Extension of shift-invariant systems in $L^2(\mathbb{R})$ to frames

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Abstract

In this paper we show that any shift-invariant Bessel sequence with an at most countable number of generators can be extended to a tight frame for its closed linear span by adding another shift-invariant system with at most the same number of generators. We show that in general this result is optimal, by providing examples where it is impossible to obtain a tight frame by adding a smaller number of generators. An alternative construction (which avoids the technical complication of extracting the square root of a positive operator) yields an extension of the given Bessel sequence to a pair of dual frame sequences.

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1 Introduction

Frames are known to be a very useful tool in signal processing and other areas where a convenient expansion in terms of basis-like elements is needed. In fact, the so-called *tight frames* immediately lead to such an expansion. General frames also lead to a signal expansion, but it is only useful in practice if a convenient *dual frame* can be found. The purpose of this paper is to consider extensions of shift-invariant Bessel sequences in $L^2(\mathbb{R})$ to tight frames, respectively, dual frame pairs. We show that any shift-invariant Bessel sequence with an at most countable number of generators can be extended to a tight frame for its closed linear span by adding another shift-invariant system with at most the same number of generators. In practice it might be inconvenient that the construction involves extracting the square root of a positive operator. We therefore also describe a slightly modified construction, which avoids this technical issue and yields an extension of the given Bessel sequence to a pair of dual frame sequences.

The paper is motivated by the results by Li and Sun in [10], where it is shown that any Bessel sequence in a separable Hilbert space can be extended to a tight frame by adding some elements (see [3] by Casazza and Leonhard for a finite-dimensional version). In particular, any Gabor frame can be expanded to a tight frame by adding another Gabor system with just one window function. Here, we consider the similar question for shift-invariant systems. From [5] (or see Proposition 7.1.10 in [4]) we know that a shift-invariant system with one generator can at most be a frame for a proper subspace of $L^2(\mathbb{R})$. We therefore first consider the question of extending shift-invariant Bessel systems to frames for its closed linear space. The question of extending the system to a frame for all of $L^2(\mathbb{R})$ is handled by adding a shift-invariant system with more than one generator.

The structure of this paper is as follows. In the rest of this section we will review the necessary parts from frame theory. In Section 2 we present the results about extensions of a shift-invariant Bessel system to frames for the corresponding subspace. A concrete example illustrating the results is given. Finally, Section 3 deals with extensions to a frame for all of $L^2(\mathbb{R})$.

In the rest of this Section we review the needed facts from frame theory. Let H denote a separable Hilbert space. A sequence $\{f_i\}_{i\in I}$ in H is called a *frame* if there exist constants $A, B > 0$ such that, for any $f \in H$,

$$A \parallel f \parallel^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B \parallel f \parallel^2 . \tag{1}$$

A and B are called lower and upper frame bounds respectively. The sequence $\{f_i\}_{i\in I}$ is a Bessel sequence if at least the upper bound in (1) is satisfied. A frame is said to be *tight* if we can choose $A = B$.

For any frame $\{f_i\}_{i\in I}$ there exists at least one dual frame, i.e., a frame $\{g_i\}_{i\in I}$ for which

$$f = \sum_{i\in I} \langle f, g_i\rangle f_i, \forall f \in H. \tag{2}$$

In order to apply frames in practice it is essential to be able to construct a dual frame explicitly. For the case of a tight frame $\{f_i\}_{i\in I}$ with bound A, it is known that the sequence $\{A^{-1}f_i\}_{i\in I}$ is a dual frame, so the representation (2) takes the convenient form

$$f = \frac{1}{A}\sum_{i\in I}\langle f, f_i\rangle f_i, \forall f \in H. \tag{3}$$

A frame $\{f_i\}_{i\in I}$ necessarily spans all of H. In this paper we will encounter sequences $\{f_i\}_{i\in I}$ that do not necessarily span all of H, but form frames for the space $\overline{\text{span}}\{f_i\}_{i\in I}$. A sequence $\{f_i\}_{i\in I}$ which is a frame for the space $\overline{\text{span}}\{f_i\}_{i\in I}$ is called a *frame sequence.* For more information about frames we refer to the book [4].

