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Linear Independence of Time-Frequency Translates in $\mathbb{R}^d$

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Abstract We establish the linear independence of time-frequency translates for functions f on $\mathbb{R}^d$ having one-sided decay $\lim_{x\in H,\ |x|\to\infty} |f(x)|e^{c|x|\log|x|} = 0$ for all $c > 0$, which do not vanish on an affine half-space $H \subset \mathbb{R}^d$.

Keywords Gabor system · Exponential decay · HRT conjecture · Linear independence · Time-frequency translates

Mathematics Subject Classification 26A99 · 42C40 · 11K70

1 Introduction

The Heil–Ramanathan–Topiwala (HRT) conjecture [12] states that time-frequency translates of a non-zero square integrable function f on $\mathbb{R}^d$ are linearly independent.[1] There have been a few partial results on this conjecture, mostly focusing on finding conditions on $\Lambda \subset \mathbb{R}^d \times \mathbb{R}^d$ which guarantee that time-frequency translates

$$\mathcal{G}(f, \Lambda) := \left\{ M_a T_b f = e^{2\pi i a\cdot} f(\cdot - b) : (a, b) \in \Lambda \right\}$$

[1] The original HRT conjecture was only for $\mathbb{R}$, but the question is also open for higher dimensions.

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along Λ are linearly independent [3,5–7,16]. Other interesting results related to the HRT conjecture can be found in [1,10,11]. In [4], the authors found a one-sided decay condition that guarantees that arbitrary time-frequency shifts are linearly independent.

The goal of this paper is to generalize the main result of the authors [4] to higher dimensions. We point out that the generalization to higher dimensions of linear independence does not always follow expectations. Using the Fourier transform it is easy to see that in $\mathbb{R}$, translates of L^p functions are linearly independent for $1 \leq p < \infty$. However, the situation in $\mathbb{R}^d$ is quite different, as all translates of L^p functions in $\mathbb{R}^d$ are linearly independent if and only if $p \leq \frac{2d}{d-1}$, by the results of Edgar and Rosenblatt [8,19].

The main theorem of this paper can be formulated as follows.

Theorem 1.1 *Let H be an affine half-space in $\mathbb{R}^d$, i.e., $H = \{x \in \mathbb{R}^d : \langle x, v\rangle > a\}$ for some $v \in \mathbb{R}^d \setminus \{0\}$ and $a \in \mathbb{R}$. Let $f : \mathbb{R}^d \to \mathbb{C}$ be a Lebesgue measurable function which does not vanish almost everywhere on H. Assume that for all $c > 0$,*

$$\lim_{x\in H,\ |x|\to\infty} |f(x)|e^{c|x|\log|x|} = 0. \tag{1.1}$$

Then, the set $\mathcal{G}(f, \mathbb{R}^{2d})$ of time-frequency translates of f is linearly independent. That is, $\mathcal{G}(f, \Lambda)$ is linearly independent for any $\Lambda \subset \mathbb{R}^{2d}$.

Note that, unlike the one-dimensional case, we must make the additional assumption that a function f does not vanish on a half-space. This is because in one dimension, functions which vanish on a tail trivially have linearly independent time-frequency shifts. However, if there is a function $f \in L^2(\mathbb{R})$ with linearly dependent time-frequency shifts, then $\mathbf{1}_{[-1,0]} \otimes f \in L^2(\mathbb{R}^2)$ vanishes on a half-space $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$ and has linearly dependent time-frequency shifts. However, it is possible to remove the assumption that f does not vanish on a half-space in Theorem 1.1 provided we weaken the corresponding conclusion. This is shown in Theorem 3.5.

There are several new ingredients employed in extending the one-dimensional result [4, Theorem1.1] to the higher-dimensional Theorem 1.1. First, we prove a generalization of the Montgomery–Vaughan inequality [17] to higher dimensions, Theorem 2.1, using the theory of Beurling–Selberg extremal functions for Euclidean balls developed by Holt and Vaaler [13]. We also show a higher-dimensional analogue of the Turán–Nazarov inequality, Theorem 2.2, from the corresponding one-dimensional result [18]. Using this we extend the lower bound estimate on products of trigonometric polynomials from [4] to higher dimensions. Then, we establish the key sufficient condition for the linear independence of time-frequency translates, Theorem 3.1, using the concept of an extended half-space, which induces a total order on $\mathbb{R}^d$. We also introduce the notion of directional quasi-norm that enables us to prove Theorem 3.3, which provides a sharper version of Theorem 1.1.

2 Useful Facts

In this section, we recall generalizations of the Montgomery–Vaughan inequality and the Turán–Nazarov inequality to $\mathbb{R}^d$. We will also provide proofs of the exact inequalities that we need in our development. In Theorem 2.1 we assume that the dimension d is fixed; all constants are allowed to (implicitly) depend on d.

Theorem 2.1 (Holt, Vaaler) *Fix $d \in \mathbb{N}$. For every $\delta > 0$, there exists $R > 0$ such that whenever a trigonometric polynomial*

$$u(x) = \sum_{j=1}^{m} c_j e^{2\pi i \langle a_j, x\rangle}, \qquad c_j \in \mathbb{C},\ a_j \in \mathbb{R}^d, \tag{2.1}$$

satisfies $\min\{|a_j - a_k| : j \neq k\} \geq \delta$, *we have*

$$\frac{|\mathbf{B}_R(y)|}{2} \sum_{j=1}^{m} |c_j|^2 \leq \int_{\mathbf{B}_R(y)} \big|u(x)\big|^2\, dx \qquad \textit{for all } y \in \mathbb{R}^d. \tag{2.2}$$

Here, $|\mathbf{B}_R(y)|$ is d-dimensional Lebesgue measure of the Euclidean ball $\mathbf{B}_R(y)$ of radius R centered at y.

