SMOOTH ORTHOGONAL PROJECTIONS ON SPHERE

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Abstract. We construct a decomposition of the identity operator on the sphere $\mathbb{S}^d$ as a sum of smooth orthogonal projections subordinate to an open cover of $\mathbb{S}^d$. We give applications of our main result in the study of function spaces and Parseval frames on the sphere.

1. Introduction

The goal of this paper is to give a construction of a decomposition of the identity operator on the sphere as a sum of smooth orthogonal projections with desired localization properties. Our construction is reminiscent of the ubiquitous smooth partition of unity subordinate to an open cover of a manifold. However, partitions of unity do not give rise (in any obvious way) to a decomposition into projections which are desirable in the study of function spaces and Parseval frames.

Smooth projections on the real line were introduced in a systematic way by Auscher, Weiss, and Wickerhauser [1] in their study of local sine and cosine bases of Coifman and Meyer [5] and smooth wavelets. The standard procedure of tensoring can be used to extend their construction to the Euclidean space $\mathbb{R}^d$. In this paper we explain how smooth projections can be defined on a simplest non-Euclidean manifold, i.e, on the sphere $\mathbb{S}^d$. Our main result, Theorem 5.1, can be reformulated in the following way.

Theorem 1.1. *Suppose $\mathcal{U}$ is an open cover of $\mathbb{S}^d$. Then, there exists a family of operators $\{P_U\}_{U\in\mathcal{U}}$ defined pointwise for functions on $\mathbb{S}^d$ such that:*

(i) all but finitely many projections P_U are zero,

(ii) each P_U is localized on an open set U, i.e., for any $f : \mathbb{S}^d \to \mathbb{R}$ we have

$$P_U f(x) = 0 \qquad \text{for } x \in \mathbb{S}^d \setminus U,$$

(iii) each P_U considered as an operator $P_U : L^2(\mathbb{S}^d) \to L^2(\mathbb{S}^d)$ is an orthogonal projection and the projections $\{P_U\}_{U\in\mathcal{U}}$ give a decomposition of the identity operator $\mathbf{I}$

$$\sum_{U\in\mathcal{U}} P_U = \mathbf{I},$$

(iv) for any $r = 0, 1, \dots$ and $1 \le p < \infty$, each P_U maps boundedly $C^r(\mathbb{S}^d)$ and the Sobolev space $W^r_p(\mathbb{S}^d)$ into itself.

The paper is organized as follows. In Section 2 we introduce smooth orthogonal projections on the weighted L^2 spaces on the real line extending constructions in [1, 14]. This construction is used in Section 3 to introduce smooth latitudinal projections on the sphere.

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In Section 4 we describe a procedure of lifting an operator acting on $\mathbb{S}^{d-1}$ to a higher dimensional sphere $\mathbb{S}^d$. We show that our lifting preserves smoothness of functions away from two poles of $\mathbb{S}^d$. These results are then used in Section 5 to construct a family of smooth orthogonal projections corresponding to a partition of the sphere into patches. Finally, in Section 6 we give applications of our main result in the study of function spaces and Parseval frames on the sphere.

2. Smooth orthogonal projections on the real line and the circle

In this section we define smooth projections on the real line originally introduced by Auscher, Weiss, and Wickerhauser [1] in the process of constructing local sine and cosine bases of Coifman and Meyer [5]. These constructions are nicely explained in Sections 1.3–1.5 of the book by Hernández and Weiss [14]. Unlike the original approach in [14] we are also interested in the weighted L^2 spaces. This requires some necessary modifications of constructions in [14].

2.1. **Smooth projections on $L^2(\mathbb{R}, \psi)$.** Let $\delta > 0$ be given and fixed. Let $\psi \in C^\infty(\mathbb{R})$ be smooth and nonnegative function such that

$$\psi(t) \geq c > 0 \qquad \text{for } t \in [-\delta, \delta]. \tag{2.1}$$

The function ψ is the weight in the Hilbert space $L^2(\mathbb{R}, \psi)$ consisting of real-valued measurable functions with the inner product

$$\langle f, g \rangle_\psi = \int_{\mathbb{R}} f(t) g(t) \psi(t) dt.$$

Definition 2.1. Assume that there is a real, smooth function, $s \in C^\infty(\mathbb{R})$, such that

$$\operatorname{supp} s \subset [-\delta, +\infty) \tag{2.2}$$

and for all $t \in \mathbb{R}$

$$s^2(t) + s^2(-t) = 1.$$

For the construction of such function see [5, 14]. We define Auscher-Weiss-Wickerhauser (AWW) operator $E_\psi^\pm$ for a real-valued function g on $\mathbb{R}$ by

$$E_\psi^\pm(g)(t) = \begin{cases} g(t) & t > \delta, \\ s^2(t)g(t) \pm s(t)s(-t)\sqrt{\frac{\psi(-t)}{\psi(t)}}g(-t) & t \in [-\delta, \delta], \\ 0 & t < -\delta. \end{cases}$$

The choice of $\pm$ is referred as the polarity of $E_\psi^\pm$. If polarity is not indicated, we shall assume it is positive, i.e., $E_\psi = E_\psi^+$.

By the assumption made on the function s, we have $s^2(t) = 1$ for $t \geq \delta$ and $s(t) = 0$ for $t \leq -\delta$. Thus, by Definition 2.1 and (2.1), there exists $\epsilon > 0$ such that

$$E_\psi^\pm(g)(t) = s^2(t)g(t) \pm s(t)s(-t)\sqrt{\frac{\psi(-t)}{\psi(t)}}g(-t) \qquad \text{for } t \in [-\delta - \epsilon, \delta + \epsilon]. \tag{2.3}$$

Moreover,

$$E^{\pm}_{\psi}(g)(t) = \begin{cases} g(t) & t > \delta, \\ E^{\pm}_{\psi}(g\mathbf{1}_{[-\delta,\delta]})(t) & t \in [-\delta,\delta], \\ 0 & t < -\delta. \end{cases}$$

where $\mathbf{1}_{[-\delta,\delta]}$ is a characteristic function of an interval $[-\delta,\delta]$.

The following result is an extension of the construction in [14, Section 1.3].

Proposition 2.1. *Under the above assumptions we have that*

$$E^{\pm}_{\psi}(C^{\infty}(\mathbb{R})) \subset C^{\infty}(\mathbb{R}) \tag{2.4}$$

and $E^{\pm}_{\psi}$ is an orthogonal projection as an operator

$$E^{\pm}_{\psi} : L^2(\mathbb{R},\psi) \to L^2(\mathbb{R},\psi). \tag{2.5}$$

Proof. The property (2.4) is obvious from (2.3). To prove (2.5) we need to check that

$$(E^{\pm}_{\psi})^2 := E^{\pm}_{\psi} \circ E^{\pm}_{\psi} = E^{\pm}_{\psi} \tag{2.6}$$

and

$$\langle E^{\pm}_{\psi}(f), g\rangle_{\psi} = \langle f, E^{\pm}_{\psi}(g)\rangle_{\psi}. \tag{2.7}$$

For simplicity we prove (2.6) and (2.7) for the operator $E_{\psi} = E^{+}_{\psi}$. Let $\rho(t) := s^2(t)$ and let

$$\eta(t) := \begin{cases} s(t)s(-t)\sqrt{\frac{\psi(-t)}{\psi(t)}} & \text{for } t \in [-\delta-\epsilon, \delta+\epsilon], \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$E_{\psi}(g)(t) = \rho(t)g(t) + \eta(t)g(-t) \qquad \text{for all } t \in \mathbb{R}.$$

Consequently,

$$\begin{aligned} (E_{\psi})^2(g)(t) &= \rho(t)^2 g(t) + \rho(t)\eta(t)g(-t) + \eta(t)\rho(-t)g(-t) + \eta(t)\eta(-t)g(t) \\ &= (\rho(t)^2 + \eta(t)\eta(-t))g(t) + (\rho(t)\eta(t) + \eta(t)\rho(-t))g(-t). \end{aligned}$$