For $y \in \mathbb{R}$, the translation operator T_y acting on $f \in L^2(\mathbb{R})$ is defined by

$$(T_y f)(x) = f(x-y), \quad x \in \mathbb{R}.$$

We will consider an at most countable collection of functions $\{g_n\}_{n\in I}$ in $L^2(\mathbb{R})$ and the associated *shift-invariant system* $\{T_k g_n\}_{k\in\mathbb{Z},n\in I}$. Frame properties for such systems were first considered by Ron and Shen in [12]. The case of one generator is treated in Li and Benedetto in [1]. Note that a sequence $\{T_k g\}_{k\in\mathbb{Z}}$ at most can be a frame for a proper subspace of $L^2(\mathbb{R})$; see [5].

2 Extensions of $\{T_k g_n\}_{k\in\mathbb{Z},n\in I}$ to frame sequences

Consider an at most countable collection of functions $\{g_n\}_{n\in I}$ in $L^2(\mathbb{R})$. Throughout the note we let

$$V := \overline{\text{span}}\{T_k g_n\}_{k\in\mathbb{Z},n\in I}. \tag{4}$$

We will assume that $\{T_k g_n\}_{k\in\mathbb{Z},n\in I}$ is a Bessel sequence. Our purpose is to extend $\{T_k g_n\}_{k\in\mathbb{Z},n\in I}$ to a tight frame for V by adding another shift-invariant system $\{T_k h_n\}_{k\in\mathbb{Z},n\in I}$. We first state Theorem 3.3 from [2]. As a guide to the reader who is not familiar with the techniques in [2] we include an elementary proof for the case of one generator in the appendix.

Lemma 2.1 *Let $\{g_n\}_{n\in I}$ be an at most countable family of functions in $L^2(\mathbb{R})$. Then there exists a collection of functions $\{\varphi_n\}_{n\in I}$ such that $\{T_k\varphi_n\}_{k\in\mathbb{Z},n\in I}$ is a tight frame for V with frame bound $A = 1$.*

We are now ready to show the announced extension of a shift-invariant Bessel sequence to a tight frame.

Proposition 2.2 *Let $\{g_n\}_{n\in I}$ be an at most countable family of functions in $L^2(\mathbb{R})$ and assume that $\{T_k g_n\}_{k\in\mathbb{Z},n\in I}$ is a Bessel sequence with bound $B>0$. Then there exist functions $\{h_n\}_{n\in I}$ in V such that*

$$\{T_k g_n\}_{k\in\mathbb{Z},n\in I}\cup\{T_k h_n\}_{k\in\mathbb{Z},n\in I}$$

is a tight frame for V with bound B.

Proof. Consider the linear operator

$$S_1 : L^2(\mathbb{R})\to L^2(\mathbb{R}),\ S_1 f := \sum_{k\in\mathbb{Z},n\in I}\langle f, T_k g_n\rangle T_k g_n. \tag{5}$$

Then S_1 is bounded, and the operator

$$S_2 : V\to V,\ S_2 := BI - S_1$$

is positive, i.e. $S_2\geq 0$. Thus, S_2 has a unique positive square root, to be denoted by $S_2^{1/2}$. We can write any $f\in V$ as

$$f=\frac{1}{B}\left[S_1 f+(BI-S_1)f\right]=\frac{1}{B}\left[S_1 f+S_2^{1/2}S_2^{1/2}f\right].$$

Using the frame decomposition associated with the frame sequence $\{T_k\varphi_n\}_{k\in\mathbb{Z},n\in I}$ from Lemma 2.1 (on the element $S_2^{1/2}f$) we arrive at

$$\begin{aligned} f &= \frac{1}{B}\left[S_1 f+S_2^{1/2}\sum_{k\in\mathbb{Z},n\in I}\langle S_2^{1/2}f, T_k\varphi_n\rangle T_k\varphi_n\right] \\ &= \frac{1}{B}\left[\sum_{k\in\mathbb{Z},n\in I}\langle f, T_k g_n\rangle T_k g_n+\sum_{k\in\mathbb{Z},n\in I}\langle f, S_2^{1/2}T_k\varphi_n\rangle S_2^{1/2}T_k\varphi_n\right]. \end{aligned}$$

It is a standard result that the operator S_2 commutes with the translation operators T_k. Since $S_2^{1/2}$ is a limit of polynomials in S_2 in the strong operator topology, it follows that also $S_2^{1/2}$ commutes with T_k. Thus, for any $f\in V$ we arrive at

$$f=\frac{1}{B}\left[\sum_{k\in\mathbb{Z},n\in I}\langle f, T_k g_n\rangle T_k g_n+\sum_{k\in\mathbb{Z},n\in I}\langle f, T_k S_2^{1/2}\varphi_n\rangle T_k S_2^{1/2}\varphi_n\right].$$