Proof This is an immediate corollary to [13, Theorem 4]. We briefly include the relevant facts for completeness; all references are to [13] and notation is from [13, Theorem 1]. Let ξ, δ, and ν be real numbers with $\delta > 0$ and $\nu > -1$. Define $u_\nu(\xi, \delta)$ to be the infimum of

$$\frac{1}{2} \int_{-\infty}^{\infty} \big(T(x) - S(x)\big) |x|^{2\nu+1}\, dx,$$

where the infimum is over all pairs of entire functions S and T of exponential type at most $2\pi\delta$ such that

$$S(x) \leq \operatorname{sgn}(x - \xi) \leq T(x) \qquad \text{for all } x \in \mathbb{R}.$$

By the estimate [13, p. 204], there is a constant A, depending only on ν, such that

$$u_\nu(\xi, \delta) \leq \delta^{-1} \xi^{2\nu+1} \left(1 + \frac{A}{\xi^2 \delta^2}\right) \qquad \text{whenever } \xi\delta \geq 1. \tag{2.3}$$

By [13, Theorem 4] applied for $\nu = (d-2)/2$ we have

$$\omega_{d-1} \left((2\nu+2)^{-1} R^{2\nu+2} - u_\nu(R, \delta) \right) \sum_{j=1}^{m} |c_j|^2 \leq \int_{\mathbf{B}_R(y)} \big|u(x)\big|^2\, dx. \tag{2.4}$$

Here, ω_{d-1} is the surface area of the unit sphere $S^{d-1} \subset \mathbb{R}^d$. We note that [13, Theorem 4] requires balls to be centered at 0. However, in the special case when $d = 2\nu + 2$, it holds more generally for every ball since the exponent in [13, (1.28)] vanishes and translations correspond to unimodular modifications of coefficients c_j, $j = 1, \ldots, m$.

Choose $R > 0$ such that $\delta R \geq 1$ and

$$\frac{1}{\delta R}\left(1 + \frac{A}{R^2\delta^2}\right) < \frac{1}{2d}.$$

It follows by (2.3) that

$$\begin{aligned}(2\nu+2)^{-1}R^{2\nu+2} - u_\nu(R,\delta) &\geq (2\nu+2)^{-1}R^{2\nu+2} - \delta^{-1}R^{2\nu+1}\left(1+\frac{A}{R^2\delta^2}\right)\\ &\geq R^{2\nu+1}\left(\frac{R}{d} - \frac{R}{2d}\right) = \frac{R^d}{2d}.\end{aligned}$$

Combining this with (2.4) and the fact that $|\mathbf{B}_R(y)| = R^d\omega_{d-1}/d$ yields (2.2). □

We will also need a higher-dimensional analogue of the Turán–Nazarov inequality [18]. A similar result for $\mathbb{Z}^d$-periodic trigonometric polynomials u, which corresponds to the case when $a_j \in \mathbb{Z}^d$ in (2.1), was considered by Fontes-Merz [9]. Theorem 2.2 also appears in [2, Lemma 12], but without a proof that we provide below.

Theorem 2.2 (Higher-dimensional Turán–Nazarov inequality) *Let u be a trigonometric polynomial of order m as in* (2.1). *Let E be any measurable subset of positive measure of a ball $\mathbf{B}_R(y) \subset \mathbb{R}^d$, $R > 0$, $y \in \mathbb{R}^d$. There exists an absolute and dimensionless constant A such that*

$$\sup_{x\in\mathbf{B}_R(y)} |u(x)| \leq \left(d2^dA\frac{|\mathbf{B}_R(y)|}{|E|}\right)^{m-1}\sup_{x\in E}|u(x)|. \tag{2.5}$$

Proof Recall that one-dimensional Turán–Nazarov inequality [18] guarantees the existence of an absolute constant A such that for any univariate trigonometric polynomial

$$\tilde{u}(r) = \sum_{j=1}^{m}\tilde{c}_j e^{2\pi i\tilde{a}_j r}, \qquad \tilde{c}_j \in \mathbb{C},\ \tilde{a}_j \in \mathbb{R}, \tag{2.6}$$

and any measurable subset $\tilde{E}$ of positive measure of an interval $I \subset \mathbb{R}$ we have

$$\sup_{r\in I}|\tilde{u}(r)| \leq \left(\frac{A|I|}{|\tilde{E}|}\right)^{m-1}\sup_{r\in\tilde{E}}|\tilde{u}(r)|. \tag{2.7}$$

