ON WILSON BASES IN $L^2(\mathbb{R}^d)$[*]

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Abstract. A Wilson system is a collection of finite linear combinations of time frequency shifts of a square integrable function. It is well known that, starting from a tight Gabor frame for $L^2(\mathbb{R})$ with redundancy 2, one can construct an orthonormal Wilson basis for $L^2(\mathbb{R})$ whose generator is well localized in the time-frequency plane. In this paper we use the fact that a Wilson system is a shift-invariant system to explore its relationship with Gabor systems. Specifically, we show that one can construct d-dimensional orthonormal Wilson bases starting from tight Gabor frames of redundancy 2^k, where $k = 1, 2, \ldots, d$. These results generalize most of the known results about the existence of orthonormal Wilson bases.

Key words. frame, Gabor system, orthonormal basis, shift-invariant system, symplectic, time-frequency analysis, Wilson system

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1. Introduction. One of the goals in signal processing and time-frequency analysis is to find convenient series expansions of functions in $L^2(\mathbb{R}^d)$. Examples of such series expansions include Gabor (also called Weyl–Heisenberg) frames. In order to describe these systems we introduce the translation operator T_λ and the modulation operator M_γ:

$$T_\lambda f(x) = f(x - \lambda), \quad M_\gamma f(x) = e^{2\pi i \langle x, \gamma \rangle} f(x), \quad f \in L^2(\mathbb{R}^d), \, \lambda, \gamma \in \mathbb{R}^d.$$

A Gabor system generated by the window function $g \in L^2(\mathbb{R}^d)$ is the set of functions given by $\{M_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$, where Λ and Γ are lattices in $\mathbb{R}^d$. Since modulation is a translation in the frequency domain, the operation $M_\gamma T_\lambda$ is called a time-frequency shift. Now, a Gabor frame for $L^2(\mathbb{R}^d)$ is a system of the form $\{M_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ for which there exist constants $a, b > 0$ such that

$$(1) \qquad a \|f\|^2 \le \sum_{\lambda \in \Lambda, \gamma \in \Gamma} |\langle f, M_\gamma T_\lambda g \rangle|^2 \le b \|f\|^2 \quad \text{for all} \quad f \in L^2(\mathbb{R}^d).$$

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In the case that $\{M_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ satisfies (1), there exists a function $h \in L^2(\mathbb{R}^d)$ such that

$$(2) \qquad f = \sum_{\lambda\in\Lambda,\gamma\in\Gamma} \langle f, M_\gamma T_\lambda g\rangle M_\gamma T_\lambda h \quad \text{for all } f \in L^2(\mathbb{R}^d)$$

with unconditionally L^2-convergence. Whenever the product of the volume of the two fundamental domains of the full-rank lattices Λ and Γ is strictly less than one, there exist nice window functions $g \in L^2(\mathbb{R})$, e.g., in the Schwartz class or the Feichtinger algebra, such that the Gabor system $\{M_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ is a frame [16].

In many applications in engineering and mathematics, it is desirable not only to have smooth and localized generators $g \in L^2(\mathbb{R}^d)$ but also *orthogonal* expansions. However, for Gabor frames this is not possible. Indeed, the famous Balian–Low theorem [3, 4, 5, 10, 12, 23] states that if a Gabor system is an orthonormal basis or a Riesz basis for $L^2(\mathbb{R}^d)$, then g cannot have rapid decay in both time and frequency.

Yet, in 1991, Daubechies, Jaffard, and Journé [11], inspired by work of Wilson [28], were able to construct an orthonormal basis of (linear combinations of) time-frequency shifts of a univariate function $g \in L^2(\mathbb{R})$ with good time and frequency localization. The so-called Wilson systems considered in [11] are given as

$$\mathcal{W}(g) = \{T_n g\}_{n\in\mathbb{Z}} \cup \left\{ \tfrac{1}{\sqrt{2}} T_n \left(M_m + (-1)^m M_{-m}\right) g \right\}_{n\in\mathbb{Z}, m\in\mathbb{N}}$$

$$\cup \left\{ \tfrac{1}{\sqrt{2}} T_n T_{1/2} (M_m - (-1)^m M_{-m}) g \right\}_{n\in\mathbb{Z}, m\in\mathbb{N}}.$$

From this definition it is clear that, except from the pure translations $\{T_n g\}_{n\in\mathbb{Z}}$, the Wilson systems produce a bimodular covering of the frequency line, in the sense that each element of the system has two peaks in its power spectrum $|\hat{g}|^2$, assuming the window function is sufficiently localized in frequency. This should be compared with the unimodular Gabor system, where each element of $\{M_\gamma T_\lambda g\}_{\lambda\in\Lambda,\gamma\in\Gamma}$ has a single peak in the power spectrum. As the following main result of [11] shows, Wilson systems do not suffer from the restrictions of the Balian–Low theorem.

THEOREM 1.1 (see [11]). *Let $g \in L^2(\mathbb{R})$ be such that $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$ and $\|g\|_2 = 1$. Then the Gabor system $\{M_m T_{n/2} g\}_{m,n\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$ if, and only if, the Wilson system $\mathcal{W}(g)$ is an orthonormal basis for $L^2(\mathbb{R})$.*

The construction of Wilson bases using Theorem 1.1 can be illustrated by the following examples.

Example 1.
(a) Consider the function $g(x) = \cos(\tfrac{1}{2}\pi x)\mathbb{1}_{[-1,1]}(x)$. One can easily show, e.g., by Lemma 3.4, that $\{M_{m/2} T_n g\}_{m,n\in\mathbb{Z}}$ is a tight Gabor frame with frame bound $A = 2$ and $\|g\|_2 = 1$. Moreover, $g(x) = \overline{g(-x)}$ for all $x \in \mathbb{R}$ and $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$ for all $\omega \in \mathbb{R}$. Thus the Wilson system generated by this function is an orthonormal basis for $L^2(\mathbb{R})$.
(b) Consider the function $g(x) = (\sqrt{-|x|+1})\mathbb{1}_{[-1,1]}(x)$. As above one can easily show that $\{M_{m/2} T_n g\}_{m,n\in\mathbb{Z}}$ is a tight Gabor frame with frame bound $A = 2$ and $\|g\|_2 = 1$. Moreover, $g(x) = \overline{g(-x)}$ and $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$ for all $x, \omega \in \mathbb{R}$. Thus the Wilson system generated by this function is an orthonormal basis for $L^2(\mathbb{R})$.

An important point of Theorem 1.1, which is also illustrated by the above two examples, is that starting from a tight Gabor frame with redundancy 2, it is possible

to construct an orthonormal Wilson basis for $L^2(\mathbb{R})$ whose generator is well localized in time and frequency, e.g., the generator can be chosen to be a Schwartz class function or a $\mathcal{C}^\infty$ function with compact support. It easily follows that a tensor product of this orthonormal Wilson basis will lead to an orthonormal basis for $L^2(\mathbb{R}^d)$. But beyond this method, it is not known how to construct multidimensional orthonormal Wilson bases for $L^2(\mathbb{R}^d)$.

Tensoring Wilson bases to $L^2(\mathbb{R}^d)$ has several undesirable side effects. First, the basis functions of a tensored Wilson basis are 2^d-modular, and hence they give rise to a 2^d-modular covering of the frequency domain, akin to the situation of separable wavelets in $L^2(\mathbb{R}^d)$. Gabor frames $\{M_\gamma T_\lambda g\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ are unimodular in all dimensions and hence give rise to a unimodular covering of the frequency domain. Bimodular coverings are in most applications as good as unimodular coverings, in particular if the signals of interest $f \in L^2(\mathbb{R}^d)$ are real-valued. However, 2^d-modular tensor coverings have a curse of dimensionality since, e.g., symmetric peaks of the power spectrum of real-valued signals will leak out to 2^{d-1} other locations in frequency. Second, they are associated with highly redundant Gabor frames of redundancy 2^d. Third, the generating function has to, naturally, be a separable function of the form $g_1(x_1) \ldots g_d(x_d)$. Our goal in this paper is to construct Wilson orthonormal bases in higher dimensions that do not suffer from these tensoring artifacts. Using the frame theory of shift-invariant systems [6, 9, 18, 20, 24], we construct orthonormal Wilson bases for $L^2(\mathbb{R}^d)$ starting from tight Gabor frames of redundancy 2^k, $k = 1, \ldots, d$. This provides us with a family of Wilson orthonormal bases with 2^k-modular covering of the frequency domain with k ranging from 1 to d. When $k = d$ we recover the tensored Wilson bases, but when $k = 1$ we obtain a bimodular Wilson orthonormal basis for $L^2(\mathbb{R}^d)$. As we will see, the latter construction of bimodular Wilson orthonormal bases is in many ways superior to the tensored Wilson system.

Our results also shed new light on univariate (as well as multivariate) Wilson systems. We show that, whenever one of the two is well-defined, the frame operators of Gabor and Wilson systems are identical up to scalar multiplication. We present the view that Wilson system share several properties with the adjoint of the Gabor system. First, Gabor and Wilson systems satisfy a duality principle: the Gabor system is a frame if and only if the Wilson system is a Riesz basis, and we provide frame bounds. Second, Wilson systems satisfy a density-type theorem: if a Wilson system is a frame or a tight frame, then it is automatically a Riesz basis or an orthonormal basis, respectively.

The rest of the paper is organized as follows. In section 2 we recall a number of elementary facts about the symplectic matrices and their role in time-frequency analysis. Section 3 presents necessary results from the theory of shift-invariant systems and concerns bimodular Wilson orthonormal bases for $L^2(\mathbb{R}^d)$ constructed from redundancy 2 tight Gabor frames. In particular, the main result of this section is Theorem 3.1, which generalizes Theorem 1.1. Furthermore, even for $d = 1$ the results of section 3 yield a more general statement than Theorem 2.1 stated in section 2. Finally, in section 4, we consider Wilson orthonormal bases (Riesz bases) generated from tight (nontight) Gabor frames of redundancy 2^k for $k = 0, 1, 2, \ldots, d$. In particular, the main results of section 4 are Theorem 4.5 and Proposition 4.8. Theorem 4.5 is a generalization of Theorem 3.1. However, in order to improve readability and understanding we keep the proof of Theorem 3.1 as a model case for the more technical Theorem 4.5.

2. Wilson systems and symplectic matrices. In this section we collect some facts about symplectic matrices, which are needed for the proof of Corollary 3.2 and Proposition 4.8.

But first, we recall the most general known result concerning univariate Wilson bases due to Kutyniok and Strohmer [22]. Similar results can be found in the paper by Wojdyłło [30]. We point out that the lattice used to define the tight Gabor frame in Theorem 2.1 is the image of $\mathbb{Z}^2$ under a symplectic matrix.

THEOREM 2.1 (see [22]).   *Let $a, b, c > 0$ and $g \in L^2(\mathbb{R})$ be given. If $ab = 1/2$, $\hat{g}(\omega)e^{2\pi i c \omega^2/b} = \overline{\hat{g}(\omega)}$, and $\|g\|_2 = 1$, then the following assertions are equivalent:*

(i)   *The Gabor system $\{T_{na+mc}M_{mb}g\}_{m,n\in\mathbb{Z}}$ is a tight frame for $L^2(\mathbb{R})$.*

(ii)   *The Wilson system*

$$\mathcal{W}(g) = \{T_{2na}g\}_{n\in\mathbb{Z}} \cup \left\{ \tfrac{1}{\sqrt{2}}T_{2na}(T_{mc}M_{mb} + (-1)^m T_{-mc}M_{-mb})g \right\}_{n\in\mathbb{Z},m\in\mathbb{N}}$$

$$\cup \left\{ \tfrac{1}{\sqrt{2}}T_{2na}T_a(T_{mc}M_{mb} - (-1)^m T_{-mc}M_{-mb})g \right\}_{n\in\mathbb{Z},m\in\mathbb{N}}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

In Theorem 2.1 it is a slight abuse of language to speak of $\{T_{na+mc}M_{mb}g\}_{m,n\in\mathbb{Z}}$ as a Gabor system; however, since it is unitarily equivalent with the Gabor system $\{M_{mb}T_{na+mc}g\}_{m,n\in\mathbb{Z}}$, these systems share all frame theoretic properties, and we will not make any distinction between such systems in the remainder of this paper.

