$. □

Towards showing that $C^*(\mathbb{Z}, X, h)_Y$ is large

Recall: X is an infinite compact metric space, $h \colon X \to X$ is a minimal homeomorphism, $Y \subset X$ is compact, and $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$.

Lemma

Let $U \subset X$ be a nonempty open subset and let $n \in \mathbb{Z}$. Then there exist $f, g \in C(X)_+$ such that

$$f|_{h^n(Y)} = 1, \quad 0 \leq f \leq 1, \quad \operatorname{supp}(g) \subset U, \quad \text{and} \quad f \precsim_{C^*(\mathbb{Z}, X, h)_Y} g.$$

This lemma is one of the main steps. It is straightforward if one only asks that $f \precsim_{C^*(\mathbb{Z}, X, h)} g$. Getting $f \precsim_{C^*(\mathbb{Z}, X, h)_Y} g$ for both positive n and negative n is a key step in showing $C^*(\mathbb{Z}, X, h)_Y$ a large subalgebra of $C^*(\mathbb{Z}, X, h)$.

Sketch of proof

Lemma

Let $U \subset X$ be a nonempty open subset and let $n \in \mathbb{Z}$. Then there exist $f, g \in C(X)_+$ such that

$$f|_{h^n(Y)} = 1, \quad 0 \leq f \leq 1, \quad \operatorname{supp}(g) \subset U, \quad \text{and} \quad f \precsim_{C^*(\mathbb{Z},X,h)_Y} g.$$

We first prove this when $n = 0$ and, for simplification, when $Y = \{y_0\}$.

Recall:

$$Y_n = \begin{cases} \bigcup_{j=0}^{n-1} h^j(Y) & n > 0 \\ \varnothing & n = 0 \\ \bigcup_{j=1}^{-n} h^{-j}(Y) & n < 0 \end{cases}$$

and ($E(au^{-n})$ is the coefficient of u^n in the formal series for a)

$$C^*(\mathbb{Z},X,h)_Y = \{a \in C^*(\mathbb{Z},X,h) \colon E(au^{-n}) \in C_0(X \setminus Y_n) \text{ for all } n \in \mathbb{Z}\}.$$

For $Y = \{y_0\}$, the requirement is that $E(au^{-n})(h^j(y_0)) = 0$ for $j = 0, 1, \ldots, n-1$ if $n \geq 0$, and for $j = -1, -2, \ldots, n$ if $n < 0$.

Sketch of proof: $n = 0$ and $Y = \{y_0\}$

Since the forward orbit of y_0 is dense, there is $N \in \mathbb{Z}_{>0}$ such that $h^N(y_0) \in U$. Then $y_0 \in h^{-N}(U)$. Set

$$W = h^{-N}(U) \setminus \{h^{-1}(y_0),\, h^{-2}(y_0),\, \ldots,\, h^{-N}(y_0)\},$$

and observe that $y_0 \in W$. Therefore there is $f \in C(X)_+$ such that

$$\operatorname{supp}(f) \subset W, \quad 0 \leq f \leq 1, \quad \text{and} \quad f(x) = 1.$$

Further define $g = f \circ h^{-N}$. Then $\operatorname{supp}(g) \subset U$. Let $u \in C^*(\mathbb{Z},X,h)$ be (as usual) the standard unitary. Set $a = f^{1/2}u^{-N}$. Since f vanishes on $h^{-1}(y_0),\, h^{-2}(y_0),\, \ldots,\, h^{-N}(y_0)$, the characterization of $C^*(\mathbb{Z},X,h)_{\{y_0\}}$ implies that $a \in C^*(\mathbb{Z},X,h)_{\{y_0\}}$. So in $C^*(\mathbb{Z},X,h)_{\{y_0\}}$ we have

$$g = f \circ h^{-N} = a^*a \sim_{C^*(\mathbb{Z},X,h)_{\{y_0\}}} aa^* = f.$$

This completes the proof for $n = 0$ and $Y = \{y_0\}$.