Let $z_0 \in \mathbf{B}_R(y)$ be a point that achieves the maximum of $|u|$, i.e.,

$$|u(z_0)| = \sup_{x\in\mathbf{B}_R(y)}|u(x)|.$$

For any direction $\omega \in S^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ define a ray section of E by

$$E_\omega = \left\{r \in [0,\infty) : z_0 + r\omega \in E\right\}.$$

Let σ be $(d-1)$-dimensional Lebesgue measure on S^{d-1}. By the spherical integration formula we have

$$
\begin{aligned}
|E| = \int_{\mathbb{R}^d} \mathbf{1}_E(x)dx &= \int_{S^{d-1}} \int_0^\infty \mathbf{1}_E(z_0 + r\omega) r^{d-1} dr d\sigma(\omega) \\
&= \int_{S^{d-1}} \int_0^{2R} \mathbf{1}_{E_\omega}(r) r^{d-1} dr d\sigma(\omega) \\
&\leq (2R)^{d-1} \int_{S^{d-1}} |E_\omega| d\sigma(\omega) \leq (2R)^{d-1} \sigma(S^{d-1}) \operatorname*{ess\,sup}_{\omega\in S^{d-1}} |E_\omega|.
\end{aligned}
$$

Since $|\mathbf{B}_R(y)| = R^d \sigma(S^{d-1})/d$, there exists $\omega_0 \in S^{d-1}$ such that E_{ω_0} is Lebesgue measurable and

$$
\frac{|E_{\omega_0}|}{2R} \geq \frac{1}{d2^d} \frac{|E|}{|\mathbf{B}_R(y)|}. \tag{2.8}
$$

Define a univariate trigonometric polynomial $\tilde{u}$ by

$$
\tilde{u}(r) = u(z_0 + r\omega_0) = \sum_{j=1}^m \tilde{c}_j e^{2\pi i \tilde{a}_j r}, \qquad \text{where } \tilde{c}_j = c_j e^{2\pi i \langle a_j, z_0\rangle}, \tilde{a}_j = \langle a_j, \omega_0 \rangle.
$$

Applying (2.7) for $\tilde{u}$ and $\tilde{E} = E_{\omega_0} \subset [0, 2R]$, by (2.8) we have

$$
\begin{aligned}
\sup_{x\in\mathbf{B}_R(y)} |u(x)| = |u(z_0)| \leq \sup_{r\in[0,2R]} |\tilde{u}(r)| &\leq \left(A \frac{2R}{|E_{\omega_0}|}\right)^{m-1} \sup_{r\in E_{\omega_0}} |\tilde{u}(r)| \\
&\leq \left(d2^d A \frac{|\mathbf{B}_R(y)|}{|E|}\right)^{m-1} \sup_{x\in E} |u(x)|.
\end{aligned}
$$

This proves (2.5). □

As a consequence of Theorems 2.1 and 2.2 we obtain the following generalization of [3, Proposition 2.2].

Proposition 2.3 *Let u be a non-zero trigonometric polynomial as in (2.1). Let $R > 0$. Then there exists a constant $C > 0$, depending only on u and R, such that*

$$
\sup_{x\in\mathbf{B}_R(y)} |u(x)| \geq C \qquad \textit{for all } y \in \mathbb{R}^d. \tag{2.9}
$$

Proof Let $\delta = \min\{|a_j - a_k| : j \neq k\} > 0$. Let $R_0 > 0$ be the corresponding radius as in Theorem 2.1. Then,

$$
\frac{1}{2} \sum_{j=1}^m |c_j|^2 \leq \frac{1}{|\mathbf{B}_{R_0}(y)|} \int_{\mathbf{B}_{R_0}(y)} \left|u(x)\right|^2 dx \leq \sup_{x\in\mathbf{B}_{R_0}(y)} |u(x)|^2. \tag{2.10}
$$

This shows (2.9) when $R \geq R_0$. If $R < R_0$, then by Theorem 2.2, there exists a constant $c > 0$ such that

$$\sup_{x \in \mathbf{B}_R(y)} |u(x)| \geq c \sup_{x \in \mathbf{B}_{R_0}(y)} |u(x)|.$$

By (2.10) this again shows (2.9). □

3 Proof of the Main Theorem

We start by introducing a technical sufficient condition (3.1) for the linear independence of time-frequency translates of a measurable function, which generalizes the one-dimensional condition in [4]. This is a main ingredient in the proof of our main result, Theorem 1.1. Define the space of all Lebesgue measurable functions on the real line by

$$\mathcal{M} = \{f : \mathbb{R}^d \to \mathbb{C} \text{ is Lebesgue measurable}\}.$$

As is customary, we shall identify functions in $\mathcal{M}$ which are equal almost everywhere.

Definition 3.1 Given an orthonormal basis $\{v_j\}_{j=1}^d$ of $\mathbb{R}^d$ define an *extended half-space* by

$$H^{\{v_j\}} = \bigcup_{j=1}^{d} \{x \in \mathbb{R}^d : \langle x, v_j \rangle > 0 \text{ and } \langle x, v_i \rangle = 0 \text{ for all } i = 1, \dots, j-1\}.$$

Note that the extended half-space $H^{\{v_j\}}$ essentially coincides with the open half-space

$$H = \{x \in \mathbb{R}^d : \langle x, v_1 \rangle > 0\} \subset H^{\{v_j\}} \subset \bar{H} = \{x \in \mathbb{R}^d : \langle x, v_1 \rangle \geq 0\}.$$

Indeed, $H^{\{v_j\}} \setminus H \subset \{x \in \mathbb{R}^d : \langle x, v_1 \rangle = 0\}$ has measure zero.