From the assumption on s

$$\rho(t)^2 + \eta(t)\eta(-t) = s^2(t)(s^2(t) + s^2(-t)) = s^2(t)$$

and

$$\rho(t)\eta(t) + \eta(t)\rho(-t) = \eta(t).$$

Thus,

$$(E_{\psi})^2(g)(t) = E_{\psi}(g)(t).$$

Now we turn to prove (2.7). We have for $f, g \in L^2(\mathbb{R},\psi)$

$$\begin{aligned} \langle E_{\psi}(f), g\rangle_{\psi} &= \int_{\delta}^{\infty} f(t)g(t)\psi(t)dt + \int_{-\delta}^{\delta} (\rho(t)f(t) + \eta(t)f(-t))g(t)\psi(t)dt \\ &= \int_{\delta}^{\infty} f(t)g(t)\psi(t)dt + \int_{-\delta}^{\delta} (\rho(t)g(t) + \frac{\eta(-t)\psi(-t)}{\psi(t)}g(-t))f(t)\psi(t)dt \\ &= \langle f, E_{\psi}(g)\rangle_{\psi}. \end{aligned}$$

This finishes the proof of the proposition. $\square$

Remark 2.1. Note that Proposition 2.1 holds true if we replace the real line $\mathbb{R}$ by any interval $[a, b]$ such that

$$[-\delta, \delta] \subset [a, b].$$

This observation will be used in Section 3, where instead of $L^2(\mathbb{R}, \psi)$ we consider the Hilbert space $L^2([a, b], \psi)$ of real-valued measurable functions on $[a, b]$ with the inner product

$$\langle f, g\rangle_\psi = \int_a^b f(t)g(t)\psi(t)dt.$$

Indeed, for $g : [a, b] \to \mathbb{R}$, we may set $E_\psi(g) = E_\psi(\tilde{g})|_{[a,b]}$, where $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is an arbitrary extension of g. This is well-defined since for any function f on $\mathbb{R}$ we have $E_\psi(f\chi_{[a,b]}) = E_\psi(f)\chi_{[a,b]}$.

2.2. **Smooth projections on $\mathbb{S}^1$.** The construction of smooth projections on $\mathbb{R}$ can be easily translated to the setting of the circle $\mathbb{S}^1$. We shall concentrate our attention to the unweighted case. The weighted case can be dealt in a similar way as on $\mathbb{R}$.

Let $[\alpha, \beta] \subset \mathbb{R}$, and $\delta > 0$ be such that

$$2\delta < \beta - \alpha. \tag{2.8}$$

Fix a nonnegative C^∞ function ψ satisfying $\psi(t) \geq c > 0$ for $t \in [\alpha - \delta, \beta + \delta]$. Let P_α be an AWW projection onto the interval $[\alpha, \infty)$ given by $P_\alpha = T_\alpha E_{\psi_\alpha} T_{-\alpha}$, where T_α is a translation by α and $\psi_\alpha = T_{-\alpha}\psi$. Since $\psi_\alpha(t) \geq c > 0$ for $t \in [-\delta, \delta]$, in light of Proposition 2.1, P_α is an orthogonal projection on $L^2(\mathbb{R}, \psi)$ such that

$$P_\alpha f(t) = \begin{cases} 0 & t \leq \alpha - \delta, \\ f(t) & t \geq \alpha + \delta. \end{cases}$$

Furthermore, if $[\alpha - \delta, \alpha + \delta] \subset [a, b]$, then for any function f on $\mathbb{R}$,

$$P_\alpha(f\mathbf{1}_{[a,b]}) = (P_\alpha f)\chi_{[a,b]}.$$

Hence, by Remark 2.1 we can treat P_α as an orthogonal projection on $L^2([a, b], \psi)$.

In a similar way, let P^β be an AWW projection onto the interval $(-\infty, \beta]$ given by $P^\beta = T_\beta(\mathbf{I} - E_{\psi_\beta})T_{-\beta}$, where $\mathbf{I}$ is the identity and $\psi_\beta = T_{-\beta}\psi$. Using (2.8) it can be easily seen that P_α and P^β commute, see [14, Section 1.3]. Define an AWW projection onto the interval $[\alpha, \beta]$ by

$$P_{[\alpha,\beta]} = P_\alpha P^\beta = P^\beta P_\alpha.$$

For simplicity, let us now assume that the weight $\psi \equiv 1$. A simple calculation using (2.8) shows the following explicit formula for $P_{[\alpha,\beta]}$

$$P_{[\alpha,\beta]} f(t) = \begin{cases} 0 & t < \alpha - \delta, \\ s^2(t-\alpha)f(t) + s(t-\alpha)s(\alpha - t)f(2\alpha - t) & t \in [\alpha - \delta, \alpha + \delta], \\ f(t) & t \in (\alpha + \delta, \beta - \delta), \\ s^2(\beta - t)f(t) - s(t-\beta)s(\beta - t)f(2\beta - t) & t \in [\beta - \delta, \beta + \delta], \\ 0 & t > \beta + \delta. \end{cases} \tag{2.9}$$

Definition 2.2. Let $Q = \{\Psi_1(t) = (\sin t, \cos t) : t \in [\alpha, \beta]\}$ be an arc in $\mathbb{S}^1$ such that (2.8) holds and

$$2\delta < 2\pi - (\beta - \alpha). \tag{2.10}$$

Define a *smooth orthogonal projection* P_Q on $\mathbb{S}^1$ by

$$P_Q f(\xi) = P_{[\alpha,\beta]}(f \circ \Psi_1)(t), \qquad \text{where } \xi = \Psi_1(t),\ t \in [\alpha - \delta, 2\pi + \alpha - \delta).$$

Then we have the following variant of Proposition 2.1 justifying the name for P_Q.

Theorem 2.1. *Suppose that $Q \subset \mathbb{S}^1$ is an arc as in Definition 2.2. Then,*

$$P_Q(C^\infty(\mathbb{S}^1)) \subset C^\infty(\mathbb{S}^1)$$

and P_Q is an orthogonal projection as an operator

$$P_Q : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1).$$

Moreover, if arcs $Q_1, \ldots, Q_n$, each as in Definition 2.2, form a partition of $\mathbb{S}^1$, then

$$\sum_{j=1}^{n} P_{Q_j} = \mathbf{I}. \tag{2.11}$$

In particular, $P_{Q_i} P_{Q_j} = 0$ whenever $1 \le i \ne j \le n$.

Proof. The first part of the theorem is a consequence of Proposition 2.1 and (2.9). The moreover part follows from [14, Theorem 1.3.15] which states that for any two adjacent intervals $[\alpha, \beta]$ and $[\beta, \gamma]$ satisfying (2.8) we have

$$P_{[\alpha,\beta]} + P_{[\beta,\gamma]} = P_{[\alpha,\gamma]}.$$

Hence, for two adjacent arcs Q, Q' such that Q, Q', and $Q \cup Q'$ satisfy Definition 2.2, we have

$$P_Q + P_{Q'} = P_{Q \cup Q'} \qquad \text{and} \qquad P_{\mathbb{S}^1 \setminus Q} = \mathbf{I} - P_Q.$$

These identities yield the decomposition (2.11). $\square$

2.3. **Range of smooth projections.** We also have the following generalization of [14, Theorem 1.3.20], which characterizes the image of $L^2(\mathbb{R}, \psi)$ under the orthogonal projection E_ψ from Definition 2.1. To show this note that we have

$$E_\psi(g)(t) = b_R(t) \left(s(t)\sqrt{\psi(t)}g(t) + s(-t)\sqrt{\psi(-t)}g(-t) \right) \qquad \text{for } t \in [-\delta - \epsilon, \delta + \epsilon], \tag{2.12}$$

where $b_R(t) = s(t)/\sqrt{\psi(t)}$. In a similar way we have

$$\begin{aligned} (\mathbf{I} - E_\psi)(g)(t) =& (1 - s^2(t))g(t) - s(t)s(-t)\sqrt{\frac{\psi(-t)}{\psi(t)}}g(-t) \\ =& s^2(-t)g(t) - s(t)s(-t)\sqrt{\frac{\psi(-t)}{\psi(t)}}g(-t) \\ =& b_L(t) \left(s(-t)\sqrt{\psi(t)}g(t) - s(t)\sqrt{\psi(-t)}g(-t) \right) \qquad \text{for } t \in [-\delta - \epsilon, \delta + \epsilon], \end{aligned} \tag{2.13}$$

where $b_L(t) = s(-t)/\sqrt{\psi(t)}$. Note that function b_R and B_L are initially well-defined only for $t \in \mathbb{R}$ such that $\psi(t) \neq 0$. If $t \in \mathbb{R}$ is such that $\psi(t) = 0$, then we can assign the value for $b_R(t)$ in any way want, say $b_R(t) = b_L(t) = 0$.