Letting $h_n := S_2^{1/2}\varphi_n \in V$, it follows that

$$||f||^2 = \langle f, f\rangle = \frac{1}{B}\left[\sum_{k\in\mathbb{Z},n\in I} |\langle f, T_k g_n\rangle|^2 + \sum_{k\in\mathbb{Z},n\in I} |\langle f, T_k h_n\rangle|^2\right].$$

This completes the proof. $\square$

In particular, a Bessel sequence $\{T_k g\}_{k\in\mathbb{Z}}$ generated by one function can be extended to a tight frame sequence by adding shifts of just one function h. In the following example, we calculate the function h explicitly in a concrete case. A more general result will appear in Theorem 2.6.

Example 2.3 Consider the orthonormal system $\{T_k\chi_{[0,1]}\}_{k\in\mathbb{Z}}$. Then an application of the general Example 5.1.10 in [4] shows the following:

- The system $\{T_k\chi_{[0,1]} + T_{k+1}\chi_{[0,1]}\}_{k\in\mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$ with bound $B = 4$;
- $\{T_k\chi_{[0,1]} + T_{k+1}\chi_{[0,1]}\}_{k\in\mathbb{Z}}$ is not a frame sequence.
- $V := \overline{\text{span}}\{T_k\chi_{[0,1]} + T_{k+1}\chi_{[0,1]}\}_{k\in\mathbb{Z}} = \overline{\text{span}}\{T_k\chi_{[0,1]}\}_{k\in\mathbb{Z}}$.

Note that $T_k\chi_{[0,1]} + T_{k+1}\chi_{[0,1]} = T_k(\chi_{[0,1]} + T_1\chi_{[0,1]}) = T_k\chi_{[0,2]}$. Letting $g := \chi_{[0,2]}$, we therefore have an example of a Bessel sequence $\{T_k g\}_{k\in\mathbb{Z}}$ which does not form a frame sequence.

We will now construct a function $h \in V$ such that $\{T_k g\}_{k\in\mathbb{Z}} \cup \{T_k h\}_{k\in\mathbb{Z}}$ is a tight frame for V with bound 4. It is clear that functions in V are constant on intervals $[k, k+1], k \in \mathbb{Z}$. Thus, we can write any candidate for h as

$$h = \sum_{j\in\mathbb{Z}} a_j T_j \chi_{[0,1]}$$

for some coefficients $a_j \in \mathbb{C}$. It turns out that real-valued solutions exist, so we will assume that $a_j \in \mathbb{R}$. Now, if $\{T_k g\}_{k\in\mathbb{Z}} \cup \{T_k h\}_{k\in\mathbb{Z}}$ is a tight frame for V with bound 4, we have for any $f \in V$ that

$$\begin{aligned} 4f &= \sum_{k\in\mathbb{Z}} \langle f, T_k g\rangle T_k g + \sum_{k\in\mathbb{Z}} \langle f, T_k h\rangle T_k h \\ &= \sum_{k\in\mathbb{Z}} \langle f, T_k\chi_{[0,2]}\rangle T_k\chi_{[0,2]} + \sum_{k\in\mathbb{Z}} \langle f, T_k \sum_{j\in\mathbb{Z}} a_j T_j\chi_{[0,1]}\rangle T_k \sum_{j\in\mathbb{Z}} a_j T_j \chi_{[0,1]}. \quad (6) \end{aligned}$$

Considering the function $f := \chi_{[0,1]}$, some elementary calculations show that the sequence $\{a_j\}_{j\in\mathbb{Z}}$ must satisfy the following conditions:

$$\begin{aligned} 4 &= 2+\sum_{k\in\mathbb{Z}} a_k^2; \\ 0 &= 1+\sum_{k\in\mathbb{Z}} a_k a_{k-1}; \\ 0 &= 1+\sum_{k\in\mathbb{Z}} a_k a_{k+1}; \\ 0 &= \sum_{k\in\mathbb{Z}} a_k a_{k+j}, j \in \mathbb{Z}\setminus\{0,\pm1\}. \end{aligned}$$

This set of equations is solved by $a_0 = 1, a_1 = -1$ and $a_k = 0$ for $k \neq 0, 1$. Thus, the function

$$h = \chi_{[0,1]} - T_1\chi_{[0,1]}$$

satisfies (6) for the function $f = \chi_{[0,1]}$. It follows by standard manipulations that (6) therefore also holds for any function of the type $T_j\chi_{[0,1]},\ j \in \mathbb{Z}$, and therefore for any $f \in \mathrm{span}\{T_j\chi_{[0,1]}\}_{j\in\mathbb{Z}}$. By continuity, we conclude that (6) holds for all $f \in V$, as desired. □

With Proposition 2.2 at hand it is natural to ask for the *minimal* numbers of generators we have to add to a shift-invariant system in order to obtain a tight frame. We will now show that the result in Proposition 2.2 is optimal, in the sense that we can not expect to obtain a tight frame by adding a smaller number of generators than stated in the Proposition. In order to show this we will use techniques from [2].