Theorem 3.1 *Let $H^{\{v_j\}}$ be an extended half-space, and $H = \{x \in \mathbb{R}^d : \langle x, v_1 \rangle > a\}$, $a \in \mathbb{R}$, be an affine half-space, and $f \in \mathcal{M}$. Suppose that for any non-zero trigonometric polynomial u, any finite subset $B = \{b_1, \dots, b_n\} \subset H^{\{v_j\}}$, and any $M > 0$, the set*

$$E = E_{u,M,B} = \left\{ x \in H : |u(x) f(x)| > M \sum_{i=1}^{n} |f(x + b_i)| \right\} \tag{3.1}$$

has positive measure. Then, $\mathcal{G}(f, \mathbb{R}^{2d})$ is linearly independent.

Proof Suppose for the sake of contradiction that there exist $b_1, \dots, b_N \in \mathbb{R}^d$ and trigonometric polynomials $u_1, \dots, u_N$ such that

$$\sum_{i=1}^{N} u_i(x) f(x - b_i) = 0 \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

The extended half-space $H^{\{v_j\}}$ induces a total order $\prec$ on $\mathbb{R}^d$ given by

$$x \prec y \iff y - x \in H^{\{v_j\}}.$$

This is a consequence of the observation that two extended half-spaces $H^{\{v_j\}}$ and $-H^{\{v_j\}} = H^{\{-v_j\}}$ form a partition of $\mathbb{R}^d \setminus \{0\}$. Hence, without loss of generality we can assume that

$$b_1 \prec \cdots \prec b_N. \tag{3.2}$$

Moreover, we can also assume that $||u_i||_\infty \le 1$ for all $i = 1, \dots, N$.

We shall prove that our hypothesis (3.1) implies that there exist sets of positive measure $Q_1, \dots, Q_N \subset H$ such that the matrix

$$M_N = \begin{pmatrix} u_1(x_1) f(x_1 - b_1) & u_2(x_1) f(x_1 - b_2) & \cdots & u_N(x_1) f(x_1 - b_N) \\ u_1(x_2) f(x_2 - b_1) & u_2(x_2) f(x_2 - b_2) & \cdots & u_N(x_2) f(x_2 - b_N) \\ \vdots & & & \vdots \\ u_1(x_N) f(x_N - b_1) & u_2(x_N) f(x_N - b_2) & \cdots & u_N(x_N) f(x_N - b_N) \end{pmatrix}$$

has non-zero determinant for almost all $(x_1, \dots, x_N) \in Q_1 \times \cdots \times Q_N$. This contradicts our hypothesis that the sum of the rows of M are zero almost everywhere.

For each $1 \le n \le N$, and $(x_1, \dots, x_n) \in (\mathbb{R}^d)^n$, we consider the principal $n \times n$ submatrix of M_N given by

$$M_n = M_n(x_1, \dots, x_n) = \begin{pmatrix} u_1(x_1) f(x_1 - b_1) & \cdots & u_n(x_1) f(x_1 - b_n) \\ \vdots & & \vdots \\ u_n(x_n) f(x_n - b_1) & \cdots & u_n(x_n) f(x_n - b_n) \end{pmatrix}.$$

We will show by induction the existence of sets of positive measure $Q_1, \dots, Q_n \subset \mathbb{R}^d$ and positive constants $c_1, \dots, c_n$ and $\delta_1, \dots, \delta_n$ such that

$$|f(x - b_j)| \le c_n \qquad \text{for a.e. } x \in \bigcup_{i=1}^{n} Q_i, \ \ j = 1, \dots, n, \tag{3.3}$$

$$|\det M_n(x_1, \dots, x_n)| \ge \delta_n \qquad \text{for a.e. } (x_1, \dots, x_n) \in Q_1 \times \cdots \times Q_n. \tag{3.4}$$

The base case $n = 1$ follows trivially from the presence of strict inequality in (3.1). Suppose that (3.3) and (3.4) hold for some $1 \le n < N$. Let Σ be the set of all permutations of $\{1, \dots, n+1\}$ such that $\sigma(n+1) \ne n+1$. Then, for any $(x_1, \dots, x_n, x_{n+1}) \in Q_1 \times \cdots \times Q_n \times \mathbb{R}^d$,

$$
\begin{aligned}
&|\det M_{n+1}(x_1,\ldots,x_n,x_{n+1})| \\
&\geq |u_{n+1}(x_{n+1})f(x_{n+1}-b_{n+1})\det M_n(x_1,\ldots,x_n)| \\
&\quad - \left|\sum_{\sigma\in\Sigma}\prod_{k=1}^{n+1}u_{\sigma(k)}(x_k)f(x_k-b_{\sigma(k)})\right| \qquad (3.5)\\
&\geq \delta_n|u_{n+1}(x_{n+1})f(x_{n+1}-b_{n+1})| - n!(c_n)^n\sum_{i=1}^{n}|f(x_{n+1}-b_i)|.
\end{aligned}
$$

The last estimate is a consequence of breaking the sum over $\sigma\in\Sigma$ with $\sigma(n+1)=i$, where $1\leq i\leq n$. By our hypothesis (3.1), the set

$$
E=\left\{x_{n+1}\in H : |u_{n+1}(x_{n+1}+b_{n+1})f(x_{n+1})| > M\sum_{i=1}^{n}|f(x_{n+1}+(b_{n+1}-b_i))|\right\},
$$

where $M=2n!(c_n)^n/\delta_n$, has positive measure. This is because $b_{n+1}-b_i\in H^{\{v_j\}}$ for $i=1,\ldots,n$ by (3.2). We momentarily set $Q_{n+1}=b_{n+1}+E$. Then, by (3.5) we have that for almost every $(x_1,\ldots,x_{n+1})\in Q_1\times\cdots\times Q_{n+1}$,

$$
|\det M_{n+1}(x_1,\ldots,x_{n+1})| \geq \frac{\delta_n}{2}|u_{n+1}(x_{n+1})f(x_{n+1}-b_{n+1})| > 0.
$$

Thus, by restricting to a (positive measure) subset of Q_{n+1} if necessary, we can find two constants $c_{n+1},\delta_{n+1}>0$ such that (3.3) and (3.4) hold, as desired. This completes the proof of Theorem 3.1. □

In order to establish Theorem 1.1 we will need the following lemma about products of trigonometric polynomials, which is a consequence of the Turán–Nazarov inequality. Lemma 3.2 is a straightforward generalization of the one-dimensional result [4, Lemma 3.5].