In addition to the translation operator T_λ and the modulation operator M_γ introduced in section 1, we define the following operators on $L^2(\mathbb{R}^d)$. For $C \in \mathrm{GL}_\mathbb{R}(d)$, we define the *dilation* by C:

$$D_C f(x) = |\det C|^{1/2} f(Cx).$$

For a real-valued, symmetric, $d \times d$ matrix M, we define the *chirp-multiplication* by M:

$$S_M f(x) = e^{\pi i \langle x, Mx \rangle} f(x).$$

The *Fourier transform* is defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i \langle \omega, x \rangle}\, dx,$$

which extends to all of $L^2(\mathbb{R}^d)$ by density. One readily shows that all the mentioned operators are unitary operators on $L^2(\mathbb{R}^d)$ with

$$(T_a)^{-1} = (T_a)^* = T_{-a}, \qquad (M_b)^{-1} = (M_b)^* = M_{-b},$$
$$(D_C)^{-1} = (D_C)^* = D_{C^{-1}}, \quad (S_M)^{-1} = (S_M)^* = S_{-M}$$

and $\mathcal{F}^{-1}f(\omega) = \mathcal{F}^*f(\omega) = \mathcal{F}f(-\omega)$ for $f \in L^2(\mathbb{R}^d)$.

For $\nu = (\lambda, \gamma) \in \mathbb{R}^d \times \mathbb{R}^d$, we let $\pi(\nu)$ denote the time-frequency shift operator $M_\gamma T_\lambda$. It is clear that $\pi(\nu)$ is a unitary operator on $L^2(\mathbb{R}^d)$.

The Fourier transform, dilation operator, and chirp-multiplication operator intertwine with a time-frequency shift $\pi(\nu)$, $\nu \in \mathbb{R}^d \times \mathbb{R}^d$ in the following way:

$$(3) \qquad \mathcal{F}\,\pi(\nu) = e^{2\pi i \langle \lambda, \gamma \rangle}\, \pi\!\left(\left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix} \right] \nu \right) \mathcal{F},$$

$$(4) \qquad D_C\,\pi(\nu) = \pi\!\left(\left[\begin{smallmatrix} C^{-1} & 0 \\ 0 & C^\top \end{smallmatrix} \right] \nu \right) D_C,$$

$$(5) \qquad S_M\,\pi(\nu) = e^{-\pi i \langle \lambda, M\lambda \rangle}\, \pi\!\left(\left[\begin{smallmatrix} I & 0 \\ M & I \end{smallmatrix} \right] \nu \right) S_M.$$

Because of these relations we associate to the Fourier transform, dilation, and chirp multiplication operator the following $2d \times 2d$-matrices:

$$(6) \qquad \mathcal{F} \longleftrightarrow \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad D_C \longleftrightarrow \begin{bmatrix} C^{-1} & 0 \\ 0 & C^\top \end{bmatrix}, \quad S_M \longleftrightarrow \begin{bmatrix} I & 0 \\ M & I \end{bmatrix},$$

where I is the $d \times d$ identity matrix. The three matrices in (6) play an important role in the theory of symplectic matrices:

DEFINITION 2.2. *A matrix $A \in GL_\mathbb{R}(2d)$ is a symplectic matrix if*

$$A^\top J A = J \quad \text{with } J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The set of all symplectic matrices is denoted by $Sp(d)$.

THEOREM 2.3 (see [13, 15]). *All symplectic matrices can be written as a (non-unique) finite composition of matrices of the form as in* (6).

We have that $\mathrm{Sp}(1) = \mathrm{SL}_\mathbb{R}(2)$, while for $d \geq 2$ the symplectic matrices $\mathrm{Sp}(d)$ are a proper subgroup of $\mathrm{SL}_\mathbb{R}(2d)$. It is advantageous to write symplectic matrices as block matrices of the form

$$A = \begin{bmatrix} K & L \\ Q & R \end{bmatrix},$$

where K, L, Q, and R are real valued, $d \times d$ matrices. One can show that the following statements are equivalent:
  (i) $A = \begin{bmatrix} K & L \\ Q & R \end{bmatrix} \in \mathrm{Sp}(d)$.
  (ii) $K^\top Q$ and $L^\top R$ are symmetric matrices and $K^\top R - Q^\top L = I$.
  (iii) $KL^\top$ and $QR^\top$ are symmetric matrices and $KR^\top - LQ^\top = I$.
We mention the following important decompositions of symplectic matrices into products of matrices of the form as in (6).

Example 2. Let $A = \begin{bmatrix} K & L \\ Q & R \end{bmatrix} \in \mathrm{Sp}(d)$.
  (i) If $K \in \mathrm{GL}_\mathbb{R}(d)$, then

$$A = \begin{bmatrix} I & 0 \\ QK^{-1} & I \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & (K^\top)^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -K^{-1}L & I \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

  (ii) If $L \in \mathrm{GL}_\mathbb{R}(d)$, then

$$A = \begin{bmatrix} I & 0 \\ RL^{-1} & I \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & (L^\top)^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ L^{-1}K & I \end{bmatrix}.$$

  (iii) If $Q \in \mathrm{GL}_\mathbb{R}(d)$, then

$$A = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -KQ^{-1} & I \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & (Q^\top)^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -Q^{-1}R & I \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

  (iv) If $R \in \mathrm{GL}_\mathbb{R}(d)$, then

$$A = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -LR^{-1} & I \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & (R^\top)^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ R^{-1}Q & I \end{bmatrix}.$$

Note that this list of examples does not cover all $A \in \mathrm{Sp}(d)$ as there exist symplectic matrices for which each of their block components K, L, Q, and R has zero determinant. To each matrix A in Example 2, we associate a unitary operator via the relations in (6).

Example 3. To $A = \begin{bmatrix} K & L \\ Q & R \end{bmatrix} \in \mathrm{Sp}(d)$ we associate the following operators:

(i) If $K \in \mathrm{GL}_{\mathbb{R}}(d)$, then

$$A \longleftrightarrow S_{QK^{-1}} \circ D_{K^{-1}} \circ \mathcal{F} \circ S_{-K^{-1}L} \circ \mathcal{F}^{-1}.$$

(ii) If $L \in \mathrm{GL}_{\mathbb{R}}(d)$, then

$$A \longleftrightarrow S_{RL^{-1}} \circ D_{L^{-1}} \circ \mathcal{F} \circ S_{L^{-1}K}.$$

(iii) If $Q \in \mathrm{GL}_{\mathbb{R}}(d)$, then

$$A \longleftrightarrow \mathcal{F}^{-1} \circ S_{-KQ^{-1}} \circ D_{Q^{-1}} \circ \mathcal{F} \circ S_{-Q^{-1}R} \circ \mathcal{F}^{-1}.$$

(iv) If $R \in \mathrm{GL}_{\mathbb{R}}(d)$, then

$$A \longleftrightarrow \mathcal{F}^{-1} \circ S_{-LR^{-1}} \circ D_{R^{-1}} \circ \mathcal{F} \circ S_{R^{-1}Q}.$$

More generally, given any matrix $A \in \mathrm{Sp}(d)$ there exists a unitary operator $\mu(A)$ acting on $L^2(\mathbb{R}^d)$ such that

$$(7) \qquad \mu(A)\pi(\nu) = \varphi(A, \nu) \cdot \pi(A\nu)\mu(A),$$

where $\varphi(A, \cdot)$ maps vectors $\nu \in \mathbb{R}^{2d}$ into the complex plane with $|\varphi| = 1$. Moreover, $\mu(A)$ can be written as a composition of the Fourier transform, suitable dilations, and chirp-multiplications. For $A \in \mathrm{Sp}(d)$ as in Example 2 an operator $\mu(A)$ that satisfies (7) is given by the associations as in Example 3.

It is not generally true that there is a unique operator $\mu(A)$ such that (7) holds. Indeed, from Examples 2 and 3, if multiple block components of A are invertible, then we have several choices of the decomposition of A and several operators $\mu(A)$ that we can associate to A so that (7) holds. There is a way to make the choice of $\mu(A)$ unique: one constructs the so-called metaplectic double cover of the symplectic group. For our results this is not of interest, and we refer to [13, 15] for more information on this. For our needs it is enough that given $A \in \mathrm{Sp}(d)$ a unitary operator $\mu(A)$ exists such that (7) holds. In specific examples one can use Examples 2 and 3 to construct such $\mu(A)$.

Using the relations between $A \in \mathrm{Sp}(d)$, time-frequency shifts, and the unitary operator $\mu(A)$ as expressed in (7) one can show the following well-known results on Gabor systems:

LEMMA 2.4. *Let Δ be a subset (e.g., a lattice) in $\mathbb{R}^{2d}$, $g \in L^2(\mathbb{R}^d)$, and $A \in \mathrm{Sp}(d)$. If $\mu(A)$ is a unitary operator acting on $L^2(\mathbb{R}^d)$ such that (7) holds, then the Gabor system $\{\pi(\nu)g\}_{\nu \in \Delta}$ is a [frame, tight frame, Riesz basis, orthonormal basis] if and only if the Gabor system $\{\pi(A\nu)\mu(A)g\}_{\nu \in \Delta}$ is a [frame, tight frame, Riesz basis, orthonormal basis]. Moreover, the [frame, Riesz] bounds are preserved.*

We wish to extend Lemma 2.4 to a more general class of systems which includes the Wilson systems that we consider in Theorem 3.1. To this end we need the following result.

LEMMA 2.5. *Let $\nu \in \mathbb{R}^{2d}$ and $A \in \mathrm{Sp}(d)$ be given. If $\mu(A)$ satisfies (7), then*

$$\mu(A)\pi(-\nu) = \varphi(A, \nu) \cdot \pi(-A\nu)\mu(A).$$

That is, the phase factor $\varphi(A, \nu)$ in (7) is invariant under the reflection $\nu \mapsto -\nu$ for all $\nu \in \mathbb{R}^{2d}$.

Proof. As $\mu(A)$ can be written as a composition of the Fourier transform, dilations, and chirp-mulitplication it is sufficient to prove the result for these three operators. Indeed, for $C \in \mathrm{GL}_\mathbb{R}(d)$ and $M \in \mathrm{Sym}_\mathbb{R}(d)$ we find from (3), (4), and (5) that

$$\mathcal{F}\,\pi(-\nu) = e^{2\pi i \langle -\lambda, -\gamma \rangle}\,\pi\big(-\big[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\big]\nu\big)\,\mathcal{F} = e^{2\pi i \langle \lambda, \gamma \rangle}\,\pi\big(-\big[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\big]\nu\big)\,\mathcal{F},$$

$$D_C\,\pi(-\nu) = \pi\big(-\big[\begin{smallmatrix} C^{-1} & 0 \\ 0 & C^\top \end{smallmatrix}\big]\nu\big)\,D_C,$$

$$S_M\,\pi(-\nu) = e^{-\pi i \langle -\lambda, -M\lambda \rangle}\,\pi\big(-\big[\begin{smallmatrix} I & 0 \\ M & I \end{smallmatrix}\big]\nu\big)\,S_M = e^{-\pi i \langle \lambda, M\lambda \rangle}\,\pi\big(-\big[\begin{smallmatrix} I & 0 \\ M & I \end{smallmatrix}\big]\nu\big)\,S_M.$$

In particular, this shows that

$$\varphi\big(\big[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\big], \nu\big) = \varphi\big(\big[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\big], -\nu\big) = e^{2\pi i \langle \lambda, \gamma \rangle},$$

$$\varphi\big(\big[\begin{smallmatrix} C^{-1} & 0 \\ 0 & C^\top \end{smallmatrix}\big], \nu\big) = \varphi\big(\big[\begin{smallmatrix} C^{-1} & 0 \\ 0 & C^\top \end{smallmatrix}\big], -\nu\big) = 1,$$

$$\varphi\big(\big[\begin{smallmatrix} I & 0 \\ M & I \end{smallmatrix}\big], \nu\big) = \varphi\big(\big[\begin{smallmatrix} I & 0 \\ M & I \end{smallmatrix}\big], -\nu\big) = e^{-\pi i \langle \lambda, M\lambda \rangle}. \qquad \square$$

We now immediately have the following extension of Lemma 2.4.