Let $\mathcal{T} : L^2(\mathbb{R}) \to L^2(\mathbb{T}, \ell^2(\mathbb{Z}))$ be a fiberization mapping given by

$$\mathcal{T}f(\xi) = (\hat{f}(\xi + k))_{k\in\mathbb{Z}} \qquad \text{for } f \in L^2(\mathbb{R}),\ \xi \in \mathbb{T}.$$

Here, we identify the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with its fundamental domain $\mathbb{T} = [0, 1)$. By Plancherel's Theorem $\mathcal{T}$ is an isometric isomorphism.

Now, consider an at most countable family of functions $\{g_n\}_{n\in I}$ in $L^2(\mathbb{R})$. For any $\xi \in \mathbb{T}$, define a subspace

$$J(\xi) = \overline{\mathrm{span}}\{\mathcal{T}g_n(\xi) : n \in I\} \subset \ell^2(\mathbb{Z}).$$

The mapping $\xi \mapsto J(\xi)$ is called a (measurable) range function associated to $\{g_n\}_{n\in I}$. By [2, Theorem 2.3] we have the following result.

Theorem 2.4 *Let $\{g_n\}_{n\in I}$ be an at most countable family of functions in $L^2(\mathbb{R})$, and $0 < A \le B < \infty$. Then, $\{T_k g_n\}_{k\in\mathbb{Z},n\in I}$ is a frame sequence with bounds A and B (or a Bessel sequence with bound $B > 0$), if and only if $\{\mathcal{T} g_n(\xi)\}_{n\in I}$ is a frame for $J(\xi)$ with uniform bounds A and B for a.e. $\xi \in \mathbb{T}$ (resp., a Bessel sequence with bound $B > 0$).*

The example below illustrates the announced optimality of Proposition 2.2.

Example 2.5 Let $I = \{1, \dots, N\}$ for some $N \in \mathbb{N}$ or $I = \mathbb{N}$. For each $n \in I$ define $g_n \in L^2(\mathbb{R})$ by $\hat{g}_n(\xi) = (\xi - n)\chi_{[n,n+1]}(\xi)$. Observe that for $\xi \in [0, 1)$,

$$\mathcal{T} g_n(\xi) = (\hat{g}_n(\xi + k))_{k\in\mathbb{Z}} = ((\xi + k - n)\chi_{[n,n+1]}(\xi + k))_{k\in\mathbb{Z}} = \xi e_n, \tag{7}$$

where $\{e_n\}$ is the canonical basis in $\mathbb{C}^N$ (resp. $\ell^2(\mathbb{N})$). Thus, $\{\mathcal{T} g_n(\xi)\}_{n\in I}$ is a scalar multiple (by ξ) of an orthonormal basis for the space $J(\xi) = \overline{\operatorname{span}}\{e_n : n \in I\}$, which defines a range function associated to $\{g_n\}_{n\in I}$. By Theorem 2.4, $\{g_n\}_{n\in I}$ is a Bessel sequence with bound 1. However, $\{g_n\}_{n\in I}$ is not a frame sequence, since the lower frame bound of $\{\mathcal{T} g_n(\xi)\}_{n\in I}$ approaches 0 as $\xi \to 0$. Furthermore, we claim that $\{g_n\}_{n\in I}$ can not be extended to a frame by adding fewer generators than that of $\{g_n\}_{n\in I}$. For simplicity we shall consider only the case where I finite; the case $I = \mathbb{N}$ follows by a trivial modification.

