Lecture 4: Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property

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- Lecture 1 (26 July 2014): Actions of Finite Groups on C*-Algebras and Introduction to Crossed Products.
- Lecture 2 (27 July 2014): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 3 (28 July 2014): Crossed Products by Actions with the Rokhlin Property.
- Lecture 4 (29 July 2014): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 5 (30 July 2014): Examples and Applications.

A rough outline of all five lectures

- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.

The Rokhlin property

Recall the Rokhlin property (with exact permutation of the projections): Let A be a unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the Rokhlin property if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Theorem

Let A be a unital AF algebra. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. Then $C^*(G, A, \alpha)$ is AF.

Crossed products of AF algebras by Rokhlin actions

Theorem

Let A be a unital AF algebra. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. Then $C^*(G, A, \alpha)$ is AF.

The idea of the proof: It was enough to consider a finite set $S \subset C^*(G, A, \alpha)$ of the form $S = F \cup \{u_g \colon g \in G\}$, with $F \subset A$ finite and $u_g \in C^*(G, A, \alpha)$ the standard unitary corresponding to $g \in G$. We chose a family $(e_g)_{g \in G}$ in A of Rokhlin projections for α, F, and $\delta > 0$ (depending on ε). That is,

1. $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \delta$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

We took $D_0 = \bigoplus_{g \in G} e_g A e_g \subset A$, which is a G-invariant unital subalgebra of A containing F to within ε. Then $D = C^*(G, D_0, \alpha) \subset C^*(G, A, \alpha)$ is a unital subalgebra which contains F to within ε and exactly contains u_g for $g \in G$. Thus D contains S to within ε. Also, $D \cong M_n(e_1 A e_1)$, which is AF. So finite sets in $C^*(G, A, \alpha)$ can be approximated by AF subalgebras, and hence by finite dimensional subalgebras. Thus $C^*(G, A, \alpha)$ is AF.

A distillation of the proof

Proposition

Let A be a unital C*-algebra. Let G be a finite group; set $n = \mathrm{card}(G)$. Let $\alpha \colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. Let $S \subset C^*(G, A, \alpha)$ be a finite set and let $\varepsilon > 0$. Then there exist a projection $p \in A$ and a unital subalgebra $D \subset C^*(G, A, \alpha)$ such that:

1. $D \cong M_n(pAp)$.
2. $\mathrm{dist}(b, D) < \varepsilon$ for all $b \in S$.

Proposition

Let B be a unital C*-algebra. Suppose that there is a unital AF algebra A such that, for every finite set $S \subset B$ and every $\varepsilon > 0$, there exist a projection $p \in A$, $n \in \mathbb{N}$, and a unital subalgebra $D \subset B$ such that:

1. $D \cong M_n(pAp)$.
2. $\mathrm{dist}(b, D) < \varepsilon$ for all $b \in S$.

Then B is AF.

Other structural consequences of the Rokhlin property

Informally, we have:

1. Let A be a unital C*-algebra. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. Then $C^*(G, A, \alpha)$ can be locally approximated by matrix algebras over corners of A.
2. Let B be a unital C*-algebra which can be locally approximated by matrix algebras over corners of AF algebras. Then B is AF.

The recognition that the original proof could be interpreted this way is due to Hiroyuki Osaka.

There are other properties besides "AF" for which (2) works. We list some results on the next slide.

Other structural consequences (continued)

Crossed products by actions of finite groups with the Rokhlin property preserve various other classes of C*-algebras. In many cases, the proofs follow the idea of the previous slide. Some examples of such classes:

1. Simple unital C*-algebras.
2. Various classes of unital but not necessarily simple countable direct limit C*-algebras using semiprojective building blocks, such as AI algebras and AT algebras. (With Osaka.)
3. Simple unital AH algebras with slow dimension growth and real rank zero. (With Osaka.)
4. D-absorbing separable unital C*-algebras for a strongly self-absorbing C*-algebra D. (Hirshberg-Winter.)
5. Separable nuclear unital C*-algebras whose quotients all satisfy the Universal Coefficient Theorem. (With Osaka.)
6. Separable unital approximately divisible C*-algebras. (Hirshberg-Winter.)
7. Unital C*-algebras with the ideal property and unital C*-algebras with the projection property. (With Pasnicu.)

Freeness and the Rokhlin property

A free action of a finite group on the Cantor set X has the Rokhlin property. (That is, the corresponding action on $C(X)$ has the Rokhlin property.) We saw this in Lecture 3.