LEMMA 2.6. *Let J be an index set. For each $j \in J$, let Δ_j be a subset (e.g., a lattice) in $\mathbb{R}^{2d}$ and let $g_j \in L^2(\mathbb{R}^d)$. Moreover take $A \in \mathrm{Sp}(d)$ and let $\{c_{\nu,j}\}_{\nu \in \Delta_j, j \in J}$ and $\{d_{\nu,j}\}_{\nu \in \Delta_j, j \in J}$ be sequences in $\mathbb{C}$. Suppose that $\mu(A)$ is a unitary operator acting on $L^2(\mathbb{R}^d)$ such that (7) holds. Then the system*

$$\bigcup_{j \in J} \big\{ \big(c_{\nu,j}\pi(\nu) + d_{\nu,j}\pi(-\nu)\big)g_j \big\}_{\nu \in \Delta_j}$$

is a [frame, tight frame, Riesz basis, orthonormal basis] *if and only if the system*

$$\bigcup_{j \in J} \big\{ \big(c_{\nu,j}\pi(A\nu) + d_{\nu,j}\pi(-A\nu)\big)\mu(A)g_j \big\}_{\nu \in \Delta_j}$$

is a [frame, tight frame, Riesz basis, orthonormal basis]. *Moreover, the* [frame, Riesz] *bounds are preserved.*

3. Bimodular Wilson systems in higher dimensions. In this section we consider bimodular Wilson orthonormal bases for $L^2(\mathbb{R}^d)$ that are generated by non-separable functions g. Our main result in this section is Theorem 3.1 stated below. We use boldface $\frac{1}{2}$ to denote the constant vector $(1/2, \ldots, 1/2) \in \mathbb{R}^d$, by $\frac{1}{2} + \mathbb{Z}^d$ we understand the set $\{\frac{1}{2} + z : z \in \mathbb{Z}^d\}$, and we define $(-1)^{|n|} = (-1)^{n_1 + n_2 + \cdots + n_d}$ for vectors $n \in \mathbb{Z}^d$.

THEOREM 3.1. *Let g be a function in $L^2(\mathbb{R}^d)$ and let N be a subset of $\mathbb{Z}^d$ such that $N \cap (-N) = \emptyset$ and $N \cup (-N) \cup \{0\} = \mathbb{Z}^d$. Consider the Gabor system*

$$\mathcal{G}(g) = \{T_\lambda M_\gamma g\}_{\lambda \in \mathbb{Z}^d \cup (\mathbf{1/2} + \mathbb{Z}^d), \gamma \in \mathbb{Z}^d}$$

and the Wilson system

$$\mathcal{W}(g) = \{T_\lambda g\}_{\lambda \in \mathbb{Z}^d} \cup \Big\{ \tfrac{1}{\sqrt{2}}T_\lambda(M_\gamma + (-1)^{|\gamma|}M_{-\gamma})g \Big\}_{\lambda \in \mathbb{Z}^d, \gamma \in N}$$

$$\cup \Big\{ \tfrac{1}{\sqrt{2}}T_{\lambda + \frac{1}{2}}(M_\gamma - (-1)^{|\gamma|}M_{-\gamma})g \Big\}_{\lambda \in \mathbb{Z}^d, \gamma \in N}.$$

Suppose that $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$. Then the following holds:

(i) *The Gabor system $\mathcal{G}(g)$ is a Bessel sequence with bound b if and only if the Wilson system $\mathcal{W}(g)$ is a Bessel sequence with bound $b/2$. In either (and hence both cases) the Gabor frame operator $S_\mathcal{G}$ and the Wilson frame operator $S_\mathcal{W}$ satisfy*

$$S_\mathcal{G} = 2S_\mathcal{W}.$$

(ii) *The Gabor system $\mathcal{G}(g)$ is a frame with bounds $2a$ and $2b$ for $L^2(\mathbb{R}^d)$ if and only if the Wilson system $\mathcal{W}(g)$ is a Riesz basis with bounds a and b for $L^2(\mathbb{R}^d)$.*

(iii) *The Gabor system $\mathcal{G}(g)$ is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound $a = 2$ if and only if the Wilson system $\mathcal{W}(g)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.*

The simple relationship between frame operators of the Gabor system and the Wilson system in Theorem 3.1(i) seems not have been noticed before in the literature, even in dimension one. Indeed, Auscher [1] proves a Walnut-type representation of an operator R defined as $S_\mathcal{G} - 2S_\mathcal{W}$, and Gröchenig calls its commutator properties mysterious in [16]. From Theorem 3.1 it is now clear that R is in fact the zero operator.

Statement (ii) of Theorem 3.1 is less surprising, but it shows an interesting duality principle akin to the duality principle of Gabor systems and their adjoint systems. The "only if"-assertion in (ii) is [16, Corollary 8.5.6] for $d = 1$, albeit without bounds. Part (iii) of Theorem 3.1 generalizes Theorem 1.1 to higher dimensions in a non-trivial way.

In the the following example we show that the standard construction procedure of "nice" generators g of univariate Wilson bases (see, e.g., [11]) carries over to bimodular multivariate Wilson bases in Theorem 3.1(iii).

Example 4. Take $g \in L^2(\mathbb{R}^d)$ so that $\mathcal{G}(g) = \{T_\lambda M_\gamma g\}_{\lambda \in \mathbb{Z}^d \cup (\mathbf{1/2} + \mathbb{Z}^d), \gamma \in \mathbb{Z}^d}$ is a Bessel system. If we consider $\mathcal{G}(g)$ as a critically sampled, multiwindow Gabor system $\{T_\lambda M_\gamma g_i\}_{\lambda, \gamma \in \mathbb{Z}^d, i=1,2}$ with two generators $g_1 = g$ and $g_2 = (-1)^{|\gamma|} g(\cdot - \frac{1}{2})$, it follows by [16, Theorem 8.3.1] that, for $\alpha \geq 0$,

$$(8) \qquad ZS^\alpha f(x, \omega) = \left(|Zg(x, \omega)|^2 + \left| Zg(x - \tfrac{1}{2}, \omega) \right|^2 \right)^\alpha Zf(x, \omega),$$

where Z and S denote the Zak transform and the frame operator, respectively. Here we tacitly used that $\{T_\lambda M_\gamma g_2\}_{\lambda, \gamma \in \mathbb{Z}^d}$ and $\{T_\lambda M_\gamma T_{\mathbf{1/2}}\, g\}_{\lambda, \gamma \in \mathbb{Z}^d}$ have identical frame operators. If $\mathcal{G}(g)$ is a frame, i.e., if $S_\mathcal{G}$ is invertible, then (8) also holds for $\alpha < 0$.

From (8) it is clear that for window functions g in the Wiener space $W(\mathbb{R}^d)$, the Gabor system $\{T_\lambda M_\gamma g\}_{\lambda \in \mathbb{Z}^d \cup (\mathbf{1/2} + \mathbb{Z}^d), \gamma \in \mathbb{Z}^d}$ is a frame precisely when

$$\underset{x, \omega \in [0,1)^d}{\operatorname{ess\,inf}} \left(|Zg(x, \omega)|^2 + \left| Zg(x - \tfrac{1}{2}, \omega) \right|^2 \right) > 0.$$

Let $g \in W(\mathbb{R}^d)$ be such a window function satisfying the symmetry condition $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$. Define $h = S^{-1/2}g = Z^{-1}qZg$, where $q = (|Zg|^2 + |ZT_{\mathbf{1/2}}\, g|^2)^{-1/2} \in L^\infty([0,1)^{2d})$. We remark that (8) implies preservation of symmetry under the action of the frame operator:

$$(9) \qquad \hat{g} \text{ real-valued} \Leftrightarrow \widehat{Zg} \text{ real-valued} \Leftrightarrow \widehat{S^\alpha g} \text{ real-valued.}$$

Hence, $\hat{h}(\omega) = \overline{\hat{h}(\omega)}$. Since $\{T_\lambda M_\gamma h\}_{\lambda \in \mathbb{Z}^d \cup (\mathbf{1/2} + \mathbb{Z}^d), \gamma \in \mathbb{Z}^d}$ is a Parseval frame, we conclude, by Theorem 3.1, that the Wilson system generated by $\sqrt{2}\, h \in W(\mathbb{R}^d)$ [2, Theorem 6], [21, Corollary 3.1] is an orthonormal basis for $L^2(\mathbb{R}^d)$. Note that if g is in the

Feichtinger algebra $S_0(\mathbb{R}^d)$ or the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, then so is h, respectively, [17, Corollary 4.5].

From Lemma 2.6 we get the following result.

COROLLARY 3.2. *Let A be a matrix in* $\mathrm{Sp}(d)$, *let $\mu(A)$ be a unitary operator on* $L^2(\mathbb{R}^d)$ *such that* (7) *holds, and let N be a subset of $\mathbb{Z}^d$ as in Theorem* 3.1. *For any* $g \in L^2(\mathbb{R}^d)$ *the symplectic Wilson system*

$$\mathcal{W}_s(g) = \big\{\pi(A\lambda)\mu(A)g\big\}_{\lambda \in \mathbb{Z}^d \times \{0\}^d}$$

$$\cup \Big\{\tfrac{1}{\sqrt{2}}\pi(A\lambda)\big(\pi(A\gamma) + (-1)^{|\gamma|}\pi(-A\gamma)\big)\mu(A)g\Big\}_{\lambda \in \mathbb{Z}^d \times \{0\}^d, \gamma \in \{0\}^d \times N}$$

$$\cup \Big\{\tfrac{1}{\sqrt{2}}\pi(A\lambda)\pi(A\lambda^*)\big(\pi(A\gamma) - (-1)^{|\gamma|}\pi(-A\gamma)\big)\mu(A)g\Big\}_{\lambda \in \mathbb{Z}^d \times \{0\}^d, \gamma \in \{0\}^d \times N},$$

where $\lambda^* = \{1/2\}^d \times \{0\}^d$, is a [frame, Riesz basis, orthonormal basis] *if and only if the Wilson system $\mathcal{W}(g)$ in Theorem* 3.1 *has the same property. Moreover, the* [frame, Riesz] *bounds of the two systems are the same.*

Take $d = 1$. If we let $a > 0$ be a given positive number and let $c \in \mathbb{R}_0^+$ be some nonnegative number, then we can define the symplectic matrix with associated operator $\mu(A)$ (such that (7) holds)

$$(10) \qquad A = \begin{bmatrix} 2a & c \\ 0 & 1/2a \end{bmatrix} \quad \text{and} \quad \mu(A) = D_{1/2a} \circ \mathcal{F} \circ S_{-c/2a} \circ \mathcal{F}^{-1}.$$

With these choices Theorem 3.1(iii) combined with Corollary 3.2 yields the result from Kutyniok and Strohmer stated in Theorem 2.1. From section 2 it is clear that any matrix A with determinant one can be used in the construction of symplectic Wilson bases in $L^2(\mathbb{R})$.

The rest of this section is devoted to proving Theorem 3.1. But first, we need some preliminary results about shift-invariant systems. The theory presented in Definition 3.3, Lemma 3.4 and Proposition 3.5 has been considered specifically for Gabor systems in, e.g., [20, 25] and more generally, for generalized-shift invariant systems, in [18, 26].

DEFINITION 3.3. *Let Γ be a countable index set and let $\{g_\gamma\}_{\gamma \in \Gamma} \subset L^2(\mathbb{R}^d)$. For a full-rank lattice $\Lambda = Q\mathbb{Z}^d$, where $Q \in \mathrm{GL}_d(\mathbb{R})$, the dual lattice or the annihilator is given by $\Lambda^\perp = (Q^{-1})^\top \mathbb{Z}^d$. Suppose that*

$$(11) \qquad \sum_{\gamma \in \Gamma} |\hat{g}_\gamma(\omega)|^2 < \infty \qquad \text{for a.e. } \omega \in \mathbb{R}^d.$$

For the shift-invariant system $\{T_\lambda g_\gamma\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ we define its autocorrelation *functions $\{t_\alpha\}_{\alpha \in \Lambda^\perp}$ by*

$$(12) \qquad t_\alpha(\omega) := \frac{1}{|\det Q|} \sum_{\gamma \in \Gamma} \hat{g}_\gamma(\omega)\overline{\hat{g}_\gamma(\omega - \alpha)} \qquad \text{for a.e. } \omega \in \mathbb{R}^d, \ \alpha \in \Lambda^\perp.$$

By the Cauchy–Schwarz inequality and (11), the series defining $t_\alpha(\omega)$ are absolutely convergent for a.e. ω. Although the name autocorrelation function is borrowed from signal processing, such functions appear frequently in the study of shift-invariant systems. In the case when Λ is the standard lattice $\mathbb{Z}^d$, one can employ the characteri-

zation of shift-invariant frames in terms of fiberization operators [7, Theorem 2.3] and equivalently by dual Gramians of Ron and Shen [24]. By scaling these results hold for shift-invariant systems with respect to an arbitrary (full rank) lattice $\Lambda = Q\mathbb{Z}^d \subset \mathbb{R}^d$; see [8, section 2.4]. Indeed, the dual Gramian corresponding to the shift-invariant system $\{T_\lambda g_\gamma\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ is the infinite matrix

$$\tilde{G}(\omega) = \left(\frac{1}{|\det Q|} \sum_{\gamma \in \Gamma} \hat{g}_\gamma(\omega + k)\overline{\hat{g}_\gamma(\omega + l)} \right)_{k,l \in \Lambda^\perp} = (t_{k-l}(\omega + k))_{k,l \in \Lambda^\perp} \quad \omega \in \mathbb{R}^d.$$

By [7, Theorem 2.5], $\{T_\lambda g_\gamma\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ is a Bessel sequence or frame in $L^2(\mathbb{R}^d)$ with bounds a and b if and only if the dual Gramians represent bounded or invertible operators on $\ell^2(\Lambda^\perp)$ with uniform bounds a and b for a.e. $\omega \in \mathbb{R}^d$. In particular, we have the following fact, which has been observed by many authors.