Take any functions $\{h_n\}_{n\in I'}$ such that $I' = \{1, \dots, N'\}$ for some $N' < N$. For each $\xi \in \mathbb{T}$, define the frame operator $S_2(\xi)$ corresponding to a system of fibers $\{\mathcal{T} h_n(\xi)\}_{n\in I'}$. Observe that $S_2(\xi)$ is a bounded positive operator of rank at most N'. By (7), the frame operator of $\{\mathcal{T} g_n(\xi)\}_{n\in I}$ is $S_1(\xi) = \xi P_N$, where P_N is the orthogonal projection of $\ell^2(\mathbb{Z})$ onto $\operatorname{span}\{e_1, \dots, e_N\}$. Hence, the frame operator of their union $\{\mathcal{T} g_n(\xi)\}_{n\in I} \cup \{\mathcal{T} h_n(\xi)\}_{n\in I'}$ is simply $S_1(\xi) + S_2(\xi)$. Since $N' < N$, there exists a unit norm vector $v = v(\xi) \in \operatorname{span}\{e_1, \dots, e_N\}$ such that $S_2(\xi)v = 0$. Thus,

$$\langle (S_1(\xi) + S_2(\xi))v, v\rangle = \langle \xi P_N v, v\rangle = \xi.$$

This shows that the lower frame bound of a frame sequence $\{\mathcal{T} g_n(\xi)\}_{n\in I} \cup \{\mathcal{T} h_n(\xi)\}_{n\in I'}$ is **at most** ξ, Thus, it approaches 0 as $\xi \to 0$. By Theorem 2.4, $\{T_k g_n\}_{k\in\mathbb{Z},n\in I} \cup \{T_k h_n\}_{k\in\mathbb{Z},n\in I'}$ is not a frame sequence.

This example shows the optimality of Proposition 2.2. Finally, note that the functions $\{h_n\}_{n\in I}$ given by $\hat{h}_n(\xi) = (1 - \xi + n)\chi_{[n,n+1]}(\xi)$ fulfill the conclusion of this proposition. That is, $\{T_k g_n\}_{k\in\mathbb{Z},n\in I} \cup \{T_k h_n\}_{k\in\mathbb{Z},n\in I}$ is a tight frame sequence with bound 1. □

It turns out that the result obtained un Example 2.3 via explicit calculations is a special case of a general result that can be deduced using the fiberization technique:

Theorem 2.6 *Let $\phi \in L^2(\mathbb{R})$ be such that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal sequence. Let $g = \sum_{k\in\mathbb{Z}} a_k T_k\phi$ be a finite linear combination with real coefficients a_k. Then, there exists a generator $h \in L^2(\mathbb{R})$ of the same form $h = \sum_{k\in\mathbb{Z}} b_k T_k\phi$, with all but finitely many b_k's zero, such that $\{T_k g\}_{k\in\mathbb{Z}} \cup \{T_k h\}_{k\in\mathbb{Z}}$ is a tight frame for the space $V := \overline{span}\{T_k g\}_{k\in\mathbb{Z}} = \overline{span}\{T_k\phi\}_{k\in\mathbb{Z}}$.*

Proof. Let $\Phi(\xi) = \mathcal{T}\phi(\xi) = (\widehat{\phi}(\xi + k))_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ for a.e. $\xi \in \mathbb{T}$. Since $\{T_k\phi\}_{k\in\mathbb{Z}}$ is an orthonormal sequence, we have $||\Phi(\xi)|| = 1$ a.e. Clearly, $\mathcal{T}g(\xi) = m(\xi)\Phi(\xi)$, where m is a trigonometric polynomial given by $m(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-2\pi i k\xi}$. By Theorem 2.4, $\{T_k g\}_{k\in\mathbb{Z}}$ is a frame sequence if and only if $\inf_{\xi\in\mathbb{T}} |m(\xi)| > 0$. In this case, the (optimal) frame bounds of this systems are $A = \inf_{\xi\in\mathbb{T}} |m(\xi)|^2$ and $B = \sup_{\xi\in\mathbb{T}} |m(\xi)|^2$. The presence of the exponent 2 is an artifact of the definition (1). Moreover, since $m(\xi) \neq 0$ for a.e. ξ, the range functions corresponding to the principal SI spaces generated by ϕ and g are the same. Thus, by [2, Proposition 1.5]

$$V := \overline{\text{span}}\{T_k g\}_{k\in\mathbb{Z}} = \overline{\text{span}}\{T_k\phi\}_{k\in\mathbb{Z}}.$$