Theorem 2.2. *A function $f \in E_\psi(L^2(\mathbb{R}, \psi))$ if and only if $f = b_R H$ for some $H \in L^2(\mathbb{R})$ such that H is even on $[-\delta, \delta]$. Likewise, $f \in (\mathbf{I} - E_\psi)(L^2(\mathbb{R}, \psi))$ if and only if $f = b_L H$ for some $H \in L^2(\mathbb{R})$ such that H is odd on $[-\delta, \delta]$.*

Proof. The forward direction is a consequence of (2.1), (2.12), and (2.13). To show the backward direction we have to check that $E_\psi(b_R H) = b_R H$. Since H is even on $[-\delta, \delta]$, by (2.12) we have

$$\begin{aligned} E_\psi(b_R H)(t) =& b_R(t) \left(s(t)\sqrt{\psi(t)} b_R(t) H(t) + s(-t)\sqrt{\psi(-t)} b_R(-t) H(-t) \right) \\ =& b_R(t)(s^2(t) + s^2(-t)) H(t) = b_R(t) H(t). \end{aligned}$$

In similar way we can check that $(\mathbf{I} - E_\psi)(b_L H) = b_L H$. $\square$

3. Latitudinal projections on the sphere

In this section we define smooth latitudinal projections on the k-dimensional sphere $\mathbb{S}^k \subset \mathbb{R}^{k+1}$, $k \geq 2$, using Auscher-Weiss-Wickerhauser (AWW) projections introduced in Section 2.

3.1. **H-operators.** For our purposes it is convenient to define an abstract class of H-operators on manifolds which was originally introduced in the work of Ciesielski and Figiel [3, Section 5]. The letter H stands for Hestenes [15] who considered a similar class of operators. Our definition is more restrictive than the one in [3] since we are dealing with less general classes of operators than those studied in [3]. Consequently, many of the results which required proofs in [3] follow automatically from the definition.

Definition 3.1. Let M be a σ-compact smooth Riemannian manifold (without boundary). Let $\Phi : V \to V$ be a C^∞ diffeomorphism, where $V \subset M$ is an open subset. Let $\varphi : M \to \mathbb{R}$ be a C^∞ function such that

$$\operatorname{supp} \varphi = \overline{\{x \in M : \varphi(x) \neq 0\}} \subset V.$$

We define a simple H-operator $H_{\varphi,\Phi,V}$ acting on a function $f : M \to \mathbb{R}$ by

$$H_{\varphi,\Phi,V} f(x) = \begin{cases} \varphi(x) f(\Phi(x)) & x \in V \\ 0 & x \in M \setminus V. \end{cases}$$

Let $L_0(M)$ be the space of (equivalence classes of) Lebesgue measurable functions on M equipped with the topology of convergence in measure on compact subsets of M. Clearly, a simple H-operator induces a continuous linear map of the space $L_0(M)$ into itself. We define an H-operator to be a finite combination of such simple H-operators. The space of all H-operators is denoted by $\mathcal{H}(M)$.

Observe that $\mathcal{H}(M)$ is an algebra of operators. This follows from the formula

$$H_{\varphi_2,\Phi_2,V_2} \circ H_{\varphi_1,\Phi_1,V_1} = H_{\varphi,\Phi,V},$$

where $\varphi = \varphi_2|_V \cdot (\varphi_1 \circ \Phi_2|_V)$, $\Phi = \Phi_1 \circ \Phi_2|_V$, and $V = V_2 \cap (\Phi_2)^{-1}(V_1)$. Furthermore, following [3] we shall see that H-operators are preserved under tensoring.

Definition 3.2. Suppose that M and M' are two Riemannian manifolds (without boundary) with Riemannian measures v and v', resp. Then, $M \times M'$ is also a Riemannian manifold with the Riemannian measure $v \times v'$. For any $f \in L_0(M)$ and $g \in L_0(M')$, define $f \otimes g \in L_0(M \times M')$ by

$$(f \otimes g)(x, y) = f(x)g(y) \qquad x \in M, y \in M'.$$

The following lemma and its proof are a verbatim adaptation of [3, Lemma 5.15].

Lemma 3.1. *If $H \in \mathcal{H}(M)$ and $H' \in \mathcal{H}(M')$, then there exists a unique continuous linear operator T acting on $L_0(M \times M')$ such that*

$$T(f \otimes g) = (Hf) \otimes (H'g) \qquad \text{for } f \in L_0(M), g \in L_0(M'). \tag{3.1}$$

Moreover, $T \in \mathcal{H}(M \times M')$. The operator T is said to be the tensor product $H \otimes H'$ of H-operators H and H'.

Proof. Suppose that H and H' are simple H-operators on M and M', resp. That is, $H = H_{\varphi,\Phi,V}$ and $H' = H_{\psi,\Psi,W}$ are as in Definition 3.1. Then, the operator

$$T = H_{\varphi\otimes\psi,\Phi\otimes\Psi,V\times W}$$

is a simple H-operator on $M \times M'$ satisfying (3.1). By taking linear combinations the same holds for general H-operators. To show that such T is unique it suffices to use the fact that the subspace spanned by the functions of the form $f \otimes g$ is dense in $L_0(M \times M')$. $\square$

Finally, H-operators induce bounded operators on the space $C^r(M)$ and on Sobolev spaces on M.

Definition 3.3. For $k \in \mathbb{N}$ and $f : M \to \mathbb{R}$ we denote by $\nabla^k f(x)$, $x \in M$, the covariant derivative of f of order k in some local chart. We let $|\nabla^k f|$ be its norm (which is independent of a choice of chart). The Banach space $C^r(M)$ consists of all C^r functions $f : M \to \mathbb{R}$ with the norm

$$||f||_{C^r(M)} = \sum_{k=0}^{r} \sup_{x\in M} |\nabla^k f(x)| < \infty.$$

Let v be the Riemannian measure on M. Given $1 \le p < \infty$ we define the norm

$$||f||_{W_p^r} = \sum_{k=0}^{r} \left(\int_M |\nabla^k f(x)|^p dv(x) \right)^{1/p} < \infty.$$

The Sobolev space $W_p^r(M)$ is the completion of $C^r(M)$ (or equivalently $C^\infty(M)$) with respect to the norm $||\cdot||_{W_p^r}$, see [13].

Then we have the following analogue of [3, Lemma 5.38 and Corollary 5.39].