A free action on a connected space X doesn't, since there are no nontrivial projections in $C(X)$. (We won't discuss this further, but such an action does have a "higher dimensional Rokhlin property" as defined by Hirshberg, Winter, and Zacharias.)

We consider mostly simple C*-algebras with many projections. Recall that irrational rotation algebras, UHF algebras, and Cuntz algebras all have real rank zero.

(If there are not enough projections, finite group actions are less well understood, although there has been significant recent progress. In the nonsimple case, rather little is known.)

Tracial states and the Rokhlin property

Motivation: Recall that a tracial state on a C*-algebra A is a state τ on A such that $\tau(ab) = \tau(ba)$ for all $a, b \in A$. Any tracial state τ induces a homomorphism $\tau_* \colon K_0(A) \to \mathbb{R}$, given by extending τ to $M_n(A)$ for all n, and setting $\tau_*([p]) = \tau(p)$ for any projection $p \in M_n(A)$. See the K-theory lectures, but the basic point is that if $s^* s = p$ and $ss^* = q$, then the trace condition implies $\tau(p) = \tau(q)$.

Suppose A has a unique tracial state. (This is true for both UHF algebras and irrational rotation algebras.) Let G be finite, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. In the version with exact permutation of the projections, take $\varepsilon = 1$ and $F = \varnothing$. (In fact, ε and F can be arbitrary.) We get projections e_g such that, in particular:

- $\alpha_g(e_1) = e_g$ for all $g \in G$.
- $\sum_{g \in G} e_g = 1$.

Since τ is unique, we have $\tau \circ \alpha_g = \tau$ for all $g \in G$. So $\tau(e_g) = \tau(e_1)$. It follows that

$$\tau(e_1) = \frac{1}{\mathrm{card}(G)}.$$

The Rokhlin property and A_θ

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and recall that A_θ is generated by unitaries u and v satisfying $vu = e^{2\pi i \theta} uv$. Further recall the action $\alpha \colon \mathbb{Z}_n \to \mathrm{Aut}(A_\theta)$ generated by

$$u \mapsto e^{2\pi i/n} u \quad \text{and} \quad v \mapsto v.$$

This is a noncommutative version of the free action of $\mathbb{Z}_n$ on $S^1 \times S^1$ given by rotation by $e^{2\pi i/n}$ in the first coordinate. Moreover, A_θ has many projections, like $C(X)$ when X is the Cantor set. In fact, A_θ has real rank zero. So one would hope that α has the Rokhlin property.

The Rokhlin property and A_θ (continued)

From two slides ago: If A has a unique tracial state τ, and $\alpha \colon G \to \mathrm{Aut}(A)$ has the Rokhlin property, then for any family $(e_g)_{g \in G}$ in A of Rokhlin projections (using the version of the Rokhlin property with exact permutation of the projections), we have $\tau(e_1) = \frac{1}{\mathrm{card}(G)}$.

On A_θ, we wanted $u \mapsto e^{2\pi i/n} u$ and $v \mapsto v$ to generate an action of $\mathbb{Z}_n$ with the Rokhlin property. In fact, *no* action of *any* nontrivial finite group on A_θ has the Rokhlin property! The reason is that there is no projection $e \in A_\theta$ with $\tau(e) = \frac{1}{n}$, for any $n \geq 2$. (Recall that the tracial state τ defines an isomorphism $\tau_* \colon K_0(A_\theta) \to \mathbb{Z} + \theta\mathbb{Z}$.)

For similar reasons, no action of $\mathbb{Z}_2$ on $D = \bigotimes_{n=1}^{\infty} M_3$ has the Rokhlin property. (The tracial state τ defines an isomorphism $\tau_* \colon K_0(D) \to \mathbb{Z}\left[\frac{1}{3}\right]$.)

There are more subtle obstructions to the Rokhlin property. For example, $M_2 \otimes \bigotimes_{n=1}^{\infty} M_3$ does have projections with trace $\frac{1}{2}$, but there are still no actions of $\mathbb{Z}_2$ with the Rokhlin property. (One at least needs a copy of M_{2^n} for every n.)

Consider the Cuntz algebras $\mathcal{O}_d$, for $d \in \{2, 3, \ldots, \infty\}$. The group $K_0(\mathcal{O}_d)$ is generated by $[1]$. Therefore every automorphism of $\mathcal{O}_d$ is the identity on $K_0(\mathcal{O}_d)$.