We shall prove that there exists a generator $h = \sum_{k\in\mathbb{Z}} b_k T_k\phi$, with all but finitely many b_k's zero, such that $\{T_k g\}_{k\in\mathbb{Z}} \cup \{T_k h\}_{k\in\mathbb{Z}}$ is a tight frame sequence with bound B. Indeed, we have $\mathcal{T}h(\xi) = \tilde{m}(\xi)\Phi(\xi)$, where $\tilde{m}(\xi) = \sum_{k\in\mathbb{Z}} b_k e^{-2\pi i k\xi}$. By Theorem 2.4, the required property for g and h holds if and only $\{\mathcal{T}g(\xi), \mathcal{T}h(\xi)\}$ is a tight frame sequence with bound B for a.e. $\xi \in \mathbb{T}$. Since these two vectors are multiples of $\Phi(\xi)$ this is equivalent to

$$|m(\xi)|^2 + |\tilde{m}(\xi)|^2 = B \qquad \text{for a.e. } \xi \in \mathbb{T}.$$

Observe that $M(\xi) = B - |m(\xi)|^2$ is a non-negative trigonometric polynomial with real coefficients. By the Fejér-Riesz lemma, see [6, Lemma 6.1.3] and [9], there exists a trigonometric polynomial $\tilde{m}(\xi) = \sum_{k\in\mathbb{Z}} b_k e^{-2\pi i k\xi}$ such that $M(\xi) = |\tilde{m}(\xi)|^2$ for all $\xi \in \mathbb{T}$. In turn, this polynomial yields the coefficients for the required generator h.

Finally, observe that we can allow coefficients a_k to be complex. Indeed, a complex coefficient variant of the Fejér-Riesz lemma, see [11, Theorem A.11.3], yields the polynomial $\tilde{m}$ with complex coefficients b_k. □

We can quickly deduce Example 2.3 via Theorem 2.6. Indeed, suppose that $\phi = \chi_{[0,1]}$ and $g = \chi_{[0,2]} = \chi_{[0,1]} + T_1\chi_{[0,1]}$. Then, $m(\xi) = 1 + e^{-2\pi i\xi}$ and $|m(\xi)|^2 = |1 + e^{-2\pi i\xi}|^2 = 2 + 2\cos(2\pi\xi)$. By choosing $\tilde{m}(\xi) = 1 - e^{-2\pi i\xi}$ we have $|\tilde{m}(\xi)|^2 = 2 - 2\cos(2\pi\xi)$. Thus,

$$|m(\xi)|^2 + |\tilde{m}(\xi)|^2 = 4 \qquad \text{for } \xi \in \mathbb{T}.$$

Consequently, by letting $h = \chi_{[0,1]} - T_1\chi_{[0,1]}$, $\{T_k g\}_{k\in\mathbb{Z}} \cup \{T_k h\}_{k\in\mathbb{Z}}$ is a tight frame sequence with bound 4.

In general, the construction of the function h in Proposition 2.2 involves the extraction of the square root of a positive operator. This can be avoided by constructing a pair of dual frames instead of a tight frame:

Corollary 2.7 *Let $\{g_n\}_{n\in I}$ be an at most countable family of functions in $L^2(\mathbb{R})$ and assume that $\{T_k g_n\}_{k\in\mathbb{Z},n\in I}$ is a Bessel sequence. Then there exist functions $\{\varphi_n\}_{n\in I}, \{\psi_n\}_{n\in I} \in V$ such that $\{T_k g_n\}_{k\in\mathbb{Z},n\in I} \cup \{T_k\varphi_n\}_{k\in\mathbb{Z},n\in I}$ and $\{T_k g_n\}_{k\in\mathbb{Z},n\in I} \cup \{T_k\psi_n\}_{k\in\mathbb{Z},n\in I}$ form a pair of dual frames for V. Explicitly, we can take $\{\varphi_n\}_{n\in I}$ as in Lemma 2.1 and let $\psi_n := (I - S_1)\varphi_n$ with S_1 defined as in* (5).

Proof. Let S_1 be defined by (5). For any $f \in V$, applying the frame decomposition associated with the frame sequence $\{T_k\varphi_n\}_{k\in\mathbb{Z},n\in I}$ from Lemma 2.1 we can write f as follows:

$$\begin{aligned} f &= S_1 f + (I - S_1)f \\ &= \sum_{k\in\mathbb{Z},n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in\mathbb{Z},n\in I} \langle (I - S_1)f, T_k\varphi_n\rangle T_k\varphi_n \\ &= \sum_{k\in\mathbb{Z},n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in\mathbb{Z},n\in I} \langle f, (I - S_1)^* T_k\varphi_n\rangle T_k\varphi_n. \end{aligned} \tag{8}$$