For general Y, the proof uses the previous lemma, functions $f_1, f_2, \ldots, f_l \in C(X)_+$, and elements $a_j = f_j^{1/2}u^{-n_j}$. See the lecture notes.

Sketch of proof: $n > 0$

Now suppose that $n > 0$. Choose functions f and g for the case $n = 0$, and call them f_0 and g. Since $f_0(x) = 1$ for all $x \in Y$, and since $Y \cap \bigcup_{l=1}^{n} h^{-l}(Y) = \varnothing$, there is $f_1 \in C(X)$ with $0 \leq f_1 \leq f_0$, $f_1(x) = 1$ for all $x \in Y$, and $f_1(x) = 0$ for $x \in \bigcup_{l=1}^{n} h^{-l}(Y)$. Set $v = f_1^{1/2}u^{-n}$ and $f = f_1 \circ h^{-n}$. Then $f(x) = 1$ for all $x \in h^n(Y)$ and $0 \leq f \leq 1$. The characterization of $C^*(\mathbb{Z},X,h)_Y$ implies that $v \in C^*(\mathbb{Z},X,h)_Y$. We have

$$v^*v = u^n f_1 u^{-n} = f_1 \circ h^{-n} = f \quad \text{and} \quad vv^* = f_1.$$

Using (4) on the Cuntz semigroup handout, we thus get

$$f \sim_{C^*(\mathbb{Z},X,h)_Y} f_1 \leq f_0 \precsim_{C^*(\mathbb{Z},X,h)_Y} g.$$

This completes the proof for the case $n > 0$.

Sketch of proof: $n < 0$

Finally, we consider the case $n < 0$. In this case, we have $-n-1 \geq 0$. Apply the cases already done with h^{-1} in place of h. We get $f, g \in C^*(\mathbb{Z},X,h^{-1})_{h^{-1}(Y)}$ such that $f(x) = 1$ for all $x \in (h^{-1})^{-n-1}(h^{-1}(Y)) = h^n(Y)$, such that $0 \leq f \leq 1$, such that $\operatorname{supp}(g) \subset U$, and such that $f \precsim_{C^*(\mathbb{Z},X,h^{-1})_{h^{-1}(Y)}} g$. Let $\varphi \colon C^*(\mathbb{Z},X,h^{-1}) \to C^*(\mathbb{Z},X,h)$ be the isomorphism above. Then

$$\varphi(f) = f, \quad \varphi(g) = g, \quad \text{and} \quad \varphi\big(C^*(\mathbb{Z},X,h^{-1})_{h^{-1}(Y)}\big) = C^*(\mathbb{Z},X,h)_Y.$$

Therefore $f \precsim_{C^*(\mathbb{Z},X,h)_Y} g$.

Comparison with an element of $C(X)$

The methods for following have been commonly used in connection with crossed products by discrete groups. A few examples: Elliott (1980), Archbold-Spielberg (1994), Kishimoto-Kumjian (1997).

Lemma

Let $B \subset C^*(\mathbb{Z}, X, h)$ be a unital subalgebra such that $C(X) \subset B$ and $B \cap C(X)[\mathbb{Z}]$ is dense in B. Let $a \in B_+ \setminus \{0\}$. Then there exists $b \in C(X)_+ \setminus \{0\}$ such that $b \precsim_B a$.

The key hypothesis is freeness of the action; minimality doesn't matter. The basic idea is that if $\mathrm{supp}(f) \cap h^n(\mathrm{supp}(f)) = \varnothing$, then $f(gu^n)f = 0$. See the lecture notes for the proof.

Corollary

Let $B \subset C^*(\mathbb{Z}, X, h)$ be a unital subalgebra such that $C(X) \subset B$ and $B \cap C(X)[\mathbb{Z}]$ is dense in B. Let $a \in A_+ \setminus \{0\}$ and let $b \in B_+ \setminus \{0\}$. Then there exists $f \in C(X)_+ \setminus \{0\}$ such that $f \precsim_{C^*(\mathbb{Z},X,h)} a$ and $f \precsim_B b$.