It follows that if $\mathrm{card}(G) = n$ and $\alpha\colon G \to \mathrm{Aut}(\mathcal{O}_d)$ has the Rokhlin property, then the class of a Rokhlin projection e_1 satisfies $n[e_1] = [1]$.

$K_0(\mathcal{O}_\infty) = \mathbb{Z}[1]$, so there is no action of any nontrivial finite group on $\mathcal{O}_\infty$ with the Rokhlin property.

For $\mathrm{card}(G) = d$, a Rokhlin action of G on $\mathcal{O}_d$ was described earlier (without proof). However, for example, there is no Rokhlin action of $\mathbb{Z}_2$ on $\mathcal{O}_3$, because $[1] \in K_0(\mathcal{O}_3)$ generates $K_0(\mathcal{O}_3) \cong \mathbb{Z}_2$, and so can't be written in the form $2[e]$.

There are stronger cohomological obstructions to the Rokhlin property (Izumi). In fact, the Rokhlin property is very rare.

Side comment: On A_θ, $u \mapsto e^{2\pi i/n} u$ and $v \mapsto v$ gives an action of $\mathbb{Z}_n$ with a kind of higher dimensional Rokhlin property (the version with commuting towers). However, no action of any nontrivial finite group on $\mathcal{O}_\infty$ can even have this property (with Hirshberg).

Pointwise outer actions

For simplicity of the crossed product, a much weaker condition suffices.

Definition

An action $\alpha\colon G \to \mathrm{Aut}(A)$ is said to be *pointwise outer* if, for $g \in G \setminus \{1\}$, the automorphism α_g is outer, that is, not of the form $a \mapsto \mathrm{Ad}(u)(a) = uau^*$ for some unitary u in the multiplier algebra $M(A)$ of A.

Theorem (Kishimoto)

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a simple separable C*-algebra A. Suppose that α is pointwise outer. Then $C^*_{\mathrm{r}}(G, A, \alpha)$ is simple.

Pointwise outerness is not enough

The product type action of $\mathbb{Z}_2$ generated by

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}(\mathrm{diag}(-1, 1, 1, \ldots, 1)) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_{2^n+1}.$$

is pointwise outer. However, its crossed product has "too many" tracial states. (A has a unique tracial state, but the crossed product has two extreme tracial states.)

Worse, Elliott has constructed an example of a pointwise outer action α of $\mathbb{Z}_2$ on a simple unital AF algebra A such that $C^*(\mathbb{Z}_2, A, \alpha)$ does not have real rank zero.

Pointwise outerness thus seems not to be enough for proving classifiability of crossed products.

Recall the Rokhlin property

Definition

Let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Crossed products by actions with the Rokhlin property are very well behaved, but the Rokhlin property is rare. Pointwise outerness is common, but not good enough. Therefore we look for an intermediate condition. It will be the tracial Rokhlin property. There are many actions which have this property.

The tracial Rokhlin property

Definition

Let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra $\overline{xAx}$ of A generated by x.
4. With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

The Rokhlin property corresponds to $e = 1$.

If A is finite, the last condition can be omitted. (We omit the proof.)

Subsets and exactly permuting the projections

The conditions in the definition:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra generated by x.
4. With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

As for the Rokhlin property, we need only consider finite subsets of a generating set for A.

As for the Rokhlin property, we can use equivariant semiprojectivity to replace

$$\text{“}\|\alpha_g(e_h) - e_{gh}\| < \varepsilon \text{ for all } g, h \in G\text{”}$$

(in (1)) with

$$\text{“}\alpha_g(e_h) = e_{gh} \text{ for all } g, h \in G\text{”}.$$

We do this in these lectures.

The tracial Rokhlin property (continued)

The conditions in the definition for the finite case:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

The first two conditions are the same as for the Rokhlin property.

If the algebra “has enough tracial states” (for example, for irrational rotation algebras and simple AF algebras), the element x can be omitted, and the third condition replaced by:

- With $e = \sum_{g \in G} e_g$, the projection $1 - e$ satisfies $\tau(1 - e) < \varepsilon$ for all tracial states τ on A.

Comparison: tracial rank zero

The tracial Rokhlin property was motivated by the definition of tracial rank zero (originally called “tracially AF”):

Definition

Let A be a simple separable unital C*-algebra. Then A has tracial rank zero if for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $x \in A$, there exists a nonzero projection $p \in A$ and a unital finite dimensional subalgebra $D \subset pAp$ such that:

1. $\|[a, p]\| < \varepsilon$ for all $a \in F$.
2. $\mathrm{dist}(pap, D) < \varepsilon$ for all $a \in F$.
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{xAx}$.