It is clear that $(I - S_1)^* = I - S_1$. Now, for any $h \in V$, from Lemma 7.2.1 in [4] we have

$$(I - S_1)T_k h = T_k h - S_1 T_k h = T_k h - T_k S_1 h = T_k(I - S_1)h.$$

Hence

$$(I - S_1)T_k = T_k(I - S_1).$$

Put $\psi_n := (I - S_1)\varphi_n$. Then (8) becomes

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z},n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in\mathbb{Z},n\in I} \langle f, (I-S_1)T_k\varphi_n\rangle T_k\varphi_n \\ &= \sum_{k\in\mathbb{Z},n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in\mathbb{Z},n\in I} \langle f, T_k(I-S_1)\varphi_n\rangle T_k\varphi_n \\ &= \sum_{k\in\mathbb{Z},n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in\mathbb{Z},n\in I} \langle f, T_k\psi_n\rangle T_k\varphi_n. \end{aligned}$$

This proves that $\{T_k g_n\}_{k\in\mathbb{Z},n\in I} \cup \{T_k\varphi_n\}_{k\in\mathbb{Z},n\in I}$ and $\{T_k g_n\}_{k\in\mathbb{Z},n\in I} \cup \{T_k\psi_n\}_{k\in\mathbb{Z},n\in I}$ form a pair of dual frames for V. $\square$

3 Extensions of $\{T_k g_n\}_{k\in\mathbb{Z},n\in I}$ to frames for $L^2(\mathbb{R})$

So far we have only considered extensions of a shift-invariant system to a frame for the Hilbert space spanned by the given system. The next result deals with extensions to frames for $L^2(\mathbb{R})$.

Proposition 3.1 *Suppose that $\{T_k g_n\}_{k\in\mathbb{Z},n\in I}$ is a Bessel sequence in $L^2(\mathbb{R})$ with the upper bound B. For any $\lambda \geq B$, there is a sequence $\{h_m\}_{m\in\mathbb{Z}}$ such that $\{T_k g_n\}_{k\in\mathbb{Z},n\in I} \cup \{T_k h_m\}_{m,k\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ with the bound λ.*

Proof. Let

$$S_1 f = \sum_{k\in\mathbb{Z},n\in I} \langle f, T_k g_n\rangle T_k g_n, \ \forall f \in L^2(\mathbb{R}). \tag{9}$$

Since $\{T_k g_n\}_{k\in\mathbb{Z},n\in I}$ is a Bessel sequence with bound B, S_1 is self-adjoint and positive and satisfies

$$\langle S_1 f, f\rangle \leq B\|f\|_2^2, \ \forall f \in L^2(\mathbb{R}).$$

For any $\lambda \geq B$, define $S_2 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$S_2 = \lambda I - S_1.$$

Then S_2 is positive and commutes with T_n. From Lemma 2.4.4 in [4] we know that $S_2^{1/2}$ exists, is self-adjoint and can be expressed as a limit of a sequence of polynomials in S_2. Accordingly, $S_2^{1/2}$ commutes with T_n.

Take $\{T_k\Psi_m\}_{m,k\in\mathbb{Z}}$ to be a tight frame for $L^2(\mathbb{R})$ with frame bound 1. Like in the proof of Proposition 2.2 we see that

$$\begin{aligned} S_2 f &= S_2^{1/2} S_2^{1/2} f \\ &= S_2^{1/2} \sum_{m,k\in\mathbb{Z}} \langle S_2^{1/2} f, T_k\Psi_m\rangle T_k\Psi_m \\ &= \sum_{m,k\in\mathbb{Z}} \langle f, S_2^{1/2} T_k\Psi_m\rangle S_2^{1/2} T_k\Psi_m. \\ &= \sum_{m,k\in\mathbb{Z}} \langle f, T_k S_2^{1/2}\Psi_m\rangle T_k S_2^{1/2}\Psi_m. \end{aligned}$$

Put $h_m = S_2^{1/2}\Psi_m$. Since $S_1 + S_2 = \lambda I$, it follows that

$$\{T_k g_n\}_{k\in\mathbb{Z},n\in I} \cup \{T_k h_m\}_{m,k\in\mathbb{Z}}$$

is a tight frame for $L^2(\mathbb{R})$ with frame bound λ. $\square$

We remark that most of the results of this paper hold for shift-invariant systems in $L^2(\mathbb{R}^d)$ for any dimension $d \geq 1$. This is because the theory of SI spaces trivially extends to the higher dimensional setting by replacing the lattice of shifts $\mathbb{Z}$ by $\mathbb{Z}^d$. The only result that does not immediately extends to the higher dimensional setting is Theorem 2.6, which uses in a crucial way the Fejér-Riesz lemma whose application is restricted to the one dimensional setting, see [7, 8].