LEMMA 3.4 (see [18, 20, 24]). *Let $Q \in \mathrm{GL}_d(\mathbb{R})$, $\Lambda = Q\mathbb{Z}^d$, Γ be a countable index set, and let $\{g_\gamma\}_{\gamma \in \Gamma} \subset L^2(\mathbb{R}^d)$. Then the following holds:*
(i) *If $\{T_\lambda g_\gamma\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ is a Bessel sequence with bound b, then*

$$t_0(\omega) = \sum_{\gamma \in \Gamma} |\hat{g}_\gamma(\omega)|^2 \leq b \qquad \text{for a.e. } \omega \in \mathbb{R}^d.$$

(ii) *$\{T_\lambda g_\gamma\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound a if and only if*

$$(13) \qquad t_\alpha(\omega) = a\delta_{\alpha,0} \ \text{ for all } \ \alpha \in \Lambda^\perp = \left(Q^{-1}\right)^\top \mathbb{Z}^d \ \text{ and a.e } \ \omega \in \mathbb{R}^d.$$

For a given function $t \in L^\infty(\mathbb{R}^d)$, define the multiplication operator

$$M_t f(x) = t(x)f(x) \qquad \text{for } f \in L^2\left(\mathbb{R}^d\right).$$

For the special choice of $t(x) = e^{2\pi i \langle x, \gamma \rangle}$, $\gamma \in \mathbb{R}^d$, this yields the modulation operator M_γ, which justifies our notation. Let

$$\mathcal{D} = \left\{ f \in L^2\left(\mathbb{R}^d\right) : \hat{f} \in L^\infty\left(\mathbb{R}^d\right) \ \text{and} \ \mathrm{supp}\, \hat{f} \ \text{is bounded} \right\}.$$

We will employ the following result, which gives a weak representation of the (possibly unbounded) frame operator of the shift-invariant system $\{T_\lambda g_\gamma\}_{\lambda \in \Lambda, \gamma \in \Gamma}$ on the dense subspace $\mathcal{D} \subset L^2(\mathbb{R}^d)$ in terms of autocorrelation functions.

PROPOSITION 3.5 (see [18]). *Let $Q \in \mathrm{GL}_d(\mathbb{R})$, $\Lambda = Q\mathbb{Z}^d$ and let Γ be a countable index set. Assume that $\{g_\gamma\}_{\gamma \in \Gamma} \subset L^2(\mathbb{R}^d)$ satisfies*

$$(14) \qquad\qquad\qquad \sum_{\gamma \in \Gamma} |\hat{g}_\gamma(\cdot)|^2 \in L^1_{loc}\left(\mathbb{R}^d\right).$$

Let $\{t_\alpha\}_{\alpha \in \Lambda^\perp}$, $\Lambda^\perp = (Q^{-1})^\top \mathbb{Z}^d$, be the autocorrelation functions of the shift-invariant system $\{T_\lambda g_\gamma\}_{\lambda \in \Lambda, \gamma \in \Gamma}$. Then, for any $f \in \mathcal{D}$, we have

$$(15) \quad \sum_{\gamma \in \Gamma} \sum_{\lambda \in \Lambda} |\langle f, T_\lambda g_\gamma \rangle|^2 = \sum_{\alpha \in \Lambda^\perp} \int_{\mathbb{R}^d} t_\alpha(\omega)\hat{f}(\omega - \alpha)\overline{\hat{f}(\omega)}d\omega = \sum_{\alpha \in \Lambda^\perp} \left\langle M_{t_\alpha} T_\alpha \hat{f}, \hat{f} \right\rangle.$$

Proof. Since the support of $\hat{f}$ is bounded, the sum (15) over $\Lambda^{\perp} = (Q^{-1})^{\top}\mathbb{Z}^d$ has finitely many nonzero terms. In the proof of (15) we shall employ [18, Proposition 2.4], which holds for generalized shift-invariant systems under the local integrability condition (LIC). However, for shift-invariant the LIC used in [18] is equivalent with (14). Consequently, for $f \in \mathcal{D}$,

$$(16) \qquad w_f(x) := \sum_{\gamma \in \Gamma} \sum_{\lambda \in \Lambda} \left| \langle T_x f, T_\lambda g_\gamma \rangle \right|^2, \quad x \in \mathbb{R}^d,$$

is a continuous function that coincides pointwise with the trigonometric polynomial

$$(17) \qquad \sum_{\alpha \in \Lambda^{\perp}} \hat{w}(\alpha) e^{2\pi i \langle \alpha, x \rangle}, \qquad \text{where } \hat{w}(\alpha) = \int_{\mathbb{R}^d} t_\alpha(\omega) \hat{f}(\omega - \alpha) \overline{\hat{f}(\omega)} \, d\omega.$$

Taking $x = 0$ in (17) yields (15). $\qquad \square$

LEMMA 3.6. *The annihilator $\Lambda^{\perp}$ of the lattice $\Lambda := \mathbb{Z}^d \cup (\frac{1}{2} + \mathbb{Z}^d)$ is given by*

$$\Lambda^{\perp} = \{n \in \mathbb{Z}^d : n_1 + n_2 + \cdots + n_d \in 2\mathbb{Z}\}.$$

Proof. One easily verifies that Λ is a lattice. Define now

$$H := \{n \in \mathbb{Z}^d : n_1 + n_2 + \cdots + n_d \in 2\mathbb{Z}\}.$$

Take $n \in H$ and $\lambda \in \Lambda$. If $\lambda \in \mathbb{Z}^d$, then $\langle n, \lambda \rangle \in \mathbb{Z}$. Likewise, if $\lambda = (\frac{1}{2} + k), k \in \mathbb{Z}^d$, then

$$\langle n, \lambda \rangle = \tfrac{1}{2} \underbrace{(n_1 + n_2 + \cdots + n_d)}_{\in 2\mathbb{Z}} + \langle n, k \rangle \in \mathbb{Z}.$$

This shows that $H \subset \Lambda^{\perp}$. To show the converse inclusion we observe the following. By definition we have $\mathbb{Z}^d \subset \Lambda$ and so $\Lambda^{\perp} \subset \mathbb{Z}^d$. Take any $n \in \Lambda^{\perp} \subset \mathbb{Z}^d$. Then, choosing $\lambda = \frac{1}{2} \in \Lambda$, we have

$$\langle n, \lambda \rangle = \tfrac{1}{2}(n_1 + n_2 + \ldots n_d) \in \mathbb{Z}.$$

Thus $n \in H$, which shows that $H = \Lambda^{\perp}$. $\qquad \square$

The following lemma establishes the connection between Wilson and Gabor systems via their autocorrelation functions. This is a key result for the proof of Theorem 3.1.

LEMMA 3.7. *Let $g \in L^2(\mathbb{R}^d)$ be such that $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$ and let $\Lambda = \mathbb{Z}^d \cup (\frac{1}{2} + \mathbb{Z}^d)$. Furthermore, let $\mathcal{G}(g)$ and $\mathcal{W}(g)$ be the Gabor system and the Wilson system considered in Theorem 3.1, respectively. Suppose that*

$$(18) \qquad \sum_{\gamma \in \mathbb{Z}^d} |\hat{g}(\omega - \gamma)|^2 < \infty \qquad \text{for a.e. } \omega \in \mathbb{R}^d.$$

Then the following holds:

   (i) *If the Gabor system $\mathcal{G}(g)$ is considered as a shift-invariant system with generators $\{M_\gamma g\}_{\gamma \in \mathbb{Z}^d}$ and with shifts along the lattice Λ, then its autocorrelation functions are given by*

$$t_{\alpha,\mathcal{G}}(\omega) = 2 \sum_{\gamma \in \mathbb{Z}^d} \hat{g}(\omega - \gamma) \overline{\hat{g}(\omega - \gamma - \alpha)}, \qquad \alpha \in \Lambda^{\perp}, \text{ a.e. } \omega \in \mathbb{R}^d.$$

(ii) *If the Wilson system $\mathcal{W}(g)$ is considered as a shift-invariant system with generators*

$$g, \ \left\{\tfrac{1}{\sqrt{2}}\left(M_\gamma + (-1)^{|\gamma|}M_{-\gamma}\right)g\right\}_{\gamma\in N} \quad and$$

$$\left\{\tfrac{1}{\sqrt{2}}T_{\mathbf{1/2}}\left(M_\gamma - (-1)^{|\gamma|}M_{-\gamma}\right)g\right\}_{\gamma\in N}$$

and with shifts along the lattice $\mathbb{Z}^d$, then its autocorrelation functions are given by

$$t_{\alpha,\mathcal{W}}(\omega) = \begin{cases} \sum_{\gamma\in\mathbb{Z}^d}\hat{g}(\omega-\gamma)\overline{\hat{g}(\omega-\gamma-\alpha)} & \alpha\in\Lambda^\perp, \\ 0 & \alpha\in\mathbb{Z}^d\backslash\Lambda^\perp, \end{cases} \quad a.e.\ \omega\in\mathbb{R}^d.$$

Proof. First, observe that the assumption (18) guarantees that the generators of $\mathcal{G}(g)$ and $\mathcal{W}(g)$ satisfy condition (11). Hence, their autocorrelation functions are well-defined. Then, a straightforward calculation of (12) verifies (i).

The result in (ii) needs some explanation. By Definition 3.3, for $\alpha\in\mathbb{Z}^d$ we have

$$t_{\alpha,\mathcal{W}}(\omega) = \hat{g}(\omega)\overline{\hat{g}(\omega-\alpha)}$$
$$+ \tfrac{1}{2}\sum_{\gamma\in N}\hat{g}(\omega-\gamma)\overline{\hat{g}(\omega-\gamma-\alpha)} + (-1)^{|\gamma|}\hat{g}(\omega-\gamma)\overline{\hat{g}(\omega+\gamma-\alpha)}$$

(19)
$$+ (-1)^{|\gamma|}\hat{g}(\omega+\gamma)\overline{\hat{g}(\omega-\gamma-\alpha)} + \hat{g}(\omega+\gamma)\overline{\hat{g}(\omega+\gamma-\alpha)}$$
$$+ \tfrac{1}{2}e^{2\pi i\langle\mathbf{1/2},\alpha\rangle}\sum_{\gamma\in N}\hat{g}(\omega-\gamma)\overline{\hat{g}(\omega-\gamma-\alpha)} - (-1)^{|\gamma|}\hat{g}(\omega-\gamma)\overline{\hat{g}(\omega+\gamma-\alpha)}$$
$$- (-1)^{|\gamma|}\hat{g}(\omega+\gamma)\overline{\hat{g}(\omega-\gamma-\alpha)} + \hat{g}(\omega+\gamma)\overline{\hat{g}(\omega+\gamma-\alpha)}.$$

Note the difference in the signs used in the two sums in the terms with alternating signs $(-1)^{|\gamma|}$ and the phase factor in front of the second sum. Because of this phase factor we will consider two cases: (I) $\alpha_1+\alpha_2+\cdots+\alpha_d\in 2\mathbb{Z}$ and (II) $\alpha_1+\alpha_2+\cdots+\alpha_d\in 2\mathbb{Z}+1$. By Lemma 3.6 these cases correspond to $\alpha\in\Lambda^\perp$ and $\alpha\in\mathbb{Z}^d\backslash\Lambda^\perp$, respectively. Because of $N\cup(-N)\cup\{0\} = \mathbb{Z}^d$ and $N, -N$ and $\{0\}$ are mutually disjoint sets, and (19) yields

(I) for $\alpha\in\Lambda^\perp$

$$(20)\qquad t_{\alpha,\mathcal{W}}(\omega) = \sum_{\gamma\in\mathbb{Z}^d}\hat{g}(\omega-\gamma)\overline{\hat{g}(\omega-\gamma-\alpha)} \qquad a.e.\ \omega\in\mathbb{R}^d;$$