Lemma 3.2. *Suppose that $H \in \mathcal{H}(M)$, where M is compact smooth Riemannian manifold. Then, for any $r = 0, 1, \dots$, the operator H induces a continuous linear operator*

$$H : C^r(M) \to C^r(M), \qquad \text{where } r = 0, 1, \dots, \tag{3.2}$$

$$H : W_p^r(M) \to W_p^r(M), \qquad \text{where } 1 \le p < \infty,\ r = 0, 1, \dots. \tag{3.3}$$

Proof. It suffices to consider H to be a simple H-operator $H_{\varphi,\Phi,V}$. Using the product and the chain rule, one can express derivatives of Hf at x in terms of derivatives of φ and Φ at x and f at $\Phi(x)$. Since $\operatorname{supp}\varphi$ is compact, these are bounded on V. This yields the conclusion (3.2). Likewise, using the change of variables formula and a standard density argument [3, Lemma 5.37 and Corollary 5.39] yields (3.3). $\square$

3.2. **AWW projections on the sphere.** Let

$$\Psi_k : [0,\pi]^{k-1} \times [0,2\pi] \to \mathbb{S}^k$$

be the standard spherical coordinates given by the recurrence formula

$$\begin{aligned} \Psi_1(t) &= (\sin t, \cos t), \qquad t \in [0,2\pi], \\ \Psi_{k+1}(t,x) &= (\sin(t)\xi, \cos t), \qquad (t,x) \in [0,\pi] \times ([0,\pi]^{k-1} \times [0,2\pi]), \end{aligned}$$

where $\Psi_k(x) = \xi \in \mathbb{S}^k$. For $k \geq 2$ it is useful to define a surjective function

$$\Phi_k : [0,\pi] \times \mathbb{S}^{k-1} \to \mathbb{S}^k$$

by the formula

$$\Phi_k(\vartheta,\xi) = (\xi \sin\vartheta, \cos\vartheta), \qquad \text{where } (\vartheta,\xi) \in [0,\pi] \times \mathbb{S}^{k-1}.$$

Note that Φ_k is a diffeomorphism

$$\Phi_k : (0,\pi) \times \mathbb{S}^{k-1} \to \mathbb{S}^k \setminus \{\mathbf{1}^k, -\mathbf{1}^k\},$$

where $\mathbf{1}^k = (0,\ldots,0,1) \in \mathbb{S}^k$ is the "North Pole". Let σ_k be the normalized Lebesgue measure on $\mathbb{S}^k$. Then, for any $f \in L^1(\mathbb{S}^k, d\sigma_k)$ we have the following well-known and useful identity, see [7, (1.5.4)]

$$\int_{\mathbb{S}^k} f(u)d\sigma_k(u) = \int_{\mathbb{S}^{k-1}} \int_0^\pi f \circ \Phi_k(\vartheta,\xi)(\sin(\vartheta))^{k-1}\, d\vartheta d\sigma_{k-1}(\xi). \tag{3.4}$$

For the sake of simplicity we will often write

$$f(\vartheta,\xi) = f(\Phi_k(\vartheta,\xi)), \qquad \text{for } (\vartheta,\xi) \in [0,\pi] \times \mathbb{S}^{k-1}.$$

We are now ready to introduce AWW projections on $\mathbb{S}^k$, $k \geq 2$.

Definition 3.4. Let $\vartheta^\circ \in (0,\pi)$ and $0 < \delta < \min\{\vartheta^\circ, \pi - \vartheta^\circ\}$. For $\vartheta \in [-\vartheta^\circ, \pi - \vartheta^\circ]$ and $\xi \in \mathbb{S}^{k-1}$ define

$$\begin{aligned} \Phi_k^{\xi,\circ}(\vartheta) &= \Phi_k(\vartheta^\circ + \vartheta, \xi), \\ \psi^\circ(\vartheta) &= (\sin(\vartheta + \vartheta^\circ))^{k-1}. \end{aligned}$$

We define the Auscher-Weiss-Wickerhauser (AWW) operator E^{ϑ° for functions $g : \mathbb{S}^k \to \mathbb{R}$ by

$$E^{\vartheta^\circ}(g)(\Phi_k(\vartheta,\xi)) = E_{\psi^\circ}(g \circ \Phi_k^{\xi,\circ})(\vartheta - \vartheta^\circ) \qquad \text{for } (\vartheta,\xi) \in [0,\pi] \times \mathbb{S}^{k-1},$$

where E_{ψ° is an AWW projection on the interval $[-\vartheta^\circ, \pi - \vartheta^\circ]$ with weight ψ^0.

Equivalently, we can define $E^{\vartheta^\circ} = P_{\vartheta^\circ} \otimes \mathbf{I}$ as a tensor product of two H-operators, where P_{ϑ° is an orthogonal projection on $L^2([0,\pi],\psi)$, $\psi(\vartheta) = (\sin(\vartheta))^{k-1}$, with a cut-off at $\vartheta^\circ \in (0,\pi)$, which was defined in Subsection 2.2, and $\mathbf{I}$ is the identity operator on $\mathbb{S}^{k-1}$.

By Definition 2.1 and our convention,

$$E^{\vartheta^\circ}(g)(\vartheta,\xi)=\begin{cases} g(\vartheta,\xi) & \vartheta>\vartheta^\circ+\delta,\\ 0 & \vartheta<\vartheta^\circ-\delta.\end{cases} \tag{3.5}$$

Moreover, by (2.3) there exists $\epsilon>0$ such that for $\vartheta-\vartheta^\circ\in[-\delta-\epsilon,\delta+\epsilon]$

$$\begin{aligned} E^{\vartheta^\circ}(g)(\vartheta,\xi)&=E_{\psi^\circ}(g\circ\Phi_k^{\xi,\circ})(\vartheta-\vartheta^\circ)\\ &=s^2(\vartheta-\vartheta^\circ)g(\vartheta,\xi)+s(\vartheta-\vartheta^\circ)s(\vartheta^\circ-\vartheta)\left(\frac{\sin(2\vartheta^\circ-\vartheta)}{\sin\vartheta}\right)^{(k-1)/2}g(2\vartheta^\circ-\vartheta,\xi).\end{aligned} \tag{3.6}$$

Lemma 3.3. *Let $\vartheta^\circ\in(0,\pi)$ and $0<\delta<\min\{\vartheta^\circ,\pi-\vartheta^\circ\}$. Then, $E^{\vartheta^\circ}\in\mathcal{H}(\mathbb{S}^k)$ and*

$$E^{\vartheta^\circ}:L^2(\mathbb{S}^k,d\sigma_k)\to L^2(\mathbb{S}^k,d\sigma_k) \tag{3.7}$$

is an orthogonal projection.

Proof. Using Definition 3.1, (2.2), (3.5), and (3.6) observe that E^{ϑ° is a sum of two simple H-operators. One of them is a multiplier operator, where Φ is the identity on $\mathbb{S}^k$. The other one corresponds to Φ being a longitudinal reflection around ϑ° on the set

$$V=\{\Phi_k(\vartheta,\xi):\vartheta\in(\vartheta^\circ-\delta,\vartheta^\circ+\delta),\xi\in\mathbb{S}^{k-1}\}.$$

It remains to check that E^{ϑ° induces an orthogonal projection (3.7). Since E_{ψ^0} is an orthogonal projection on $L^2([-\vartheta^\circ,\pi-\vartheta^\circ],\psi^\circ)$ we have

$$(E^{\vartheta^\circ})^2g(\vartheta,\xi)=E_{\psi^\circ}(E^{\vartheta^\circ}g\circ\Phi_k^{\xi,\circ})(\vartheta-\vartheta^\circ)=E_{\psi^\circ}(E_{\psi^\circ}(g\circ\Phi_k^{\xi,\circ}))(\vartheta-\vartheta^\circ)=E^{\vartheta^\circ}g(\vartheta,\xi).$$

Indeed, the middle step is a consequence of

$$(E^{\vartheta^\circ}g)\circ\Phi_k^{\xi,\circ}(t)=E^{\vartheta^\circ}g(\Phi_k(t+\vartheta^\circ,\xi))=E_{\psi^\circ}(g\circ\Phi_k^{\xi,\circ})(t).$$