One more lemma

Lemma

Let A be a C*-algebra, and let $S \subset A$ be a subset which generates A as a C*-algebra. Then for every finite subset $F \subset A$ and every $\varepsilon > 0$ there are a finite subset $F_0 \subset S$ and $\varepsilon_0 > 0$ such that whenever $b \in A$ satisfies $\|b\| \leq 1$ and $\|ba - ab\| < \varepsilon_0$ for all $a \in F_0$, then $\|ba - ab\| < \varepsilon$ for all $a \in F$.

The point is that if $a_1, a_2, \ldots, a_n \in A$, a is an algebraic expression in $a_1, a_2, \ldots, a_n$, and b commutes with $a_1, a_2, \ldots, a_n$ up to a small enough tolerance, then $\|ba - ab\|$ is small.

Centrally large subalgebras

Recall the definition of a centrally large subalgebra:

Definition

Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *centrally large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
4. $g \precsim_B y$ and $g \precsim_A x$.
5. $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.
6. For $j = 1, 2, \ldots, m$ we have $\|ga_j - a_jg\| < \varepsilon$.

Starting the proof that $C^*(\mathbb{Z}, X, h)_Y$ is centrally large

Since h is minimal, it is well known that A is simple and finite. Also clearly A is infinite dimensional. Set $B = C^*(\mathbb{Z}, X, h)_Y$.

We claim that it suffices to do the following. Let $m \in \mathbb{Z}_{>0}$, let $a_1, a_2, \ldots, a_m \in A$, let $\varepsilon > 0$, and let $f \in C(X)_+ \setminus \{0\}$. We find $c_1, c_2, \ldots, c_m \in A$ and $g \in C(X)$ such that:

1. $0 \leq g \leq 1$.
2. For all j, we have $\|c_j - a_j\| < \varepsilon$ and $(1 - g)c_j \in B$.
3. $g \precsim_B f$.
4. $\|gu - ug\| < \varepsilon$.

There are three points. First, since $C^*(\mathbb{Z}, X, h)$ is stably finite, we don't need a condition of the form $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$. Second, replacing the usual comparison conditions by $f \in C(X)_+ \setminus \{0\}$ and $g \precsim_B f$ is allowed by the corollary on the previous slide. Third, we don't have to use the same finite set in (4) as for (2). By the lemma on the previous slide, in (4) we can use a finite subset of a generating set; we take $C(X) \cup \{u\}$. Since $g \in C(X)$, we need only estimate $\|gu - ug\|$.

Choosing the approximations

Choose $c_1, c_2, \ldots, c_m \in C(X)[\mathbb{Z}]$ such that $\|c_j - a_j\| < \varepsilon$ for $j = 1, 2, \ldots, m$. Choose $N \in \mathbb{Z}_{>0}$ such that for $j = 1, 2, \ldots, m$ there are $c_{j,l} \in C(X)$ for $l = -N, \, -N+1, \, \ldots, \, N-1, \, N$ with

$$c_j = \sum_{l=-N}^{N} c_{j,l} u^l.$$

Choose $N_0 \in \mathbb{Z}_{>0}$ such that $\frac{1}{N_0} < \varepsilon$. Define

$$J = \big\{ -N - N_0, \, -N - N_0 + 1, \, \ldots, \, N + N_0 - 1, \, N + N_0 \big\}.$$

For the purpose of getting $g \precsim_B f$, set $U = \{x \in X \colon f(x) \neq 0\}$, and choose nonempty disjoint open sets $U_l \subset U$ for $l \in J$. For each such l, use a lemma above to choose $f_l, r_l \in C(X)_+$ such that $r_l(x) = 1$ for all $x \in h^l(Y)$, such that $0 \leq r_l \leq 1$, such that $\mathrm{supp}(f_l) \subset U_l$, and such that $r_l \precsim_B f_l$. The function g will be a sum of functions $\lambda_l g_l$ supported in $\mathrm{supp}(r_l)$, and this construction will allow us to get $g \precsim_B f$.