Lemma 3.2 *Let u be a non-zero trigonometric polynomial, let $B=\{b_1,\ldots,b_n\}\subset\mathbb{R}^d$ be a finite set, and let $R>0$. Then, there exists a constant $\eta=\eta(u,n,R)>0$ such that for any $y\in\mathbb{R}^d$ and any $k\geq 2$, there exists a measurable subset $E\subset\mathbf{B}_R(y)$ with $|E|>|\mathbf{B}_R(y)|/2$ such that*

$$
\sup_{x\in E}\sum_{i(1)=1}^{n}\cdots\sum_{i(k)=1}^{n}\left|\prod_{j=1}^{k}u\left(x+\sum_{l=1}^{j}b_{i(l)}\right)\right|^{-1}\leq e^{\eta k\log k}. \tag{3.6}
$$

Proof Recall that $\mathbf{B}_R(y)$ denotes a ball centered at $y\in\mathbb{R}^d$ with radius $R>0$. For any $b\in\mathbb{R}^d$ and $t>0$ we define

$$
E_b(t)=\{x\in\mathbf{B}_R(y) : |u(x+b)|<t\}. \tag{3.7}
$$

By Proposition 2.3 we have (2.9). Combining this with Theorem 2.2 yields

$$t \geq \sup_{x \in E_b(t)} |u(x+b)| \geq C(d2^d A)^{1-m} \left(\frac{|E_b(t)|}{|\mathbf{B}_R(y)|} \right)^{m-1}.$$

Thus,

$$|E_b(t)| \leq C' |\mathbf{B}_R(y)| t^{1/(m-1)} \qquad \text{for all } t > 0, \tag{3.8}$$

where the constant C' depends on d, R and u, but not on b or y.

For fixed $k \in \mathbb{N}$ define the set

$$\Sigma = \left\{ \sum_{i=1}^{n} \alpha_i b_i : \sum_{i=1}^{n} \alpha_i \leq k,\ \alpha_i \in \mathbb{N}_0 \right\}.$$

Since the sequence $(\alpha_1, \dots, \alpha_n, k - (\alpha_1 + \cdots + \alpha_n))$ represents a partition of k into $n+1$ blocks, we have

$$\#|\Sigma| \leq \binom{k+n}{k} \leq Ck^n. \tag{3.9}$$

For any subset $\sigma = \{\sigma(1), \dots, \sigma(k)\} \subset \Sigma$ of size k, define the function

$$f_\sigma(x) = \prod_{i=1}^{k} \frac{1}{|u(x+\sigma(i))|}.$$

Let $t > 0$. Suppose that for some $x \in \mathbf{B}_R(y)$ we have

$$\sum_{i(1)=1}^{n} \cdots \sum_{i(k)=1}^{n} \left| \prod_{j=1}^{k} u\left(x + \sum_{l=1}^{j} b_{i(l)} \right) \right|^{-1} > t. \tag{3.10}$$

By taking averages, this implies that there exists a subset $\sigma \subset \Sigma$ of size k such that $f_\sigma(x) > t/n^k$. Since f_σ is a product of k functions, at least one of them must take value greater than $(t/n^k)^{1/k}$. That is,

$$x \in \bigcup_{i=1}^{k} E_{\sigma(i)}\left(\frac{n}{t^{1/k}} \right) \subset \bigcup_{b \in \Sigma} E_b\left(\frac{n}{t^{1/k}} \right),$$

where $E_b(t)$ is given by (3.7). Thus, using (3.8) and (3.9), the Lebesgue measure of the set of points $x \in \mathbf{B}_R(y)$ satisfying (3.10) is bounded by

$$\left| \bigcup_{b \in \Sigma} E_b\left(\frac{n}{t^{1/k}} \right) \right| \leq \#|\Sigma| \max_{b \in \Sigma} \left| E_b\left(\frac{n}{t^{1/k}} \right) \right| \leq Ck^n C' |\mathbf{B}_R(y)| \frac{n^{1/(m-1)}}{t^{1/k(m-1)}}$$
$$\leq C'' |\mathbf{B}_R(y)| k^n t^{-\frac{1}{k(m-1)}}.$$

If we wish that the measure of this set does not exceed $|\mathbf{B}_R(y)|/2$, we are led to the inequality

$$t > (2C'')^{(m-1)k} k^{n(m-1)k}.$$

Thus, there exists a constant $\eta > 0$, which is independent of the choice of $k \geq 2$, such that $t = e^{\eta k \log k}$ satisfies the above bound. Consequently, the set E of points $x \in \mathbf{B}_R(y)$ such that the inequality (3.10) **fails** has measure at least $|\mathbf{B}_R(y)|/2$. This completes the proof of Lemma 3.2. □

We are now ready to state Theorem 3.3. As we will see, the main result of the paper, Theorem 1.1, follows immediately from it.