(II) for $\alpha\in\mathbb{Z}^d\backslash\Lambda^\perp$

$$(21)\qquad t_{\alpha,\mathcal{W}}(\omega) = \sum_{\gamma\in\mathbb{Z}^d}(-1)^{|\gamma|}\hat{g}(\omega-\gamma)\overline{\hat{g}(\omega+\gamma-\alpha)} \qquad a.e.\ \omega\in\mathbb{R}^d.$$

It remains to show that (21) is equal to zero. Take any $\alpha\in\mathbb{Z}^d\backslash\Lambda^\perp$. By a change of variables $\gamma\mapsto-\gamma'+\alpha$, we obtain

$$(22)\qquad t_{\alpha,\mathcal{W}}(\omega) = \sum_{\gamma'\in\mathbb{Z}^d}(-1)^{|(-\gamma'+\alpha)|}\hat{g}(\omega+\gamma'-\alpha)\overline{\hat{g}(\omega-\gamma')}$$

for a.e. $\omega\in\mathbb{R}^d$. For $\alpha\in\mathbb{Z}^d$ with $\alpha_1+\alpha_2+\cdots+\alpha_d\in 2\mathbb{Z}+1$, we note that

$$(23)\qquad (-1)^{|-\gamma'+\alpha|} = (-1)^{|-\gamma'|}(-1)^{|\alpha|} = -(-1)^{|\gamma'|}.$$

Finally, by our assumption $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$, it follows that

$$(24) \qquad \hat{g}(\omega + \gamma' - \alpha)\overline{\hat{g}(\omega - \gamma')} = \overline{\hat{g}(\omega + \gamma' - \alpha)}\hat{g}(\omega - \gamma').$$

Combining (22)–(24) yields $t_{\alpha,\mathcal{W}}(\omega) = -t_{\alpha,\mathcal{W}}(\omega)$, and hence $t_{\alpha,\mathcal{W}}(\omega) = 0$.   □

In the proof of Theorem 3.1 we will also need the following two lemmas.

LEMMA 3.8. *Let* $\{f_k\}_{k=1}^\infty \subset \mathcal{H}$ *be a tight frame for* $\mathcal{H}$ *with frame bound* a. *Then* $\{f_k\}_{k=1}^\infty$ *is an orthonormal basis for* $\mathcal{H}$ *if and only if* $a = 1$ *and* $\|f_k\|_{\mathcal{H}} = 1$ *for all* $k \in \mathbb{N}$.

LEMMA 3.9 (see [14, Theorem 3.5.12]). *Let* Δ *be a lattice in* $\mathbb{R}^d \times \mathbb{R}^d$ *and let* $g \in L^2(\mathbb{R}^d)$. *If* $\{T_\lambda M_\gamma g\}_{(\lambda,\gamma)\in\Delta}$ *is a tight frame, then the set* $\{T_\alpha M_\beta g\}$ *of all* $(\alpha, \beta) \in \mathbb{R}^d \times \mathbb{R}^d$ *for which*

$$(T_\lambda M_\gamma)(T_\alpha M_\beta) = (T_\alpha M_\beta)(T_\lambda M_\gamma) \quad \text{for all} \quad (\lambda, \gamma) \in \Delta$$

forms an orthogonal set.

We are now ready to prove the main result of this section.

Proof of Theorem 3.1. We use the setup and notation from Lemma 3.7. Suppose that either the Gabor system $\mathcal{G}(g)$ or the Wilson system $\mathcal{W}(g)$ is a Bessel sequence. It follows from Lemma 3.4(i) that $t_{0,\mathcal{G}} \in L^\infty$ or $t_{0,\mathcal{W}} \in L^\infty$, respectively. In either case, we have

$$(25) \qquad \sum_{\gamma\in\mathbb{Z}^d} |\hat{g}(\cdot - \gamma)|^2 \in L^\infty.$$

Hence, the assumption (18) in Lemma 3.7 holds, and we have the following relation between autocorrelation functions:

$$t_{\alpha,\mathcal{W}}(\omega) = \begin{cases} 2^{-1}t_{\alpha,\mathcal{G}}(\omega), & \alpha \in \Lambda^\perp, \\ 0, & \alpha \in \mathbb{Z}^d \setminus (\Lambda^\perp). \end{cases}$$

By (25), we can apply Proposition 3.5 for both $\mathcal{G}(g)$ and $\mathcal{W}(g)$. Hence, for any $f \in \mathcal{D}$,

$$\sum_{\phi\in\mathcal{G}(g)} |\langle f, \phi\rangle|^2 = \sum_{\alpha\in\Lambda^\perp} \langle M_{t_{\alpha,\mathcal{G}}} T_\alpha \hat{f}, \hat{f}\rangle = 2\sum_{\alpha\in\mathbb{Z}^d} \langle M_{t_{\alpha,\mathcal{W}}} T_\alpha \hat{f}, \hat{f}\rangle = 2 \sum_{\phi\in\mathcal{W}(g)} |\langle f, \phi\rangle|^2.$$

Now, suppose the Gabor system $\mathcal{G}(g)$ is a Bessel sequence with bound b. Then, for any $f \in \mathcal{D}$,

$$2 \sum_{\phi\in\mathcal{W}(g)} |\langle f, \phi\rangle|^2 = \langle S_\mathcal{G} f, f\rangle \leq b \|f\|^2.$$

Since $\mathcal{D}$ is dense in $L^2(\mathbb{R}^d)$, this inequality extends to all of $L^2(\mathbb{R}^d)$, which shows that $\mathcal{W}(g)$ is a Bessel sequence with bound $b/2$ and

$$(26) \qquad 2\langle S_\mathcal{W} f, f\rangle = \langle S_\mathcal{G} f, f\rangle \qquad \text{for all } f \in L^2(\mathbb{R}^d).$$

Since the frame operator is positive and self-adjoint, we obtain $S_\mathcal{G} = 2S_\mathcal{W}$. Conversely, assuming that $\mathcal{W}(g)$ is Bessel yields the same conclusion (26), which proves (i). It remains to show statements (ii) and (iii).

Assume that the Wilson system is a Riesz basis or an orthonormal basis. Then it is, in particular, also a frame or tight frame, respectively. However, from the equality

$S_{\mathcal{G}} = 2S_{\mathcal{W}}$, it is clear that the Gabor system $\mathcal{G}(g)$ is a frame with frame bounds a and b if and only if the Wilson system $\mathcal{W}(g)$ is a frame with frame bound $a/2$ and $b/2$. Hence, it follows that the Gabor system $\mathcal{G}(g)$ is a frame or tight frame, respectively.

For the converse directions in statements (ii) and (iii) we have to work a bit harder. We first prove the "only if"-direction in (iii). Assume therefore that the Gabor system $\mathcal{G}(g)$ is a tight frame with frame bound 2, and then, by (i), the Wilson system is a tight frame with frame bound 1. By Lemma 3.8, it remains to show that

$$\left\| \tfrac{1}{\sqrt{2}} \left(M_\gamma \pm (-1)^{|\gamma|} M_{-\gamma} \right) g \right\|_2 = \|g\|_2 = 1 \ \text{ for all } \gamma \in N \subset \mathbb{Z}^d.$$

To show this, it suffices to prove that $\{M_{2\gamma} g\}_{\gamma \in \mathbb{Z}^d}$ is an orthogonal system. By Lemma 3.9 this is true if the frequency shifts $\{M_{2\gamma} g\}_{\gamma \in \mathbb{Z}^d}$ commute with the time frequency shifts used in the tight Gabor frame $\mathcal{G}$, i.e.,

$$(M_{2\gamma})(T_\lambda M_\gamma) = (T_\lambda M_\gamma)(M_{2\gamma}) \ \text{ for all } \ (\lambda, \gamma) \in \Lambda \times \Gamma,$$

where

$$\Lambda = \mathbb{Z}^d \cup \left(\tfrac{1}{2} + \mathbb{Z}^d \right) \ \text{ and } \ \Gamma = \mathbb{Z}^d.$$

Indeed, by using the commutator relations $M_b T_a = e^{2\pi i \langle b, a \rangle} T_a M_b$, one finds that

$$(M_{2\gamma})(T_\lambda M_\gamma) = e^{2\pi i \langle 2\gamma, \lambda \rangle} (T_\lambda M_\gamma)(M_{2\gamma}).$$

Observe that $\Lambda \subset \tfrac{1}{2} \mathbb{Z}^d$ and thus $2\mathbb{Z}^d = 2\Gamma \subset \Lambda^\perp$. This implies that indeed

$$(M_{2\gamma})(T_\lambda M_\gamma) = (T_\lambda M_\gamma)(M_{2\gamma}) \ \text{ for all } \ (\lambda, \gamma) \in \Lambda \times \Gamma$$

and so all elements in the Wilson system $\mathcal{W}(g)$ have norm 1 and by Lemma 3.8 the system $\mathcal{W}(g)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$. We have now proved (iii).

For the proof of the "only if"-direction in (ii) we use the canonical Parseval frame argument as in [16, Corollary 8.5.6], which makes use of the result in (ii). More details will be given in the proof of Theorem 4.5. □

4. A family of Wilson systems. The simplest way of obtaining Wilson bases in higher dimensions is through tensoring. However, this gives rise to 2^d-modular covering of the frequency domain which, as discussed in the introduction, is often undesirable. Theorem 3.1 shows that in any dimension one can construct bimodular Wilson orthonormal bases in $L^2(\mathbb{R}^d)$ from certain tight Gabor frames of redundancy 2. In this section we investigate intermediate 2^k-modular covering of the frequency domain for $k = 1, \ldots, d$.

Let us start by reviewing the tensor construction for $d = 2$.

Example 5. Let $g_1, g_2 \in L^2(\mathbb{R})$ be unit norm functions that generate tight Gabor frames $\{T_{n/2} M_m g_k\}_{m,n \in \mathbb{Z}}$, $k = 1, 2$ for $L^2(\mathbb{R})$. By letting $g(x, y) := g_1(x) g_2(y)$, the Gabor system $\{T_{n/2} M_m g\}_{n \in \mathbb{Z}^2, m \in \mathbb{Z}^2}$ is a tight frame for $L^2(\mathbb{R}^2)$ with density $1/4$, i.e., redundancy 4, and frame bound 4. Moreover, the tensor product of the two associated one-dimensional Wilson systems, which has the rather complicated form (27), is an

orthonormal basis for $L^2(\mathbb{R}^2)$.

$$\{T_n g\}_{n \in \mathbb{Z}^2}$$

$$\cup \left\{ \tfrac{1}{\sqrt{2}} T_n \left(M_{(m_1,0)} + (-1)^{m_1} M_{(-m_1,0)} \right) g \right\}_{n \in \mathbb{Z}^2, m_1 \in \mathbb{N}}$$

$$\cup \left\{ \tfrac{1}{\sqrt{2}} T_n T_{\frac{1}{2}(1,0)} \left(M_{(m_1,0)} - (-1)^{m_1} M_{(-m_1,0)} \right) g \right\}_{n \in \mathbb{Z}^2, m_1 \in \mathbb{N}}$$

$$\cup \left\{ \tfrac{1}{\sqrt{2}} T_n \left(M_{(0,m_2)} + (-1)^{m_2} M_{(0,-m_2)} \right) g \right\}_{n \in \mathbb{Z}^2, m_2 \in \mathbb{N}}$$

$$\cup \left\{ \tfrac{1}{\sqrt{2}} T_n T_{\frac{1}{2}(0,1)} \left(M_{(0,m_2)} - (-1)^{m_2} M_{(0,-m_2)} \right) g \right\}_{n \in \mathbb{Z}^2, m_2 \in \mathbb{N}}$$

$$\cup \left\{ \tfrac{1}{2} T_n \left(M_{(m_1,m_2)} + (-1)^{m_1} M_{(-m_1,m_2)} \right.\right.$$
$$\left.\left. + (-1)^{m_2} M_{(m_1,-m_2)} + (-1)^{m_1+m_2} M_{-(m_1,m_2)} \right) g \right\}_{n \in \mathbb{Z}^2, m \in \mathbb{N}^2}$$

(27)
$$\cup \left\{ \tfrac{1}{2} T_n T_{\frac{1}{2}(1,0)} \left(M_{(m_1,m_2)} - (-1)^{m_1} M_{(-m_1,m_2)} \right.\right.$$
$$\left.\left. + (-1)^{m_2} M_{(m_1,-m_2)} - (-1)^{m_1+m_2} M_{-(m_1,m_2)} \right) g \right\}_{n \in \mathbb{Z}^2, m \in \mathbb{N}^2}$$

$$\cup \left\{ \tfrac{1}{2} T_n T_{\frac{1}{2}(0,1)} \left(M_{(m_1,m_2)} + (-1)^{m_1} M_{(-m_1,m_2)} \right.\right.$$
$$\left.\left. - (-1)^{m_2} M_{(m_1,-m_2)} - (-1)^{m_1+m_2} M_{-(m_1,m_2)} \right) g \right\}_{n \in \mathbb{Z}^2, m \in \mathbb{N}^2}$$

$$\cup \left\{ \tfrac{1}{2} T_n T_{\frac{1}{2}(1,1)} \left(M_{(m_1,m_2)} - (-1)^{m_1} M_{(-m_1,m_2)} \right.\right.$$
$$\left.\left. - (-1)^{m_2} M_{(m_1,-m_2)} + (-1)^{m_1+m_2} M_{-(m_1,m_2)} \right) g \right\}_{n \in \mathbb{Z}^2, m \in \mathbb{N}^2}.$$

It is natural to ask if one can generalize this tensor construction allowing a nonseparable generator g. However, it turns out that the answer to this question is negative. The fact that $g(x,y) = g_1(x)g_2(y)$ is essential. Indeed, the following example shows that one cannot avoid the separability of g.