Let $f,g\in L^2(\mathbb{S}^k)$. By (3.4), the change of variables, and Proposition 1.1 we have

$$\begin{aligned}\int_{\mathbb{S}^k}E^{\vartheta^\circ}(g)fd\sigma_k&=\int_{\mathbb{S}^{k-1}}\int_0^\pi E^{\vartheta^\circ}g(\vartheta,\xi)f(\vartheta,\xi)(\sin\vartheta)^{k-1}d\vartheta d\sigma_{k-1}(\xi)\\ &=\int_{\mathbb{S}^{k-1}}\left(\int_{-\vartheta^\circ}^{-\vartheta^\circ+\pi}E_{\psi^\circ}(g\circ\Phi_k^{\xi,\circ})(\vartheta)(f\circ\Phi_k^{\xi,\circ})(\vartheta)\psi^\circ(\vartheta)d\vartheta\right)d\sigma_{k-1}(\xi)\\ &=\int_{\mathbb{S}^{k-1}}\left(\int_{-\vartheta^\circ}^{-\vartheta^\circ+\pi}(g\circ\Phi_k^{\xi,\circ})(\vartheta)E_{\psi^\circ}(f\circ\Phi_k^{\xi,\circ})(\vartheta)\psi^\circ(\vartheta)d\vartheta\right)d\sigma_{k-1}(\xi)\\ &=\int_{\mathbb{S}^k}gE^{\vartheta^\circ}(f)d\sigma_k.\end{aligned} \tag{3.8}$$

This proves that E^{ϑ° is self-adjoint and completes the proof of the lemma. □

3.3. **Latitudinal projections.** We are now ready to introduce a family of latitudinal smooth projections on $\mathbb{S}^k$ corresponding to any partition of $[0,\pi]$. Suppose that $\{\vartheta_j^k\}_{j=0}^{n_k}$ is a partition of the interval $[0,\pi]$ and $\delta>0$ satisfies

$$\delta<\frac{1}{2}\min\{\vartheta_j^k-\vartheta_{j-1}^k:j=1,\dots,n_k\}. \tag{3.9}$$

We consider the corresponding Auscher-Weiss-Wickerhauser operators $\{E_j^k\}_{j=1}^{n_k-1}$ on the sphere $\mathbb{S}^k$ given by $E_j^k = E^{\vartheta_j^k}$ as in Definition 3.4. In addition, we let $E_0^k = \mathbf{I}$ and $E_{n_k}^k = \mathbf{0}$, where $\mathbf{I}$ and $\mathbf{0}$ are identity and zero operators, resp. Note that

$$E_j^k \circ E_i^k = E_i^k \circ E_j^k = E_{\max\{i,j\}}^k. \tag{3.10}$$

Indeed, for $j = 1, \ldots, n_k - 1$ we have

$$E_j^k f(\vartheta, \xi) = \begin{cases} 0 & \text{for } (\vartheta, \xi) \in [0, \vartheta_j^k - \delta] \times \mathbb{S}^{k-1}, \\ E_j^k (f \mathbf{1}_{[\vartheta_j^k - \delta, \vartheta_j^k + \delta] \times \mathbb{S}^{k-1}})(\vartheta, \xi) & \text{for } (\vartheta, \xi) \in [\vartheta_j^k - \delta, \vartheta_j^k + \delta] \times \mathbb{S}^{k-1}, \\ f(\vartheta, \xi) & \text{for } (\vartheta, \xi) \in [\vartheta_j^k + \delta, \pi] \times \mathbb{S}^{k-1}. \end{cases} \tag{3.11}$$

Thus, if $i < j$, then $E_i^k E_j^k f = E_j^k f$ by (3.11) and

$$\operatorname{supp} E_j^k f \subset [\vartheta_j^k - \delta, \pi] \times \mathbb{S}^{k-1} \subset [\vartheta_i^k + \delta, \pi] \times \mathbb{S}^{k-1}.$$

Likewise, by (3.11)

$$E_j^k E_i^k f = E_j^k (\mathbf{1}_{[\vartheta_j^k - \delta, \pi] \times \mathbb{S}^{k-1}} E_i^k f) = E_j^k (\mathbf{1}_{[\vartheta_j^k - \delta, \pi] \times \mathbb{S}^{k-1}} f) = E_j^k f.$$

Finally, the case $i = j$ was proved in Lemma 3.3. This proves (3.10).

Definition 3.5. Given a partition $\{\vartheta_j^k\}_{j=0}^{n_k}$ of the interval $[0, \pi]$ as above we define a family of *latitudinal projections* $\{U_j^k\}_{j=1}^{n_k}$ by

$$U_j^k = E_{j-1}^k - E_j^k, \quad j = 1, \ldots, n_k,$$

Here, $E_j^k = E^{\vartheta_j^k}$ is given as in Definition 3.4 for all $j = 1, \ldots, n_k - 1$, $E_0^k = \mathbf{I}$, and $E_{n_k}^k = \mathbf{0}$.

Lemma 3.4. *The operators $\{U_j^k\}_{j=1}^{n_k}$ from Definition 3.5 have the following properties:*

(i) Each operator U_j^k is localized around the latitudinal strip $\Phi_k([\vartheta_{j-1}^k, \vartheta_j^k] \times \mathbb{S}^{k-1})$. That is, for any function $f : \mathbb{S}^k \to \mathbb{R}$ and $x \in \mathbb{S}^k$ we have for $j = 1, \ldots, n_k$

$$U_j^k f(\vartheta, \xi) = f(\vartheta, \xi) \quad \textit{for } (\vartheta, \xi) \in [\vartheta_{j-1}^k + \delta, \vartheta_j^k - \delta] \times \mathbb{S}^{k-1}, \tag{3.12}$$

and

$$U_j^k (\mathbf{1}_{[\vartheta_{j-1}^k - \delta, \vartheta_j^k + \delta] \times \mathbb{S}^{k-1}} f) = \mathbf{1}_{[\vartheta_{j-1}^k - \delta, \vartheta_j^k + \delta] \times \mathbb{S}^{k-1}} U_j^k f = U_j^k f, \tag{3.13}$$

with $\vartheta_{j-1}^k - \delta$ and $\vartheta_j^k + \delta$ replaced by 0 and π when $j = 1$ and $j = n_k$, resp.

(ii) Each U_j^k considered as an operator

$$U_j^k : L^2(\mathbb{S}^k, d\sigma_k) \to L^2(\mathbb{S}^k, d\sigma_k)$$

is an orthogonal projection. The projections $\{U_j^k\}$ give a decomposition of $L^2(\mathbb{S}^{d-1})$ into orthogonal subspaces

$$\sum_{j=1}^{n_k} U_j^k = \mathbf{I}, \tag{3.14}$$

where $\mathbf{I}$ is the identity operator on $L^2(\mathbb{S}^k, \sigma_k)$. In particular, we have mutual orthogonality $U_j^k \circ U_i^k = 0$ for any $i \neq j$.

(iii) Each U_j^k belongs to $\mathcal{H}(\mathbb{S}^k)$. In particular, for any $r = 0, 1, \ldots$ and $1 \leq p < \infty$, U_j^k induces bounded operators on $C^r(\mathbb{S}^k)$ and on the Sobolev space $W_p^r(\mathbb{S}^k)$.