Theorem 3.3 *Let $\{v_j\}_{j=1}^d$ be an orthonormal basis of $\mathbb{R}^d$. Define the corresponding directional quasi-norm as a mapping $N : \mathbb{R}^d \to [0, \infty)$ given for $x \in \mathbb{R}^d$ by*

$$N(x) = \sum_{j=1}^{d} \langle x, v_j \rangle_+, \qquad \textit{where } y_+ = \max(y, 0). \tag{3.11}$$

Let $f : \mathbb{R}^d \to \mathbb{C}$ be a Lebesgue measurable function that does not vanish almost everywhere on an affine half-space $H = \{x \in \mathbb{R}^d : \langle x, v_1 \rangle > a\}$, $a \in \mathbb{R}$. Assume that f satisfies for all $c > 0$,

$$\lim_{t \to \infty} e^{ct \log t} \sup_{x \in H, N(x) > t} |f(x)| = 0. \tag{3.12}$$

Then, the set $\mathcal{G}(f, \mathbb{R}^{2d})$ of time-frequency translates of f is linearly independent.

In the proof of Theorem 3.3 we will need the following lemma.

Lemma 3.4 *Let $B = \{b_1, \dots, b_n\}$ be a finite subset of an extended half-space $H^{\{v_j\}}$. Then, there exists $\delta > 0$ such that for any $k \in \mathbb{N}$ and any choice of $i(l) \in \{1, \dots, n\}$, $l = 1, \dots, k$ we have*

$$N\left(\sum_{l=1}^{k} b_{i(l)}\right) > \delta k.$$

Proof We shall proceed by induction on the dimension d. Lemma 3.4 is trivially true when $d = 1$. Assume by inductive hypothesis that it is true in the dimension $d - 1$. Without loss of generality, we can assume that elements of B are arranged in increasing order $\prec$ as in the proof of Theorem 3.1, i.e., $b_1 \prec \cdots \prec b_n$. Let $s = 1, \dots, n$ be the largest index such that $\langle b_s, v_1 \rangle = 0$. If such s does not exist, then we let $s = 0$. Observe that

$$0 < \langle b_{s+1}, v_1 \rangle \leq \cdots \leq \langle b_n, v_1 \rangle. \tag{3.13}$$

On the other hand, the elements $\{b_1,\dots,b_s\}$ lie in the subspace $\operatorname{span}\{v_2,\dots,v_d\}$, which we can identify with $\mathbb{R}^{d-1}$. Since $H^{\{v_j\}_{j=2}^d}$ is an extended half-space in $\mathbb{R}^{d-1}$, by the inductive hypothesis there exists $\delta>0$ such that

$$N\Bigg(\sum_{l=1,\ i(l)\le s}^{k} b_{i(l)}\Bigg) > \delta(k-k_0), \tag{3.14}$$

where k_0 is the number of $l=1,\dots,k$ such that $i(l)>s$. Moreover, we have

$$N\Bigg(\sum_{l=1}^{k} b_{i(l)}\Bigg) \ge N\Bigg(\sum_{l=1,\ i(l)\le s}^{k} b_{i(l)}\Bigg) - k_0 dC, \tag{3.15}$$

where $C=\max\{|b_i| : i=1,\dots,n\}$. Indeed, (3.15) follows easily from

$$\Bigg\langle \sum_{l=1}^{k} b_{i(l)}, v_j\Bigg\rangle_+ \ge \Bigg\langle \sum_{l=1,\ i(l)\le s}^{k} b_{i(l)}, v_j\Bigg\rangle_+ - k_0 C \qquad \text{for all } j=1,\dots,d.$$

Combining (3.14) and (3.15) we have

$$N\Bigg(\sum_{l=1}^{k} b_{i(l)}\Bigg) > \delta(k-k_0) - k_0 dC \ge \frac{\delta}{2}k, \qquad \text{if } k_0\le \varepsilon k, \tag{3.16}$$

where $\varepsilon=\delta/(2\delta+2dC)$. However, if $k_0>\varepsilon k$, then by (3.13) we have

$$\begin{aligned} N\Bigg(\sum_{l=1}^{k} b_{i(l)}\Bigg) &\ge \Bigg\langle \sum_{l=1}^{k} b_{i(l)}, v_1\Bigg\rangle_+ \ge \Bigg\langle \sum_{l=1,\ i(l)>s}^{k} b_{i(l)}, v_1\Bigg\rangle \\ &\ge k_0\langle b_{s+1}, v_1\rangle \ge \varepsilon\langle b_{s+1}, v_1\rangle k. \end{aligned} \tag{3.17}$$

Combining (3.16) with (3.17) completes the proof of Lemma 3.4. □

Proof of Theorem 3.3 Let $H=\{x\in\mathbb{R}^d : \langle x,v\rangle > a\}$ be an affine half-space, where $v\in\mathbb{R}^d\setminus\{0\}$ and $a\in\mathbb{R}$. Choose any orthonormal basis $\{v_j\}_{j=1}^d\subset\mathbb{R}^d$ such that $v_1=v/|v|$. By Theorem 3.1 it suffices to show that for any trigonometric polynomial $u\ne 0$, any finite subset $B=\{b_1,\dots,b_n\}\subset H^{\{v_j\}}$ and any $M>0$, the set $E_{u,M,B}$ given by (3.1) has positive measure.