In both definitions, the strong version (the Rokhlin property, or local approximation by finite dimensional C*-algebras) is supposed to hold only after cutting down by a “large” projection.

An example for the tracial Rokhlin property

Here is an example for which it is fairly easy to see that the tracial Rokhlin property holds but the Rokhlin property doesn't hold.

Take $v_k \in M_{3^k}$ to be the unitary

$$v_k = \operatorname{diag}\big(1,\, 1,\, \ldots,\, 1,\, -1,\, -1,\, \ldots,\, -1\big),$$

in which the diagonal entry 1 occurs $\frac{1}{2}(3^k + 1)$ times and the diagonal entry -1 occurs $\frac{1}{2}(3^k - 1)$ times. Let α be the order 2 automorphism

$$\alpha = \bigotimes_{k=1}^{\infty} \operatorname{Ad}(v_k) \quad \text{of} \quad A = \bigotimes_{k=1}^{\infty} M_{3^k}.$$

It generates an action of $\mathbb{Z}_2$ (also called α). Note that A is just the 3^∞ UHF algebra.

We consider the tracial Rokhlin property below, but we can see right away that α does not have the Rokhlin property: there is no action at all of $\mathbb{Z}_2$ on this algebra which has the Rokhlin property.

An example for the tracial Rokhlin property (continued)

$$v_k = \operatorname{diag}\big(1,\, 1,\, \ldots,\, 1,\, -1,\, -1,\, \ldots,\, -1\big),$$

with -1 occurring $r(k) = \frac{1}{2}(3^k - 1)$ times and 1 occurring $r(k) + 1 = \frac{1}{2}(3^k + 1)$ times, and α is the $\mathbb{Z}_2$ action on the 3^∞ UHF algebra generated by $\bigotimes_{n=1}^{\infty} \operatorname{Ad}(v_k)$.

v_k is unitarily equivalent to the block unitary (in which subscripts indicate matrix sizes)

$$w_k = \begin{pmatrix} 0 & 1_{r(k)} & 0 \\ 1_{r(k)} & 0 & 0 \\ 0 & 0 & 1_1 \end{pmatrix} \in M_{3^k}.$$

(This is the direct sum of many copies of the fact, used earlier, that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are unitarily equivalent.) So α is conjugate to $\beta = \bigotimes_{n=1}^{\infty} \operatorname{Ad}(w_k)$.

$$w_k = \begin{pmatrix} 0 & 1_{r(k)} & 0 \\ 1_{r(k)} & 0 & 0 \\ 0 & 0 & 1_1 \end{pmatrix} \in M_{3^k} \quad \text{and} \quad \beta = \bigotimes_{n=1}^{\infty} \operatorname{Ad}(w_k).$$

For $F \subset A$ finite, we have to find two projections e_0 and e_1 in A such that:

1. The action approximately exchanges e_0 and e_1.
2. e_0 and e_1 approximately commute with all elements of F.
3. $1 - e_0 - e_1$ is "small", here, the (unique) tracial state τ on A gives $\tau(1 - e_0 - e_1) < \varepsilon$.

We can assume that there is n such that $F \subset A_n = \bigotimes_{k=1}^{n} M_{3^k}$. We can increase n, so also assume $3^{-n-1} < \varepsilon$. Set

$$p_0 = \begin{pmatrix} 1_{r(n+1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0_1 \end{pmatrix} \in M_{3^{n+1}} \quad \text{and} \quad p_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{r(n+1)} & 0 \\ 0 & 0 & 0_1 \end{pmatrix} \in M_{3^{n+1}}.$$

Then $w_{n+1} p_0 w_{n+1}^* = p_1$, $w_{n+1} p_1 w_{n+1}^* = p_0$, and the normalized trace of $1 - p_0 - p_1$ is $\dfrac{1}{2r(n+1)+1} = 3^{-(n+1)} < \varepsilon$.

An example for the tracial Rokhlin property (continued)

We had, in $M_{3^{n+1}}$,

$$w_{n+1} p_0 w_{n+1}^* = p_1 \quad \text{and} \quad w_{n+1} p_1 w_{n+1}^* = p_0,$$

and the normalized trace of $1 - p_0 - p_1$ is less than ε.