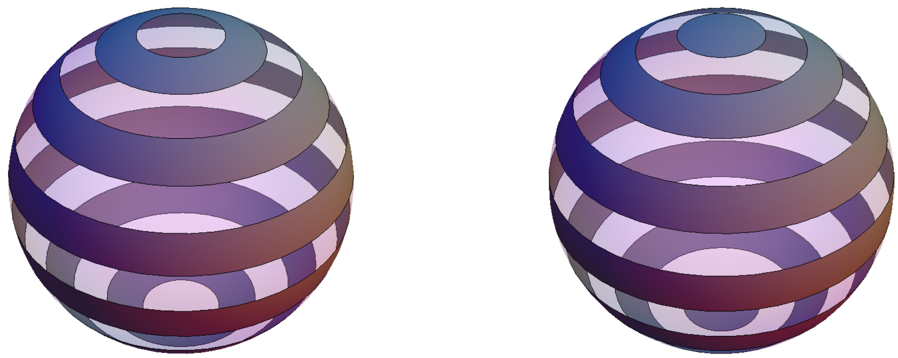

FIGURE 1. Supports of latitudinal projections U_j^2, $j = 1, \dots, 13$, on the sphere $\mathbb{S}^2$

Proof. (i) follows from Definition 3.5, Lemma 3.3, and (3.11). To prove (ii) note that by (3.10) for $i < j$ we have

$$U_j^k \circ U_i^k = (E_{j-1}^k - E_j^k) \circ (E_{i-1}^k - E_i^k) = E_{j-1}^k - E_{j-1}^k - E_j^k + E_j^k = 0.$$

The case $i > j$ is similar. Moreover, by (3.10) we also have that $U_j^k \circ U_j^k = U_j^k$. On the other hand, by (3.8) each U_j^k is self-adjoint:

$$\int_{\mathbb{S}^k} U_j^k f(x) g(x) d\sigma_k(x) = \int_{\mathbb{S}^k} (E_{j-1}^k - E_j^k) f(x) g(x) d\sigma_k(x) = \int_{\mathbb{S}^k} f(x) U_j^k g(x) d\sigma_k(x).$$

Moreover, (3.14) follows from Definition 3.5 by telescoping, thus proving the property (ii). Finally, (iii) follows immediately from Lemma 3.2 since each $U_j^k \in \mathcal{H}(\mathbb{S}^k)$. This completes the proof of the lemma. □

4. LIFTING OF H-OPERATORS ON THE SPHERE

In this section we introduce the procedure of lifting an operator acting on functions on $\mathbb{S}^{k-1}$ to a higher dimensional sphere $\mathbb{S}^k$. Despite that lifting does not preserve the property of H-operators due to singularities at the poles, this property can be recovered after composing with suitable longitudinal projections.

Definition 4.1. Suppose that T is an H-operator on $\mathbb{S}^{k-1}$, i.e., $T \in \mathcal{H}(\mathbb{S}^{k-1})$. We define the lifted operator $\hat{T}$ acting on functions $f : \mathbb{S}^k \setminus \{\mathbf{1}^k, -\mathbf{1}^k\} \to \mathbb{R}$ using the relation

$$\hat{T}(f)(t,\xi) = \begin{cases} T(f^t)(\xi) & (t,\xi) \in (0,\pi) \times \mathbb{S}^{k-1}, \\ 0 & t = 0 \quad \text{or} \quad t = \pi. \end{cases} \tag{4.1}$$

where

$$f^t(\xi) = f(t,\xi), \quad (t,\xi) \in (0,\pi) \times \mathbb{S}^{k-1} \approx \mathbb{S}^k \setminus \{\mathbf{1}^k, -\mathbf{1}^k\}.$$

The following lemma shows that the lifting preserves the property of an H-operator (away from the two poles) and an orthogonal projection from a lower to a higher dimensional sphere.

Lemma 4.1. *Assume that $T \in \mathcal{H}(\mathbb{S}^{k-1})$. Then, $\hat{T} \in \mathcal{H}(\mathbb{S}^k \setminus \{\mathbf{1}^k, -\mathbf{1}^k\})$ satisfies a commutation relation*

$$\hat{T} \circ U_j^k f = U_j^k \circ \hat{T} f, \qquad \text{for all } f \in L_0(\mathbb{S}^k),\ 1 \le j \le n_k. \tag{4.2}$$

Moreover, the operators $\hat{T} \circ U_j^k$ belong to $\mathcal{H}(\mathbb{S}^k)$ for any $2 \le j \le n_k - 1$.

In addition, if T induces an orthogonal projection $T : L^2(\mathbb{S}^{k-1}) \to L^2(\mathbb{S}^{k-1})$, then $\hat{T} : L^2(\mathbb{S}^k) \to L^2(\mathbb{S}^k)$ is also an orthogonal projection.

Proof. Observe that the lifted operator $\hat{T}$ can be identified with the tensor product $\mathbf{I} \otimes T$ of the identity operator $\mathbf{I}$ on $L_0(0,\pi)$ and an H-operator T on $L_0(\mathbb{S}^{k-1})$, using the identification $(0,\pi) \times \mathbb{S}^{k-1} \approx \mathbb{S}^k \setminus \{\mathbf{1}^k, -\mathbf{1}^k\}$. Consequently, by Lemma 3.1 we have $\hat{T} \in \mathcal{H}(\mathbb{S}^k \setminus \{\mathbf{1}^k, -\mathbf{1}^k\})$.

To show (4.2), let $f \in L_0(\mathbb{S}^k)$ be such that

$$f(t,\xi) = f_1(t) f_2(\xi) = (f_1 \otimes f_2)(t,\xi), \quad (t,\xi) \in (0,\pi) \times \mathbb{S}^{k-1}, \tag{4.3}$$

where $f_1 \in L_0(0,\pi)$ and $f_2 \in L_0(\mathbb{S}^{k-1})$. Then, by (3.6) and (4.1) we have that for $(t,\xi) \in (0,\pi) \times \mathbb{S}^{k-1}$,

$$\hat{T} \circ U_j^k (f_1 \otimes f_2)(t,\xi) = T(f_2)(\xi) U_j^k (f_1 \otimes \mathbf{1}_{\mathbb{S}^{k-1}})(t,\xi).$$

Likewise, by the linearity of T

$$U_j^k \circ \hat{T}(f_1 \otimes f_2)(t,\xi) = U_j^k((f_1 \otimes T(f_2)))(t,\xi) = T(f_2)(\xi) U_j^k (f_1 \otimes \mathbf{1}_{\mathbb{S}^{k-1}})(t,\xi).$$

Since linear combinations of measurable functions with separated variables are dense among measurable functions on the product space, i.e., functions f of the form (4.3) are are dense in $L_0(\mathbb{S}^k)$, by a density argument we obtain (4.2).

Now we shall prove that $\hat{T} \circ U_j^k \in \mathcal{H}(\mathbb{S}^k)$ for any $2 \le j \le n_k - 1$. Take a function $\varphi \in C^\infty(\mathbb{S}^{k-1})$ such that

$$\varphi(t,\xi) = 1 \qquad \text{for all } (t,\xi) \in (\vartheta_1^k - \delta, \vartheta_{n_k-1}^k + \delta) \times \mathbb{S}^{k-1},$$

and $\operatorname{supp} \varphi$ is a compact subset of $\mathbb{S}^k \setminus \{\mathbf{1}^k, -\mathbf{1}^k\}$. Then, the multiplication operator S defined by $Sf = \varphi f$ belongs to $\mathcal{H}(\mathbb{S}^k)$. Since, the range of S consists of functions vanishing in some neighborhood of the two poles $\{\mathbf{1}^k, -\mathbf{1}^k\}$, the composition $\hat{T} \circ S$ also belongs to $\mathcal{H}(\mathbb{S}^k)$. Consequently, $\hat{T} \circ S \circ U_j^k \in \mathcal{H}(\mathbb{S}^k)$. However, in light of (3.13) this composition coincides with $\hat{T} \circ U_j^k$ for any $2 \le j \le n_k - 1$. Hence, we obtain the required conclusion.