On the contrary, suppose that for some choice of u, B, and $M>0$ we have

$$|f(x)| \le M\sum_{i=1}^{n}\frac{|f(x+b_i)|}{|u(x)|} \qquad \text{for a.e. } x\in H. \tag{3.18}$$

By recursion, (3.18) implies that

$$|f(x)| \le M^k \sum_{i(1)=1}^{n} \cdots \sum_{i(k)=1}^{n} \left| f\left(x + \sum_{l=1}^{k} b_{i(l)}\right)\right| \prod_{j=1}^{k} \left| u\left(x + \sum_{l=1}^{j-1} b_{i(l)}\right)\right|^{-1}. \quad (3.19)$$

By the non-vanishing hypothesis on f, there exists a constant $\varepsilon > 0$, such that the set $\{x \in H : |f(x)| > \varepsilon\}$ has positive measure. By the Lebesgue differentiability theorem applied to that set, there exists a ball $\mathbf{B}_R(y) \subset H$ and such that

$$|\{x \in \mathbf{B}_R(y) : |f(x)| > \varepsilon\}| > |\mathbf{B}_R(y)|/2. \quad (3.20)$$

Observe that the quasi-norm N defined by (3.11) satisfies the triangle inequality $N(x+z) \le N(x) + N(z)$. Thus, for any $x \in \mathbf{B}_R(y)$ and $z \in \mathbb{R}^d$,

$$N(x+z) \ge N(z) - N(-x) \ge N(z) - \sqrt{d}(R + |y|).$$

By Lemma 3.4 and the Cauchy–Schwarz inequality

$$N(x) \le \sum_{j=1}^{d} |\langle x, v_j\rangle| \le \sqrt{d}\left(\sum_{j=1}^{d} |\langle x, v_j\rangle|^2\right)^{1/2} = \sqrt{d}|x|, \quad (3.21)$$

there exists $\delta > 0$ such that

$$N\left(x + \sum_{l=1}^{k} b_{i(l)}\right) \ge \delta k - \sqrt{d}(R + |y|) \ge \delta k/2 \qquad \text{for } k \ge k_0 := 2\sqrt{d}(R + |y|)/\delta.$$

Thus, for any $x \in \mathbf{B}_R(y)$ and $k \ge k_0$,

$$\left| f\left(x + \sum_{l=1}^{k} b_{i(l)}\right)\right| \le \sup_{z \in H,\ N(z) > \delta k/2} |f(z)|. \quad (3.22)$$

Here, we used the following fact: $x \in H$ and $b \in H^{\{v_j\}} \implies x + b \in H$.

Combining (3.19) and (3.22) with Lemma 3.2 yields a subset $E_k \subset \mathbf{B}_R(y)$ with $|E_k| > |\mathbf{B}_R(y)|/2$ such that

$$|f(x)| \le M^k e^{\eta k \log k} \sup_{z \in H,\ N(z) > \delta k/2} |f(z)| \qquad \text{for } x \in E_k. \quad (3.23)$$

By (3.20) the set E_k must non-trivially intersect with the set $\{x \in \mathbf{B}_R(y) : |f(x)| > \varepsilon\}$. Hence, we conclude that

$$\sup_{z \in H,\ N(z) > \delta k/2} |f(z)| \ge \varepsilon M^{-k} e^{-\eta k \log k} \qquad \text{for } k \ge k_0.$$

This contradicts our decay hypothesis (3.12) and completes the proof of Theorem 3.3. $\square$

As an immediate consequence of Theorem 3.3 we can deduce Theorem 1.1.

Proof of Theorem 1.1 Suppose that a function f satisfies the decay condition (1.1). That is, for all $c > 0$,

$$\lim_{t\to\infty} e^{ct\log t} \sup_{x\in H, |x|>t} |f(x)| = 0.$$

Combining this with (3.21) implies the weaker decay condition (3.12). Consequently, Theorem 3.3 yields the desired conclusion. $\square$

We end by presenting a decay condition that is a more direct generalization of the main theorem in [4]. Indeed, the condition (3.24) is automatically satisfied on the real line since any finite subset $B \subset \mathbb{R}$ can be arranged in increasing order. Then, depending on the sign of $v \in \mathbb{R}$, the decay condition (3.25) corresponds to one-sided limit as $x \to \infty$ or $x \to -\infty$. Thus, Theorem 3.5 implies the main result in [4].

Theorem 3.5 *Let $B = \{b_1, b_2, \dots, b_n\}$ be a finite subset of $\mathbb{R}^d$. Suppose there exists a vector $v \in \mathbb{R}^d$ and $1 \le j_0 \le n$ such that*

$$\langle v, b_{j_0}\rangle < \langle v, b_j\rangle \quad \text{for all } 1 \le j \le n,\ j \ne j_0. \tag{3.24}$$

Let $f : \mathbb{R}^d \to \mathbb{C}$ be a non-zero Lebesgue measurable function satisfying the directional decay condition

$$\lim_{x\in\mathbb{R}^d,\ \langle x,v\rangle\to\infty} |f(x)| e^{c\langle x,v\rangle \log(\langle x,v\rangle)} = 0 \quad \text{for all } c > 0. \tag{3.25}$$

Then, if $u_1, \dots, u_n$ are trigonometric polynomials such that

$$\sum_{j=1}^{n} u_j(x) f(x + b_j) = 0 \quad \text{for a.e. } x \in \mathbb{R}^d, \tag{3.26}$$

then $u_{j_0} = 0$. In particular, if $\langle v, b_i\rangle \ne \langle v, b_j\rangle$ for all $i \ne j$, then $\mathcal{G}(f, \mathbb{R}^d \times (-B))$ is linearly independent.