Example 6. Consider $\{T_{n/2} M_m g\}_{m,n \in \mathbb{Z}^2}$, where $g \in L^2(\mathbb{R}^2)$ is such that $\hat{g} = \tfrac{1}{2}\mathbb{1}_D$ with
$$D = \left\{ (x,y) \in \mathbb{R}^2 : 0 \le x \le 4, 0 \le y \le 2, -2 + x \le y \le x \right\}.$$
Note that $\|g\|_2 = 1$. One can easily show that this function generates a tight Gabor frame with density $1/4$ and frame bound 4. However, the Wilson system in (27) is not an orthonormal basis. To see this, we apply Lemma 3.4, which gives a characterization when the shift-invariant system (27) is a Parseval frame. In particular, if $\alpha = (1,1)$, then a rather heavy calculation of autocorrelation functions of the Wilson system (27) shows that the necessary condition is that

$$t_\alpha(\omega) = \sum_{m \in \mathbb{Z}^2} (-1)^{m_1+m_2} \overline{\hat{g}(\omega - m)} \hat{g}(\omega + m - \alpha) = 0 \qquad \text{a.e. } \omega \in \mathbb{R}^2.$$

However, one finds that

$$\sum_{m \in \mathbb{Z}^2} (-1)^{m_1+m_2} \overline{\hat{g}(\omega - m)} \hat{g}(\omega + m - \alpha) = \frac{1}{2} \quad \text{for } \omega \in \Omega,$$

where $\Omega = \{(x,y) \in \mathbb{R}^2 \ : \ 1 \le x \le 4, 1 \le y \le 2, -2 + x \le y \le x\}$. Hence, the Wilson system in (27) with g given as above is not an orthonormal basis for $L^2(\mathbb{R}^2)$.

Example 5 suggests that if one assumes that a function $g \in L^2(\mathbb{R}^d)$ is separable in all its variables, or more generally separable in the sense of Definition 4.3, then one can formulate a generalization of Theorem 3.1. In the rest of this section we prove that this is the case. But first, we introduce some necessary concepts.

DEFINITION 4.1. *For a vector $\sigma \in \mathbb{Z}^d$ we define the reflection operator*

$$R_\sigma : \mathbb{R}^d \to \mathbb{R}^d, \ \ R_\sigma : x \mapsto \left((-1)^{\sigma_1} x_1, (-1)^{\sigma_2} x_2, \dots, (-1)^{\sigma_d} x_d\right).$$

On phase-space we define the reflection operator to act by reflecting each component

$$\widetilde{R}_\sigma : \mathbb{R}^{2d} \to \mathbb{R}^{2d}, \ \ \widetilde{R}_\sigma : (x, \omega) \mapsto (R_\sigma x, R_\sigma \omega), \ \ x, \omega \in \mathbb{R}^d.$$

Clearly, R_σ is the identity for $\sigma \in 2\mathbb{Z}^d$. Hence, the reflection operators R_σ form a group $\mathbb{Z}^d/(2\mathbb{Z}^d)$, which is identified with its coset representatives $\{0,1\}^d$. For a fixed subgroup $G \subset \mathbb{Z}^d/(2\mathbb{Z}^d)$, we define the orbit of a point $x \in \mathbb{R}^d$ under G to be the set

$$\mathrm{orbit}(x) := \left\{ y \in \mathbb{R}^d \ : \ y = R_\sigma x, \sigma \in G \right\}.$$

DEFINITION 4.2. *Define the support of $\sigma \in \mathbb{Z}^d/(2\mathbb{Z}^d)$ by*

$$\mathrm{supp}\,\sigma = \{i \in [d] : \sigma_i \equiv 1 \mod 2\}, \qquad where \ [d] = \{1, \dots, d\}.$$

We say that a subgroup $G \subset \mathbb{Z}^d/(2\mathbb{Z}^d)$ is separable if there exist generators $\sigma^1, \dots, \sigma^k$ of G such that
$$\mathrm{supp}\,\sigma^i \cap \mathrm{supp}\,\sigma^j = \emptyset \qquad for \ i \ne j.$$

It follows that a separable group G is uniquely determined by a collection of nonempty disjoint sets $S_i = \mathrm{supp}\,\sigma^i \subset [d]$, $i = 1, \dots, k$. In general, the set $S_0 = [d] \setminus \bigcup_{i=1}^k S_i$ might be nonempty.

DEFINITION 4.3. *For any subset $S = \{s_1 < \cdots < s_m\} \subset [d]$, let $P_S : \mathbb{R}^d \to \mathbb{R}^m$ be the coordinate projection given by*

$$P_S(x_1, \dots, x_d) = (x_{s_1}, \dots, x_{s_m}), \qquad x \in \mathbb{R}^d.$$

We say that a function $g : \mathbb{R}^d \to \mathbb{C}$ is separable with respect to a separable group G if there there exist functions $g_i : \mathbb{R}^{|S_i|} \to \mathbb{C}$, $i = 0, \dots, k$, such that

$$(28) \qquad g(x) = \prod_{i=0}^k g_i \circ P_{S_i}(x) \qquad for \ x \in \mathbb{R}^d.$$

We also need the following elementary lemma.

LEMMA 4.4. *Let $G \subset \mathbb{Z}^d/(2\mathbb{Z}^d)$ be a separable group as in Definition 4.2. Define the lattice*

$$\Lambda = \bigcup G = \bigcup_{\sigma \in G} \left(\sigma + 2\mathbb{Z}^d\right).$$

Then G and its dual group $\widehat{G}$ can be identified as

$$(29) \qquad G \cong \Lambda / \left(2\mathbb{Z}^d\right) \qquad and \qquad \widehat{G} \cong \mathbb{Z}^d / \left(2\Lambda^\perp\right),$$

where $\Lambda^\perp$ is the dual lattice (annihilator) of Λ. The duality pairing $\langle \cdot, \cdot \rangle_*$ between elements in $\widehat{G}$ and G is given by

$$(30) \qquad \left\langle \alpha + 2\Lambda^\perp, \sigma + 2\mathbb{Z}^d \right\rangle_* = (-1)^{\langle \alpha, \sigma \rangle} \qquad \text{for } \alpha \in \mathbb{Z}^d, \sigma \in \Lambda.$$

Moreover, G is self-dual and there exists a canonical isomorphism $I : G \to \widehat{G}$ satisfying

$$(31) \qquad \left\langle I(\sigma^i), \sigma^j \right\rangle \equiv \delta_{i,j} \mod 2,$$

where σ^i, $i = 1, \ldots, k$, are generators as in Definition 4.2. In particular,

$$(32) \qquad \langle I(\sigma), h \rangle \equiv \langle \sigma, I(h) \rangle \mod 2 \qquad \text{for all } \sigma, h \in G.$$

Proof. Observe that

$$2\mathbb{Z}^d \subset \Lambda \subset \mathbb{Z}^d, \qquad 2\mathbb{Z}^d \subset 2\Lambda^\perp \subset \mathbb{Z}^d.$$

To prove (29), we can use the following general fact. If $\Gamma_1 \subset \Gamma_2$ are two (full rank) lattices in $\mathbb{R}^d$, then we have a group isomorphism

$$(33) \qquad \widehat{\Gamma_2/\Gamma_1} \cong (\Gamma_1)^\perp/(\Gamma_2)^\perp.$$

This is a consequence of the duality theorem [27, Theorem 2.1.2] since

$$\widehat{\Gamma_2/\Gamma_1} \cong \text{Ann}(\hat{\Gamma}_2, \Gamma_1) \cong \text{Ann}\left(\mathbb{R}^d/(\Gamma_2)^\perp, \Gamma_1\right) \cong (\Gamma_1)^\perp/(\Gamma_2)^\perp,$$

where $\text{Ann}(\hat{\Gamma}_2, \Gamma_1)$ denotes the annihilator of a subgroup Γ_1 in $\hat{\Gamma}_2$. Applying the above to $\Gamma_1 = 2\mathbb{Z}^d$ and $\Gamma_2 = \Lambda$ yields (29)

$$\widehat{\Lambda/(2\mathbb{Z}^d)} \cong \left(\tfrac{1}{2}\mathbb{Z}^d\right)/\Lambda^\perp \cong \mathbb{Z}^d/\left(2\Lambda^\perp\right).$$

To prove the pairing (30), note that any $\alpha + 2\Lambda^\perp$, $\alpha \in \mathbb{Z}^d$, defines a character on G by (30). Since G is assumed to be separable, we can explicitly identify the dual lattice of $\frac{1}{2}\Lambda$ as

$$(34) \qquad 2\Lambda^\perp = \left\{ \alpha \in \mathbb{Z}^d : \sum_{j \in S_i} \alpha_j \equiv 0 \mod 2 \qquad \text{for } i = 1, \ldots, k \right\}.$$

Hence, if $\alpha \notin 2\Lambda^\perp$, then (30) defines a nontrivial character on G. Thus, all characters on G must be of this form.

For every $i = 1, \ldots, k$, choose $n_i \in S_i$. Define the mapping I first on generators

$$(35) \qquad I(\sigma^i) = \delta_{n_i} + 2\Lambda^\perp \qquad \text{for } i = 1, \ldots, k$$

and then extend it to a group homomorphism $I : G \to \hat{G}$. This is well-defined since all nontrivial elements of $\widehat{G}$ have torsion 2. To show that this is an isomorphism take any nontrivial element $\sigma \in G$ of the form $\sigma = \sum_{i=1}^k c_i \sigma^i$, where $c_i = 0, 1$. Then,

$$I(\sigma) = \alpha + 2\Lambda^\perp, \qquad \text{where } \alpha = \sum_{i=1}^k c_i \delta_{n_i}.$$

Since $c_i = 1$ for some i, by (34) $\alpha \notin 2\Lambda^\perp$. Hence, I is $1-1$ and thus an isomorphism.

Finally, (31) follows immediately from (35). Likewise, by (35) we have for any $\sigma, h \in G$,

$$\langle I(\sigma), h \rangle \equiv \sum_{i=1}^{k} \sigma_{n_i} h_{n_i} \equiv \langle \sigma, I(h) \rangle \mod 2.$$

This completes the proof of the lemma.   □

In light of Lemma 4.4 we shall slightly abuse the notation by identifying elements of $\widehat{G}$ with some fixed choice of coset representatives of $\mathbb{Z}^d/(2\Lambda^\perp)$. We are now ready to formulate the main result of this section.