To prove the second part of the lemma, suppose that T induces an orthogonal projection on $L^2(\mathbb{S}^{k-1})$. To verify that $\hat{T}$ is an orthogonal projection we take any $f, g \in L^2(\mathbb{S}^k)$. Then for almost all $\vartheta \in [0,\pi]$, $f^\vartheta, g^\vartheta \in L^2(\mathbb{S}^{k-1})$. Hence,

$$\begin{aligned}
\int_{\mathbb{S}^k} \hat{T} f(x) g(x) d\sigma_k(x) &= \int_0^\pi \int_{\mathbb{S}^{k-1}} T(f^\vartheta)(\xi) g^\vartheta(\xi) d\sigma_{k-1}(\xi) (\sin(\vartheta))^{k-1} d\vartheta \\
&= \int_0^\pi \int_{\mathbb{S}^{k-1}} f^\vartheta(\xi) T(g^\vartheta)(\xi) d\sigma_{k-1}(\xi) (\sin(\vartheta))^{k-1} d\vartheta \\
&= \int_{\mathbb{S}^k} f(x) \hat{T} g(x) d\sigma_k(x).
\end{aligned}$$

Observe that

$$\hat{T} \circ \hat{T}(f)(t, \xi) = T((\hat{T}(f))^t)(\xi) = T(T(f^t))(\xi) = T(f^t)(\xi) = \hat{T}(t, \xi).$$

The second equality is a consequence of

$$(\hat{T}(f))^t(\xi) = \hat{T}(f)(t, \xi) = T(f^t)(\xi).$$

This finishes the proof of Lemma 4.1. $\square$

5. Smooth orthogonal projections on $\mathbb{S}^d$

In this section we construct smooth orthogonal projections corresponding to a partition of the sphere $\mathbb{S}^d$. We start with definitions of partitions of the sphere $\mathbb{S}^d$ into latitudinal strips and patches.

Definition 5.1. Given a partition $\{\vartheta_j^k\}_{j=0}^{n_k}$ of the interval $[0, \pi]$, i.e.,

$$0 = \vartheta_0^k < \vartheta_1^k < \cdots < \vartheta_{n_k-1}^k < \vartheta_{n_k}^k = \pi,$$

we define an associated partition into *latitudinal strips* of the sphere $\mathbb{S}^k$, $k \geq 2$, by

$$A_j^k = \Phi_k\left([\vartheta_{j-1}^k, \vartheta_j^k] \times \mathbb{S}^{k-1}\right), \qquad j = 1, \ldots, n_k.$$

In the special case when $j = 1$ or $j = n_k$, the sets A_1^k and $A_{n_k}^k$ are referred as *polar patches.*

Definition 5.2. Suppose that $\{\vartheta_j^k\}_{j=0}^{n_k}$ are partitions of the interval $[0, 2\pi]$ when $k = 1$, and of the interval $[0, \pi]$ when $k = 2, \ldots, d$, where $d \geq 1$. The associated partition into patches of the sphere $\mathbb{S}^d$ consists of two kinds of sets. The *interior patches* are given by

$$\Psi_d([\vartheta_{i_d-1}^d, \vartheta_{i_d}^d] \times \cdots \times [\vartheta_{i_1-1}^1, \vartheta_{i_1}^1]), \tag{5.1}$$

where $1 \leq i_1 \leq n_1$, and $2 \leq i_k \leq n_k - 1$ for $2 \leq k \leq d$. The *boundary patches* are given by

$$\Psi_d([\vartheta_{i_d-1}^d, \vartheta_{i_d}^d] \times \cdots \times [\vartheta_{i_{k+1}-1}^{k+1}, \vartheta_{i_{k+1}}^{k+1}] \times \Psi_k^{-1}(A_j^k)), \tag{5.2}$$

where $j = 1$ or $j = n_k$, $2 \leq k \leq d$, and $2 \leq i_l \leq n_l - 1$ for $k + 1 \leq l \leq d$. In particular, (5.2) is understood as one of the polar patches A_j^d, $j = 1, n_d$, when $k = d$. The set of all patches Q of the form(5.1) and (5.2) is denoted by $\mathcal{Q}^d$.

Observe that we can also define a partition of the sphere $\mathbb{S}^d$ into patches using the following recursion:

$$\mathcal{Q}^d = \begin{cases} \{\Psi_1([\vartheta_{j-1}^1, \vartheta_j^1]) : 1 \leq j \leq n_1\} & d = 1, \\ \{\Phi_d([\vartheta_{j-1}^d, \vartheta_j^d] \times Q) : 2 \leq j \leq n_d - 1, Q \in \mathcal{Q}^{d-1}\} \cup \{A_1^d, A_{n_d}^d\} & d \geq 2. \end{cases}$$

Any such partition of $\mathbb{S}^{d-1}$ gives rise to the corresponding family of smooth projections as defined below.

Definition 5.3. Let $\mathcal{Q}^d$ be a partition of the sphere of $\mathbb{S}^d$ into patches. We define the family of $\{P_Q\}_{Q \in \mathcal{Q}^d}$ of *smooth orthogonal projections* on $\mathbb{S}^d$ according to the following recursive procedure. If $d = 1$, then P_Q's are defined as in Definition 2.2. Suppose $d \geq 2$. If $Q = A_j^d$, $j = 1$ or $j = n_d$, is one of two polar patches, then we let $P_Q = U_j^d$. Otherwise, if

$$Q = \Phi_d([\vartheta_{j-1}^d, \vartheta_j^d] \times Q'), \tag{5.3}$$

for some $2 \leq j \leq n_d - 1$ and $Q' \in \mathcal{Q}^{d-1}$, then we let $P_Q = \hat{P_{Q'}} \circ U_j^d$.

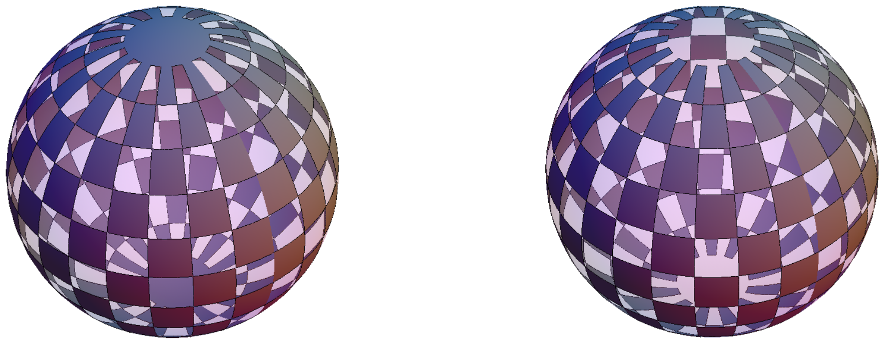

FIGURE 2. The partition $\mathcal{Q}^2$ of the sphere $\mathbb{S}^2$ into patches

The following theorem summarizes properties of the projections P_Q.

Theorem 5.1. *Let $\mathcal{Q}^d$ be the set of patches of the sphere $\mathbb{S}^d$, $d \geq 1$, as in Definition 5.2. For any $\varepsilon > 0$, there exists a family of linear operators $\{P_Q\}_{Q\in\mathcal{Q}^d}$ defined pointwise for functions on $\mathbb{S}^d$ satisfying the following properties:*

(i) Each operator P_Q is localized around the patch Q. That is, for any function $f : \mathbb{S}^d \to \mathbb{R}$ and $x \in \mathbb{S}^d$ we have

$$P_Q f(x) = \begin{cases} 0 & \text{if } d(x, Q) > \varepsilon, \\ f(x) & \text{if } d(x, \mathbb{S}^d \setminus Q) > \varepsilon. \end{cases}$$

(ii) Each P_Q considered as an operator

$$P_Q : L^2(\mathbb{S}^d, \sigma_d) \to L^2(\mathbb{S}^d, \sigma_d)$$

is an orthogonal projection. The operators $\{P_Q\}$ give a decomposition of $L^2(\mathbb{S}^d)$ into orthogonal subspaces. That is,

$$\sum_{Q\in\mathcal{Q}^d} P_Q = \mathbf{I},$$

where $\mathbf{I}$ is the identity operator. In particular, we have $P_Q \circ P_{Q'} = 0$ for $Q \neq Q' \in \mathcal{Q}^d$.