Remark 3.1 Note that unlike Theorem 1.1 we do not assume that f does not vanish on a half-space. Moreover, the decay condition (1.1) is weakened by the condition (3.25) that does not impose any decay in directions perpendicular to a vector $v \in \mathbb{R}^d$. As a consequence, the conclusion of Theorem 3.5 must also be weakened. Indeed, the following simple example shows that we cannot expect that the remaining polynomials satisfy $u_1 = \dots = u_n = 0$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x_1, x_2) = \begin{cases} e^{x_2} & \text{if } x_1 \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Then, f satisfies the hypothesis of Theorem 3.5 with $B = \{(0,0), (0,1), (-1,0)\}$ and $v = (1,0)$, but $\mathcal{G}(f, \{0\} \times B)$ is linearly dependent.

Proof of Theorem 3.5 On the contrary suppose that there exists a solution to (3.26) with a non-zero u_{j_0}. Then,

$$|f(x)| \le M \sum_{j=1,\ j\neq j_0}^{n} \frac{|f(x+b_j-b_{j_0})|}{|u_{j_0}(x)|} \qquad \text{for a.e. } x \in \mathbb{R}^d,$$

where $M = \max(||u_1||_\infty, \dots, ||u_n||_\infty)$. Hence, the same inequality as in (3.18) holds true. Define a directional quasi-norm $\tilde{N}(x) = \langle x, v\rangle_+$. By the assumption (3.24), Lemma 3.4 holds for the set $\tilde{B} = \{b_j - b_{j_0} : 1 \le j \ne j_0 \le n\}$ and quasi-norm $\tilde{N}$ in place of B and N, resp. Moreover, the same argument as in the proof of Theorem 3.3 works for the quasi-norm $\tilde{N}$ in place of the original one given by (3.11). As a result we obtain a contradiction with our hypotheses that $u_{j_0} \neq 0$, thus completing the proof of Theorem 3.5. □

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References

1. Benedetto, J., Bourouihiya, A.: Linear independence of finite Gabor systems determined by behavior at infinity. J. Geom. Anal. **25**(1), 226–254 (2015)
2. Bourgain, J., Rudnick, Z.: On the geometry of the nodal lines of eigenfunctions of the two-dimensional torus. Ann. Henri Poincaré **12**(6), 1027–1053 (2011)
3. Bownik, M., Speegle, D.: Linear independence of Parseval wavelets. Ill. J. Math. **54**(2), 771–785 (2010)
4. Bownik, M., Speegle, D.: Linear independence of time-frequency translates of functions with faster than exponential decay. Bull. London Math. Soc. **45**(3), 554–566 (2013)
5. Demeter, C.: Linear independence of time frequency translates for special configurations. Math. Res. Lett. **17**(4), 761–779 (2010)
6. Demeter, C., Gautam, Z.: On the finite linear independence of lattice Gabor systems. Proc. Am. Math. Soc. **141**(5), 1735–1747 (2013)
7. Demeter, C., Zaharescu, A.: Proof of the HRT conjecture for (2, 2) configurations. J. Math. Anal. Appl. **388**(1), 151–159 (2012)
8. Edgar, G., Rosenblatt, J.: Difference equations over locally compact abelian groups. Trans. Am. Math. Soc. **253**, 273–289 (1979)
9. Fontes-Merz, N.: A multidimensional version of Turán's lemma. J. Approx. Theory **140**(1), 27–30 (2006)
10. Gröchenig, K.: Linear independence of time frequency shifts?, preprint, Monatsch. Math. (to appear)
11. Heil, C.: Linear independence of finite Gabor systems. In: Heil, C. (ed.) Harmonic Analysis and Applications, pp. 171–206. Birkhäuser, Boston (2006)
12. Heil, C., Ramanathan, J., Topiwala, P.: Linear independence of time-frequency translates. Proc. Am. Math. Soc. **124**(9), 2787–2795 (1996)
13. Holt, J., Vaaler, J.: The Beurling-Selberg extremal functions for a ball in Euclidean space. Duke Math. J. **83**(1), 202–248 (1996)
14. Lawrence, J., Pfander, G., Walnut, D.: Linear independence of Gabor systems in finite dimensional vector spaces. J. Fourier Anal. Appl. **11**(6), 715–726 (2005)
15. Linnell P (1998) Analytic versions of the zero divisor conjecture. In: Geometry and cohomology in group theory, Durham, pp 209–248, London Mathematical Society Lecture Note Series, vol. 252, Cambridge University Press, Cambridge (1994)

16. Linnell, P.: von Neumann algebras and linear independence of translates. Proc. Am. Math. Soc. **127**(11), 3269–3277 (1999)
17. Montgomery, H.L.: Ten lectures on the interface between analytic number theory and harmonic analysis. In: CBMS Regional Conference Series in Mathematics, vol. 84 pp. xiv+220. American Mathematical Society, Providence (1994)
18. Nazarov, F.: Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type. St. Petersburg Math. J. **5**(4), 663–717 (1994)
19. Rosenblatt, J.: Linear independence of translations. J. Aust. Math. Soc. Ser. A **59**, 131–133 (1995)