Now we take

$$e_0 = 1_{A_n} \otimes p_0, \quad e_1 = 1_{A_n} \otimes p_1 \in A_n \otimes M_{3^{n+1}} = A_{n+1}.$$

Since all elements of F have the form $a \otimes 1_{M_{3^{n+1}}}$ with $a \in A_n$, these projections exactly commute with the elements of F.

Also, $\tau(1 - e_0 - e_1) < \varepsilon$ because $\tau(1 - e_0 - e_1)$ is the normalized trace of $1 - p_0 - p_1$.

Finally, on $A_n \otimes M_{3^{n+1}}$, the automorphism β has the form $\operatorname{Ad}\big(w^{(n)} \otimes w_{n+1}\big)$ (with $w^{(n)} = w_1 \otimes w_2 \otimes \cdots \otimes w_n$), so

$$\beta(e_0) = e_1 \quad \text{and} \quad \beta(e_1) = e_0.$$

This proves the tracial Rokhlin property.

An example for the tracial Rokhlin property (continued)

$$v_k = \mathrm{diag}\big(1,\, 1,\, \ldots,\, 1,\, -1,\, -1,\, \ldots,\, -1\big),$$

and α is the $\mathbb{Z}_2$-action on the 3^∞ UHF algebra generated by $\bigotimes_{n=1}^{\infty} \mathrm{Ad}(v_k)$.

The matrix size is very large, and the numbers of 1's and -1's are very close. This looks special.

Recall from Lecture 1 the action of $\mathbb{Z}_2$ on $\bigotimes_{n=1}^{\infty} M_3$ generated by $\gamma = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v)$, with $v = \mathrm{diag}(1,\, 1,\, -1)$. Rewrite γ as

$$\mathrm{Ad}(v) \otimes \big[\mathrm{Ad}(v) \otimes \mathrm{Ad}(v)\big] \otimes \big[\mathrm{Ad}(v) \otimes \mathrm{Ad}(v) \otimes \mathrm{Ad}(v)\big] \otimes \cdots = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v^{\otimes n}).$$

We have $v_1 = v$. On the diagonal of $v^{\otimes 2}$ there are $2 \cdot 2 + 1 \cdot 1 = 5$ entries equal to 1 and $2 \cdot 1 + 1 \cdot 2 = 4$ entries equal to -1. So $v^{\otimes 2}$ is unitarily equivalent to v_2. Inductively, $v^{\otimes n}$ is unitarily equivalent to v_n. So the action γ has the tracial Rokhlin property.

The tracial Rokhlin property is common

$$\gamma = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right)$$

has the tracial Rokhlin property.

In fact, it turns out to be hard to write down a product type action of $\mathbb{Z}_2$ using conjugation by matrices of the form

$$\mathrm{diag}\big(1,\, 1,\, \ldots,\, 1,\, -1,\, -1,\, \ldots,\, -1\big)$$

which doesn't have the tracial Rokhlin property. Either eventually they all have to be ± 1 (in which case the action is inner), or the matrix sizes must go to infinity. (Example: At the end of the next slide.)

Exercise: Any product type action of $\mathbb{Z}_2$ is conjugate to one of this form.

Some other actions with the tracial Rokhlin property

- The actions on irrational rotation algebras coming from finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$ have the tracial Rokhlin property.
- The action of $\mathbb{Z}_n$ on an irrational rotation algebra generated by $u \mapsto e^{2\pi i/n} u$ and $v \mapsto v$ has the tracial Rokhlin property.
- The tensor flip on any UHF algebra has the tracial Rokhlin property.
- An action of a finite group on a unital Kirchberg algebra has the tracial Rokhlin property if and only if it is pointwise outer (essentially due to Nakamura).

Except for the last, none of these actions has the Rokhlin property. In the last case, most of them don't have the Rokhlin property. Some of the proofs are complicated.

It is true (and easy) that the tracial Rokhlin property implies pointwise outerness in complete generality, but the converse is false. Counterexample without proof:

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}(\mathrm{diag}(-1, 1, 1, \ldots, 1)) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_{2^n+1}.$$

Crossed products by actions with the tracial Rokhlin property

The tracial Rokhlin property is good for understanding the structure of crossed products.

Theorem

Let A be a simple separable unital C*-algebra with tracial rank zero. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial rank zero.

This is important because tracial rank zero is a hypothesis in a major classification theorem (due to Lin).

There are examples (such as the one of Elliott mentioned above) which show that this theorem fails if one weakens the condition on the action to pointwise outerness.