Defining g

Recall: the sets $U_l \subset U$ are disjoint. The functions $f_l, r_l \in C(X)_+$ satisfy $r_l = 1$ on $h^l(Y)$, $0 \leq r_l \leq 1$, $\mathrm{supp}(f_l) \subset U_l$, and $r_l \precsim_B f_l$.

With a bit of work, we can find g_0 such that the functions $g_l = g_0 \circ h^{-l}$ satisfy $0 \leq g_l \leq r_l \leq 1$ and $g_l = 1$ on $h^l(Y)$. Then $\sum_{l \in J} g_l \precsim_B \sum_{l \in J} f_l \precsim_B f$.

Define λ_l for $l \in J$ by

$$\lambda_{-N-N_0} = 0, \quad \lambda_{-N-N_0+1} = \frac{1}{N_0}, \quad \lambda_{-N-N_0+2} = \frac{2}{N_0}, \quad \ldots,$$

$$\lambda_{-N-1} = 1 - \frac{1}{N_0}, \quad \lambda_{-N} = \lambda_{-N+1} = \cdots = \lambda_{N-1} = \lambda_N = 1,$$

$$\lambda_{N+1} = 1 - \frac{1}{N_0}, \quad \lambda_{N+2} = 1 - \frac{2}{N_0}, \quad \ldots, \quad \lambda_{N+N_0-1} = \frac{1}{N_0}, \quad \lambda_{N+N_0} = 0.$$

Set $g = \sum_{l \in J} \lambda_l g_l$. The supports of the functions g_l are disjoint, so $0 \leq g \leq 1$. Also $g \leq \sum_{l \in J} g_l \precsim_B f$.

Checking that $(1-g)c_j \in B$

We check that $(1-g)c_j \in B$. Since $1 - g$ vanishes on the sets

$$h^{-N}(Y), \, h^{-N+1}(Y), \, \ldots, \, h^{N-2}(Y), \, h^{N-1}(Y),$$

the characterization of $C^*(\mathbb{Z}, X, h)_Y$ implies that $(1-g)u^l \in B$ for $l = -N, \, -N+1, \, \ldots, \, N-1, \, N$. For $j = 1, 2, \ldots, m$, since $c_{j,l} \in C(X) \subset B$ for $l = -N, \, -N+1, \, \ldots, \, N-1, \, N$, we get

$$(1-g)c_j = \sum_{l=-N}^{N} c_{j,l} \cdot (1-g)u^l \in B.$$

Thus $(1-g)c_j \in B$.

Checking the approximate commutation relation

Finally, we check the commutation relation $\|gu - ug\| < \varepsilon$. We have

$$\begin{aligned} \|gu - ug\| &= \|g - ugu^*\| = \|g - g \circ h^{-1}\| \\ &= \left\| \sum_{l \in J} \lambda_l g_0 \circ h^{-l} - \sum_{l \in J} \lambda_l g_0 \circ h^{-l-1} \right\|. \end{aligned}$$

In the second sum in the last term, we change variables to get $\sum_{l+1 \in J} \lambda_{l-1} g_0 \circ h^{-l}$. Use $\lambda_{-N-N_0} = \lambda_{N+N_0} = 0$ and combine terms to get

$$\|gu - ug\| = \left\| \sum_{l=-N-N_0+1}^{N+N_0} (\lambda_l - \lambda_{l-1}) g_0 \circ h^{-l} \right\|.$$

The expressions $g_0 \circ h^{-l}$ are orthogonal and have norm 1, so

$$\|gu - ug\| = \max_{-N-N_0+1 \leq l \leq N+N_0} |\lambda_l - \lambda_{l-1}| = \frac{1}{N_0} < \varepsilon.$$

This finishes the proof that $C^*(\mathbb{Z}, X, h)_Y$ is large in $C^*(\mathbb{Z}, X, h)$.