A Appendix A

The full proof of Lemma 2.1 (Theorem 3.3 in [2]) is quite involved. In order for the reader to get insight into the basic idea we give an elementary proof for the case of one generator below.

Lemma A.1 *Let $g \in L^2(\mathbb{R})$ and assume that $\{T_k g\}_{k\in\mathbb{Z}}$ is a Bessel sequence with bound $B > 0$. Then there exists a function $\varphi \in V$ such that $\{T_k\varphi\}_{k\in\mathbb{Z}}$ is a tight frame for V.*

Proof. Consider the function

$$G(\gamma) := \sum_{k\in\mathbb{Z}} |\hat{g}(\gamma+k)|^2. \tag{10}$$

It is well known that G is bounded, $G(\gamma) \le B$, see, e.g., Lemma 7.1.4 in [4]. Let $N := \{\gamma \in \mathbb{R} \mid G(\gamma) = 0\}$, and define the function Φ by

$$\Phi(\gamma) := \begin{cases} \frac{\hat{g}(\gamma)}{G(\gamma)^{1/2}}, & \text{if } \gamma \notin N, \\ 0, & \text{if } \gamma \in N. \end{cases}$$

Then

$$\begin{aligned} \int_{-\infty}^{\infty} |\Phi(\gamma)|^2 \, d\gamma &= \int_{\mathbb{R}\setminus N} \left| \frac{\hat{g}(\gamma)}{G(\gamma)^{1/2}} \right|^2 d\gamma \\ &= \sum_{k\in\mathbb{Z}} \int_{[0,1]\setminus N} \left| \frac{\hat{g}(\gamma+k)}{G(\gamma+k)^{1/2}} \right|^2 d\gamma. \end{aligned}$$

Since G is 1-periodic, this implies that

$$\int_{-\infty}^{\infty} |\Phi(\gamma)|^2 \, d\gamma = \int_{[0,1]\setminus N} \frac{1}{G(\gamma)} \sum_{k\in\mathbb{Z}} |\hat{g}(\gamma+k)|^2 \, d\gamma = \int_{[0,1]\setminus N} d\gamma < 1.$$

Thus $\Phi \in L^2(\mathbb{R})$. We can therefore define a function $\phi \in L^2(\mathbb{R})$ by $\hat{\phi} := \Phi$. Clearly, $\sum_{k\in\mathbb{Z}} |\hat{\phi}(\gamma+k)|^2 = 0$ if $\gamma \in N$. For $\gamma \notin N$,

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(\gamma+k)|^2 = \sum_{k\in\mathbb{Z}} \left| \frac{\hat{g}(\gamma+k)}{G(\gamma+k)^{1/2}} \right|^2 = 1.$$

By Theorem 7.1.7 in [4] this implies that the set $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a tight frame for

$$W := \overline{\text{span}}\{T_k\phi\}_{k\in\mathbb{Z}}, \tag{11}$$

with frame bound 1. We also note that $\hat{g} = G^{1/2}\hat{\phi}$. By Lemma 7.1.11 in [4] this implies that $g \in W$. Since W is a shift-invariant and closed subspace of $L^2(\mathbb{R})$, we conclude that

$$V \subseteq W. \tag{12}$$

Let P denote the orthogonal projection of $L^2(\mathbb{R})$ onto V. Directly from the frame definition it follows that $\{PT_k\phi\}_{k\in\mathbb{Z}}$ is a tight frame for V, also with frame bound 1. We will now show that the operators P and T_n commute for all $n \in \mathbb{Z}$; taking $\varphi := P\phi$ this will conclude the proof.

First, for any function $f \in \operatorname{span}\{T_k g\}_{k\in\mathbb{Z}}$ and any $n \in \mathbb{Z}$, we have that $T_n f \in \operatorname{span}\{T_k g\}_{k\in\mathbb{Z}}$. Thus,

$$PT_n f = T_n f = T_n P f.$$

By continuity, this implies that $PT_n = T_n P$ on the subspace V. Now, if $f \in V^\perp$ and $g \in L^2(\mathbb{R})$ is arbitrary,

$$\langle PT_n f, g\rangle = \langle T_n f, Pg\rangle = \langle f, T_{-n} Pg\rangle = 0$$

because $Pg \in V$ and V is shift-invariant. Thus $PT_n f = 0$. By a similar proof, $T_n P f = 0$, which shows that $PT_n = T_n P$ on $V^\perp$, as desired. $\square$

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