THEOREM 4.5. *Let $G \subset \mathbb{Z}^d/(2\mathbb{Z}^d)$ be a separable group with k generators and thus of order 2^k. Furthermore, let g be a function in $L^2(\mathbb{R}^d)$ and let N be a subset of $\mathbb{Z}^d$ such that*

$$|N \cap \operatorname{orbit}(x)| = 1 \quad \text{for all } x \in \mathbb{Z}^d.$$

For each $\gamma \in N$, set $c_\gamma = 2^{-k}|\operatorname{orbit}(\gamma)|^{1/2}$. Consider the Gabor system

$$\mathcal{G}(g, G) = \{T_\lambda M_\gamma g\}_{\lambda \in \frac{1}{2}\Lambda, \gamma \in \mathbb{Z}^d}, \qquad \text{where } \Lambda = \cup_{\sigma \in G}\left(\sigma + 2\mathbb{Z}^d\right),$$

and the Wilson system

$$(36) \qquad \mathcal{W}(g, G) = \left\{ T_\lambda T_{\frac{1}{2}h} c_\gamma \sum_{\sigma \in G} (-1)^{\langle I(h)+\gamma, \sigma \rangle} M_{R_\sigma \gamma} g \right\}_{\lambda \in \mathbb{Z}^d, h \in G, \gamma \in N}.$$

If g is separable with respect to G and $\hat{g}(\omega) = \overline{\hat{g}(\omega)}$, then the following holds:

(i) *The Gabor system $\mathcal{G}(g, G)$ has Bessel bound b if and only if the Wilson system $\mathcal{W}(g, G)$ has Bessel bound $2^{-k}b$. In either (and hence both cases) the Gabor frame operator $S_\mathcal{G}$ and the Wilson frame operator $S_\mathcal{W}$ satisfy*

$$S_\mathcal{G} = 2^k S_\mathcal{W}.$$

(ii) *The Gabor system $\mathcal{G}(g, G)$ is a frame for $L^2(\mathbb{R}^d)$ with bounds a and b if and only if the Wilson system $\mathcal{W}(g, G)$ is a Riesz basis for $L^2(\mathbb{R}^d)$ with bounds $2^{-k}a$ and $2^{-k}b$.*

(iii) *The Gabor system $\mathcal{G}(g, G)$ is a tight frame for $L^2(\mathbb{R}^d)$ with frame bound $a = 2^k$ if and only if the Wilson system $\mathcal{W}(g, G)$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.*

Remark 1. Note that the Wilson system $\mathcal{W}(g, G)$ corresponding to the choice of G being the trivial subgroup is simply a Gabor system $\{T_\lambda M_\gamma g\}_{\lambda \in \mathbb{Z}^d, \gamma \in \mathbb{Z}^d}$. Hence, the statements of Theorem 4.5(i) are trivial for $k = 0$. In contrast, the Wilson system corresponding to the maximal group $G = \mathbb{Z}^d/(2\mathbb{Z}^d)$, where $k = d$, for appropriate choice of $N \subset \mathbb{Z}^d$, is a tensor product of one-dimensional Wilson systems as in Example 5. Moreover, observe that the choice of a subgroup G generated by $\sigma = (1, \ldots, 1)$ yields the same Wilson system as that in Theorem 3.1.

The following density-type theorem for Wilson systems is an easy consequence of Theorem 4.5.

COROLLARY 4.6. *Consider the same setup and the same assumptions as in Theorem 4.5. If $\mathcal{W}(g, G)$ is a frame for $L^2(\mathbb{R}^d)$ with bounds a and b, then $\mathcal{W}(g, G)$ is a Riesz basis for $L^2(\mathbb{R}^d)$ with bounds a and b.*

Proof. If $\mathcal{W}(g, G)$ is a frame for $L^2(\mathbb{R}^d)$ with bounds a and b, then, by Theorem 4.5(i), so is $\mathcal{G}(g, G)$ with bounds $2^k a$ and $2^k b$. The conclusion now follows from Theorem 4.5(ii).   □

Before proceeding with the proof we need to emphasize that some of the functions appearing in the Wilson system (36) are zero. Hence, they should be disregarded due to the cancellation that might happen for some choices of $h \in G$ and $\gamma \in N$. This is a consequence of the following elementary lemma.

LEMMA 4.7. *Let $\gamma \in N$ and $h \in G$. Let $G_\gamma = \{\sigma \in G : R_\sigma \gamma = \gamma\}$ be the stabilizer of γ. Consider a character $\chi \in \widehat{G}_\gamma$ given by*

$$\chi(\sigma) = (-1)^{\langle I(h) + \gamma, \sigma \rangle} \qquad for \ \sigma \in G_\gamma.$$

Then for any $\sigma_0 \in G$, the following sum over a coset of the quotient group G/G_γ satisfies

$$(37) \qquad \sum_{\sigma \in \sigma_0 + G_\gamma} (-1)^{\langle I(h) + \gamma, \sigma \rangle} = \begin{cases} \pm |G_\gamma|, & \chi \equiv 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\sigma_0 = 0$, then formula (37) follows easily from [19, Lemma (23.19)] and Lemma 4.4. The general case $\sigma_0 \in G$ introduces an additional factor ± 1, and hence (37) holds in full generality. □

A symplectic Wilson system is constructed in the following result.

PROPOSITION 4.8. *Assume the same setup as in Theorem 4.5. If a matrix $A \in \text{Sp}(d)$ is given with associated operator $\mu(A)$ such that (7) is satisfied, i.e.,*

$$\mu(A)\pi(\nu) = \varphi(A, \nu) \cdot \pi(A\nu)\mu(A) \ \ for \ all \ \ \nu \in \mathbb{R}^{2d}$$

and where $|\varphi(A, \nu)| = 1$, then the symplectic Wilson system

$$\mathcal{W}_s(g, G) = \left\{ \pi(A\lambda)\pi(A\lambda_h^*) c_\gamma \sum_{\sigma \in G} \varphi(A, \widetilde{R}_\sigma \gamma)(-1)^{\langle I(h) + \gamma, \sigma \rangle} \pi(A\widetilde{R}_\sigma \gamma)\mu(A)g : \right.$$

$$\left. \lambda \in \mathbb{Z}^d \times \{0\}^d, h \in G, \gamma \in \{0\}^d \times N \right\},$$

where $\lambda_h^* = \frac{1}{2}h \times \{0\}^d$ for $h \in G$, is a [frame, Riesz basis, orthonormal basis] if and only if the Wilson system $\mathcal{W}(g, G)$ in Theorem 4.5 has the same property. Moreover, the [frame, Riesz] bounds of the two systems are the same.

Note that for this statement the phase-factor $\varphi(A, \widetilde{R}_\sigma \gamma)$ from the relation (7) is important. This was not the case for the Wilson system considered in Theorem 3.1. If all numbers in the set $\{\varphi(A, \widetilde{R}_\sigma \gamma)\}_{\sigma \in G}$ are the same for every fixed $\gamma \in \{0\}^d \times N$, then the phase factor can be omitted from the definition of $\mathcal{W}_s(g, G)$.

The key part of the proof of Theorem 4.5 is contained in the following lemma.

LEMMA 4.9. *Consider the same setup and the same assumptions as in Theorem 4.5. Suppose the condition in (18) is satisfied, then the following holds:*

(i) *If the Gabor system $\mathcal{G}(g, G)$ is considered as a shift-invariant system with generators $\{M_\gamma g\}_{\gamma \in \mathbb{Z}^d}$ and with shifts along the lattice $(1/2)\Lambda$, then its autocorrelation functions are given by*

$$t_{\alpha, \mathcal{G}}(\omega) = 2^k \sum_{\gamma \in \mathbb{Z}^d} \hat{g}(\omega - \gamma)\overline{\hat{g}(\omega - \gamma - \alpha)}, \qquad \alpha \in 2\Lambda^\perp, a.e. \ \omega \in \mathbb{R}^d.$$

(ii) *If the Wilson system $\mathcal{W}(g, G)$ is considered a shift-invariant system with generators*

$$\left\{ T_{\frac{1}{2}h} c_\gamma \sum_{\sigma \in G} (-1)^{\langle I(h)+\gamma, \sigma \rangle} M_{R_\sigma \gamma} g \right\}_{h \in G, \gamma \in N}$$

and with shifts along the lattice $\mathbb{Z}^d$, then its autocorrelation functions are given by

$$t_{\alpha, \mathcal{W}}(\omega) = \begin{cases} \sum_{\gamma \in \mathbb{Z}^d} \hat{g}(\omega - \gamma) \overline{\hat{g}(\omega - \gamma - \alpha)}, & \alpha \in 2\Lambda^\perp, \\ 0, & \alpha \in \mathbb{Z}^d \backslash 2\Lambda^\perp, \end{cases} \quad a.e. \; \omega \in \mathbb{R}^d.$$

Proof. The statement of (i) follows immediately from the definition of autocorrelation functions and the observation that the lattice $\Lambda = \cup_{\sigma \in G} (\sigma + 2\mathbb{Z}^d)$ has density 2^{-k}.

Consider now the Wilson system as a shift-invariant system along $\mathbb{Z}^d$ with generators

$$(38) \qquad \psi_{h,\gamma} = T_{\frac{1}{2}h} c_\gamma \sum_{\sigma \in G} (-1)^{\langle I(h)+\gamma, \sigma \rangle} M_{R_\sigma \gamma} g, \quad h \in G, \; \gamma \in N.$$

Then,

$$(39) \qquad t_{\alpha, \mathcal{W}}(\omega) = \sum_{h \in G, \gamma \in N} \hat{\psi}_{h,\gamma}(\omega) \overline{\hat{\psi}_{h,\gamma}(\omega - \alpha)} \qquad \text{for } \alpha \in \mathbb{Z}^d \text{ and a.e. } \omega \in \mathbb{R}^d.$$

The Fourier transform of the generators $\psi_{h,\gamma}$ are given by

$$\hat{\psi}_{h,\gamma} = c_\gamma (-1)^{\langle h, \cdot \rangle} \sum_{\sigma \in G} (-1)^{\langle I(h)+\gamma, \sigma \rangle} T_{R_\sigma \gamma} \hat{g}.$$

Hence, by (32) the expression (39) becomes the following:

$$t_{\alpha, \mathcal{W}}(\omega) = \sum_{h \in G, \gamma \in N} |c_\gamma|^2 (-1)^{\langle h, \alpha \rangle} \sum_{\sigma, \sigma' \in G} (-1)^{\langle I(h)+\gamma, \sigma+\sigma' \rangle} T_{R_\sigma \gamma} \hat{g}(\omega) \overline{T_{R_{\sigma'} \gamma} \hat{g}(\omega - \alpha)}$$

$$= \sum_{\gamma \in N, \sigma, \sigma' \in G} |c_\gamma|^2 (-1)^{\langle \gamma, \sigma+\sigma' \rangle} T_{R_\sigma \gamma} \hat{g}(\omega) \overline{T_{R_{\sigma'} \gamma} \hat{g}(\omega - \alpha)} \sum_{h \in G} (-1)^{\langle h, \alpha+I(\sigma+\sigma') \rangle}.$$

Note that by [19, Lemma (23.19)] and Lemma 4.4 for any $\alpha \in \mathbb{Z}^d$ we have

$$(40) \qquad \sum_{h \in G} (-1)^{\langle h, \alpha+I(\sigma+\sigma') \rangle} = \begin{cases} 2^k & \text{if } \alpha + I(\sigma + \sigma') \in 2\Lambda^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

For a fixed $\alpha \in \mathbb{Z}^d$, let $\tilde{\alpha} \in G$ be such that $I(\tilde{\alpha}) = \alpha + 2\Lambda^\perp$. Hence,

$$\alpha + I(\sigma + \sigma') \in 2\Lambda^\perp \iff I(\tilde{\alpha} + \sigma + \sigma') \in 2\Lambda^\perp \iff \tilde{\alpha} + \sigma + \sigma' \in 2\mathbb{Z}^d.$$

Using (40) we continue our calculation to find that

$t_{\alpha,\mathcal{W}}(\omega)$

$$
= \sum_{\gamma \in N, \sigma, \sigma' \in G} |c_\gamma|^2 (-1)^{\langle \gamma, \sigma+\sigma' \rangle} T_{R_\sigma \gamma} \hat{g}(\omega) \overline{T_{R_{\sigma'} \gamma} \hat{g}(\omega - \alpha)} \begin{cases} 2^k & \text{if } \tilde{\alpha} + \sigma + \sigma' \in 2\mathbb{Z}^d, \\ 0 & \text{otherwise.} \end{cases}
$$

$$
= \sum_{\gamma \in N, \sigma \in G} 2^k |c_\gamma|^2 (-1)^{\langle \gamma, \tilde{\alpha} \rangle} T_{R_\sigma \gamma} \hat{g}(\omega) \overline{T_{R_{\sigma+\tilde{\alpha}} \gamma} \hat{g}(\omega - \alpha)}
$$

$$
= \sum_{\gamma \in N, \sigma \in G} \frac{|\text{orbit}(\gamma)|}{2^k} (-1)^{\langle \gamma, \tilde{\alpha} \rangle} \hat{g}(\omega - R_\sigma \gamma) \overline{\hat{g}(\omega - \alpha - R_{\sigma+\tilde{\alpha}} \gamma)}
$$

$$
= \sum_{\gamma \in \mathbb{Z}^d} (-1)^{\langle \gamma, \tilde{\alpha} \rangle} \hat{g}(\omega - \gamma) \overline{\hat{g}(\omega - \alpha - R_{\tilde{\alpha}} \gamma)}.
$$

In the last step we used the identity

$$
(-1)^{\langle R_\sigma \gamma, \tilde{\alpha} \rangle} = (-1)^{\langle \gamma, \tilde{\alpha} \rangle}
$$

and the fact that the stabilizer subgroup $G_\gamma = \{\sigma \in G : R_\sigma \gamma = \gamma\}$ has order $|G_\gamma| = |G|/|\text{orbit}(\gamma)|$. Hence, for all $\alpha \in \mathbb{Z}^d$,

$$
(41) \qquad t_{\alpha,\mathcal{W}}(\omega) = \sum_{\gamma \in \mathbb{Z}^d} (-1)^{\langle \gamma, \tilde{\alpha} \rangle} \hat{g}(\omega - \gamma) \overline{\hat{g}(\omega - \alpha - R_{\tilde{\alpha}} \gamma)} \qquad \text{a.e. } \omega \in \mathbb{R}^d.
$$

We now consider two cases: (I) $\alpha \in 2\Lambda^\perp$ and (II) $\alpha \in \mathbb{Z}^d \setminus 2\Lambda^\perp$.

In case (I) we have $\tilde{\alpha} \in 2\mathbb{Z}^d$, which implies that $R_{\tilde{\alpha}}$ is the identity, and further $(-1)^{\langle \gamma, \tilde{\alpha} \rangle} = 1$ for all $\gamma \in \mathbb{Z}^d$. Therefore, for all $\alpha \in 2\Lambda^\perp$, (41) becomes

$$
(42) \qquad t_{\alpha,\mathcal{W}}(\omega) = \sum_{\gamma \in \mathbb{Z}^d} \hat{g}(\omega - \gamma) \overline{\hat{g}(\omega - \alpha - \gamma)} \quad \text{a.e. } \omega \in \mathbb{R}^d.
$$

Next we consider case (II). Due to the assumption that g is separable with respect to G, g is of the form (28). By the Fubini theorem

$$
\|g\|_2 = \prod_{j=0}^{k} \|g_j\|_2
$$

and

$$
\hat{g}(\omega) = \prod_{j=0}^{k} \hat{g}_j \circ P_{S_j}(\omega) \qquad \text{for } \omega \in \mathbb{R}^d.
$$

Hence, we can rewrite (41) as
(43)
$$
t_{\alpha,\mathcal{W}}(\omega) = \prod_{j=0}^{k} \left(\sum_{\gamma \in \mathbb{Z}^{|S_j|}} (-1)^{\langle \gamma, P_{S_j} \tilde{\alpha} \rangle} \hat{g}_j(P_{S_j} \omega - \gamma) \overline{\hat{g}_j(P_{S_j} \omega - P_{S_j} \alpha - (-1)^{\tilde{\alpha}_{n_j}} \gamma)} \right).
$$

Case (II) implies that $\alpha \in \mathbb{Z}^d \setminus 2\Lambda^\perp$ and $\tilde{\alpha} \in \Lambda \setminus 2\mathbb{Z}^d$. Therefore, there exists $j = 1, 2, \ldots, k$ such that $P_{S_j} \tilde{\alpha}$ has all odd coordinates. By (31) this implies that $|P_{S_j} \alpha|$ is odd. Consider the jth term in the product (43), i.e.,

$$
C := \sum_{\gamma \in \mathbb{Z}^{|S_j|}} (-1)^{|\gamma|} \hat{g}_j(\omega' - \gamma) \overline{\hat{g}_j(\omega' - P_{S_j} \alpha + \gamma)}, \qquad \text{where } \omega' = P_{S_j} \omega \in \mathbb{R}^{|S_j|}.
$$

We wish to show that $C = 0$ for a.e. ω'. To this end, as in the proof of Theorem 3.1, we make use of a change of variable: $\gamma \mapsto -\gamma' + P_{S_j}\alpha$. This yields that

$$
\begin{aligned}
C &= \sum_{\gamma' \in \in \mathbb{Z}^{|S_j|}} (-1)^{|-\gamma' + P_{S_j}\alpha|} \hat{g}_j(\omega' + \gamma' - P_{S_j}\alpha)\overline{\hat{g}_j(\omega' - \gamma')} \\
&= - \sum_{\gamma' \in \mathbb{Z}^{|S_j|}} (-1)^{|\gamma'|} \hat{g}_j(\omega' - \gamma)\overline{\hat{g}_j(\omega' + \gamma - P_{S_j}\alpha)} = -C.
\end{aligned}
$$

Here, we used the fact that (31) implies that $|P_{S_j}\alpha|$ is odd and that $\hat{g}_j(\omega') = \overline{\hat{g}_j(\omega')}$ for all $\omega' \in \mathbb{R}^{|S_j|}$. We conclude that $C = 0$, and hence, for all $\alpha \in \mathbb{Z}^d \setminus (2\Lambda^\perp)$, we have that

(44) $$ t_{\alpha,\mathcal{W}}(\omega) = 0 \qquad \text{a.e. } \omega \in \mathbb{R}^d. $$

This completes the proof of Lemma 4.9. $\square$

We are now ready to give the proof of Theorem 4.5.

Proof. Assume that either the Gabor system $\mathcal{G}(g, G)$ or that the Wilson system $\mathcal{W}(g, G)$ is a Bessel sequence. Then, the same argument as in the proof of Theorem 3.1 with the use of Proposition 3.5 and Lemma 4.9 instead of Lemma 3.7 shows that for any $f \in \mathcal{D}$,

$$
\sum_{\phi \in \mathcal{G}(g,G)} |\langle f, \phi \rangle|^2 = \sum_{\alpha \in \Lambda^\perp} \langle M_{t_{\alpha,\mathcal{G}}} T_\alpha \hat{f}, \hat{f} \rangle = 2^k \sum_{\alpha \in \mathbb{Z}^d} \langle M_{t_{\alpha,\mathcal{W}}} T_\alpha \hat{f}, \hat{f} \rangle = 2^k \sum_{\phi \in \mathcal{W}(g,G)} |\langle f, \phi \rangle|^2.
$$

This implies the equality $S_\mathcal{G} = 2^k S_\mathcal{W}$, which shows (i). At the same time it shows the "if" direction of (ii) and (iii) as in the proof of Theorem 3.1.

Concerning the converse directions in statements (ii) and (iii) we proceed as follows. If the Gabor system $\mathcal{G}(g, G)$ is a tight frame with frame bound 2^k, then the Wilson system is a tight frame with frame bound 1. By Lemma 3.8 it remains to show that all nonzero generators (38) of the Wilson system (36) have norm equal to 1. The assumption that the Gabor system in (i) is a tight frame combined with Lemma 3.9 imply that the family of functions $\{M_{2\gamma}g\}_{\gamma \in \Lambda^\perp}$ is an orthogonal set. Note that

$$ R_\sigma \gamma - \gamma \in 2\mathbb{Z}^d \subset 2\Lambda^\perp \qquad \text{for any } \sigma \in G, \gamma \in \mathbb{Z}^d. $$

Consequently, for any $\gamma \in N$, the family of functions $\{M_{R_\sigma\gamma}g\}_{\sigma \in G}$ is an orthonormal set after neglecting that each function is repeated $|G_\gamma| = |G|/|\text{orbit}(\gamma)|$ times. Here, $G_\gamma = \{\sigma \in G : R_\sigma\gamma = \gamma\}$ is the stabilizer of γ. For a fixed $\gamma \in N$ and $h \in G$, consider the character $\chi \in \widehat{G}_\gamma$ given as in Lemma 4.7. If $\chi \equiv 1$, then a direct calculation using (37) shows that

$$ \|\psi_{h,\gamma}\|^2 = |c_\gamma|^2 \left\| \sum_{\sigma \in G/G_\gamma} \pm |G_\gamma| M_{R_\sigma\gamma}g \right\|^2 = |c_\gamma|^2 |G_\gamma|^2 |G/G_\gamma| = 1. $$

Otherwise, if $\chi \not\equiv 1$, then $\psi_{h,\gamma} = 0$ and these generators are vacuous. Therefore, by Lemma 3.8, the Wilson system $\mathcal{W}(g, G)$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. We have now proved (iii).

To finish the proof of (ii) we adapt the argument of [16, Corollary 8.5.6]: Let S be the frame operator of the Gabor frame $\mathcal{G}(g, G)$. Then,

$$ S^{-1/2}\mathcal{G}(g) = \mathcal{G}\left(S^{-1/2}g\right) $$

is a Parseval frame for $L^2(\mathbb{R}^d)$. We claim that, just as with the function g, the function $S^{-1/2}g$ is separable with respect to group G. Since g is separable with respect to G, we can write it in the form (28). Hence, the Gabor system $\mathcal{G}(g,G)$ is a tensor product of the Gabor systems $\{M_\gamma T_\lambda g_j \ : \ \lambda \in P_{S_j}(\frac{1}{2}\Lambda), \gamma \in P_{S_j}(\mathbb{Z}^d)\}$, $j = 0, \ldots, k$. Let T_j denote the frame operator of these Gabor systems which acts on $L^2(\mathbb{R}^{|S_j|})$. Hence, the frame operator S is a tensor product of frame operators T_j. That is, for any separable function $f \in L^2(\mathbb{R}^d)$ of the form (28) we have

$$S(f)(x) = \prod_{j=0}^{k} T_j(f_j) \circ P_{S_j}(x) \qquad \text{for } x \in \mathbb{R}^d.$$

A similar formula holds for $S^{-1/2}$. Hence, we see that $S^{-1/2}g$ is separable with respect to G. Since each frame operator T_j preserves symmetry as in (9), it also follows that $\mathcal{F}S^{-1/2}g(\omega) = \overline{\mathcal{F}S^{-1/2}g(\omega)}$. Hence $\mathcal{W}(S^{-1/2}g, G)$ is an orthonormal basis. Moreover,

$$\mathcal{W}\left(S^{-1/2}g, G\right) = S^{-1/2}\mathcal{W}(g,G).$$

But this implies that the Wilson system itself is a Riesz basis. This proves (ii). □

Remark 2. In general, choosing an arbitrary separable group G of intermediate order 2^k, $k = 1, \ldots, d-1$ leads to a huge number of distinct Wilson systems. Indeed, let $p(n)$ be the partition function that represents the number of ways of writing n as a sum of positive integers. Then, any partition of $[d] = \{1, \ldots, d\}$ leads to a separable subgroup $G \subset \mathbb{Z}^d/(2\mathbb{Z}^d)$. Hence, up to a permutation isomorphism there are $p(d)$ distinct separable groups in the dimension d. Since $p(d)$ satisfies the asymptotic growth

$$\log p(d) \sim \pi\sqrt{\frac{2}{3}}\sqrt{d} \qquad \text{as } d \to \infty,$$

and hence this number grows rapidly with the dimension d.

By tensoring the construction in Example 4 and the usual construction of Wilson bases in dimension one, it is clear that we can construct generators $g \in L^2(\mathbb{R}^d)$ of 2^k-modular Wilson bases with good time-frequency localization for *each* $k = 1, \ldots, d$. In other words, for each $k = 1, \ldots, d$, we can find a subgroup G of order 2^k such that the corresponding Wilson system has nice window functions generating an orthonormal basis. However, not every Wilson system from Theorem 4.5, i.e., not every subgroup G, has nice basis generators. As an example, consider $d = 2$ and take G to be the subgroup with coset representatives $(0,0)$ and $(1,0)$. Then $\Lambda = \mathbb{Z} \times 2\mathbb{Z}$. Being separable with respect to G means that $g(x, y) = g_1(x)g_2(y)$. Hence, the Gabor system as in Theorem 4.5(i) with $g(x, y) = g_1(x)g_2(y)$ is a tight frame for $L^2(\mathbb{R}^2)$ if and only if $\{T_{k/2}M_m g_1\}_{k,m\in\mathbb{Z}}$ and $\{T_k M_m g_2\}_{k,m\in\mathbb{Z}}$ are tight frames for $L^2(\mathbb{R})$. However, by the Balian–Low theorem g_2 cannot be well localized in time and frequency. Hence, the same conclusion holds for g.

While it is now possible by Theorem 4.5 to construct Wilson bases from Gabor frames of redundancy 2^k, $k = 1, \ldots, d$, it is still an open question, mentioned in [16], whether other redundancies are possible. Wojdyłło [29] shows that it is possible to construct redundant Wilson-type tight frames for $L^2(\mathbb{R})$ from Gabor tight frames of redundancy 3, but this approach does not provide orthogonality. It is our hope that the methods developed in this paper can be used to attack this longstanding open problem.

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