(iii) Each P_Q belongs to $\mathcal{H}(\mathbb{S}^d)$. In particular, for any $r = 0, 1, \ldots$ and $1 \leq p < \infty$, P_Q induces bounded operators on $C^r(\mathbb{S}^d)$ and on the Sobolev space $W^r_p(\mathbb{S}^d)$.

Proof. The proof is given by an induction on $d \geq 1$. The base case $d = 1$ is a consequence of Theorem 2.1.

Let $\delta > 0$ be such that (3.9) holds for all $k = 1, \ldots, n_k$. Suppose that $Q \in \mathcal{Q}^d$ is an interior patch as in (5.1). By (3.12) and (3.13) in Lemma 3.4 and the recursive definition of P_Q we have

$$P_Q f(x) = \begin{cases} 0 & \text{for } x \notin \Psi_d([\vartheta^d_{i_d-1} - \delta, \vartheta^d_{i_d} + \delta] \times \cdots \times [\vartheta^1_{i_1-1} - \delta, \vartheta^1_{i_1} + \delta]), \\ f(x) & \text{for } x \in \Psi_d([\vartheta^d_{i_d-1} + \delta, \vartheta^d_{i_d} - \delta] \times \cdots \times [\vartheta^1_{i_1-1} + \delta, \vartheta^1_{i_1} - \delta]). \end{cases} \tag{5.4}$$

A similar property holds for boundary patches Q as in (5.2). Hence, by choosing $\delta > 0$ to be sufficiently small we obtain (i).

Let us assume that the theorem holds in the dimension $d-1$. Suppose that $Q \in \mathcal{Q}^d$. If Q is one of the two polar patches, then parts (ii) and (iii) follow from the corresponding parts of Lemma 3.4. If Q is of the form (5.3), then $P_Q = \hat{P}_{Q'} \circ U^d_j$ and it suffices to apply Lemmas 3.4 and 4.1 and the inductive hypothesis on $P_{Q'}$. Finally, the decomposition formula follows from

$$\mathbf{I} = \sum_{j=1}^{n_d} U^d_j = U^d_1 + U^d_{n_d} + \sum_{j=2}^{n_d-1} \Bigg(\sum_{Q' \in \mathcal{Q}^{d-1}} \hat{P_{Q'}} \Bigg) \circ U^d_j = \sum_{Q \in \mathcal{Q}^d} P_Q.$$

Indeed, the operator inside the parenthesis equals $\mathbf{I}$ as a consequence of the inductive hypothesis and (4.1). This completes the proof of the theorem. $\square$

6. Function spaces and Parseval frames on the sphere

6.1. **Decomposition of function spaces on the sphere.** Ciesielski and Figiel [2] gave a construction of spline bases on Besov spaces $B^s_{p,q}(I^d)$ and Sobolev spaces $W^k_p(I^d)$ on the unit cube $I^d = [0,1]^d$, where $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. It was the first step of their construction of spline bases on Besov and Sobolev spaces on compact manifolds [3, 4]. Their main result gives a decomposition of Sobolev and Besov spaces on smooth compact manifolds into the corresponding function spaces on d-cubes with appropriate boundary conditions, see [3, Theorem 4.9] and [11, Proposition 29]. However, their decomposition depends on the choice of the smoothness parameter m and works only for Sobolev spaces W^k_p and Besov spaces $B^s_{p,q}$ with the smoothness parameter $|k| \le m$ and $|s| < m$. In contrast our decomposition using smooth orthogonal projections works for all values of $s \in \mathbb{R}$. For simplicity we consider only positive smoothness function spaces to avoid dealing with function spaces whose elements are distributions (rather than functions).

There are several equivalent ways of defining Sobolev spaces on manifolds. The standard method, see Definition 3.3 and the book of Hebey [13], requires Riemannian structure on a manifold. Since any two Riemannian metrics are equivalent on compact manifolds, the corresponding Sobolev spaces do not depend on the choice of the metric, see [13, Proposition 2.2]. This also explains why Ciesielski-Figiel approach [2] to Sobolev spaces on compact manifolds works without relating to a Riemannian structure. We shall recall the approach in [2] below.

Definition 6.1. A subset $Q \subset \mathbb{S}^d$ is said to be a d-cube if there exists a diffeomorphism $\phi : U \to \mathbb{R}^d$ such that $Q \subset U = \operatorname{Int} U$ and $\phi(Q) = I^d$.

Note that interior patches are cubes by the definition. Moreover, any boundary patch is contained in a cube.

Definition 6.2. Let $1 \le p < \infty$ and $r \in \mathbb{N}$. A function f belongs to Sobolev spaces on d-cube Q, $f \in W^r_p(Q)$ if $f \circ \phi^{-1} \in W^r_p(I^d)$. A norm $\| \cdot \|_{W^r_p(Q)}$ is given by

$$\|f\|_{W^r_p(Q)} = \|f \circ \phi^{-1}\|_{W^r_p(I^d)}.$$

Choosing another diffeomorphism ϕ one obtains equivalent norms, see [2, Section 3] and also [3, Lemma 2.52]. In a similar way we define Besov spaces $B^s_{p,q}(Q)$, $s>0$, $1\le p\le\infty$, $1\le q\le\infty$ with the norm

$$\|f\|_{B^s_{p,q}(Q)} = \|f\circ\phi^{-1}\|_{B^s_{p,q}(I^d)}.$$

By [3, Lemma 2.52] choosing another diffeomorphism ϕ one obtains equivalent norms.

Definition 6.3. Let $\{Q_j\}_{j=1}^N$ be a family of d-cubes in $\mathbb{S}^d$ such that

$$\mathbb{S}^d = \bigcup_{j=1}^N \operatorname{Int} Q_j. \tag{6.1}$$

We define the Sobolev space $W^r_p(\mathbb{S}^d)$, $1\le p<\infty$, $r\in\mathbb{N}$, to be the collection of all f with the norm

$$\|f\|_{W^r_p(\mathbb{S}^d)} = \sum_{j=1}^N \|f|_{Q_j}\|_{W^r_p(Q_j)} < \infty.$$

When $p=\infty$ we let $W^r_\infty(\mathbb{S}^d) = C^r(\mathbb{S}^d)$. Likewise, we define the Besov space $B^s_{p,q}(\mathbb{S}^d)$, $1\le p,q\le\infty$, $s\ge 0$ to be the collection of all f with the norm

$$\|f\|_{B^s_{p,q}(\mathbb{S}^d)} = \sum_{j=1}^N \|f|_{Q_j}\|_{B^s_{p,q}(Q_j)} < \infty.$$

Choosing another family of d-cubes (6.1) one obtains equivalent norms, see [2, Section 3]. It is well known [3] that the Besov space $B^s_{p,q}(\mathbb{S}^d)$, where $s>0$, $1\le p,q\le\infty$ can be obtained using the real interpolation method between $L^p = W^0_p$ and Sobolev space W^m_p, $m>s$,

$$B^s_{p,q}(\mathbb{S}^d) = (W^0_p(\mathbb{S}^d), W^m_p(\mathbb{S}^d))_{s/m,q}. \tag{6.2}$$

Consequently, by Lemma 3.2 an operator $H\in\mathcal{H}(\mathbb{S}^d)$ induces a bounded operator on $B^s_{p,q}(\mathbb{S}^d)$ spaces.

Definition 6.4. Let $\mathcal{F}(\mathbb{S}^d)$ be either the space $B^s_{p,q}(\mathbb{S}^d)$, $s\ge 0$, $1\le p,q\le\infty$, or $W^r_p(\mathbb{S}^d)$, $r\in\mathbb{N}$, $1\le p\le\infty$. Let $\mathcal{Q}^d$ be a partition into patches of the sphere of $\mathbb{S}^d$ as in Definition 5.2. Define the spaces

$$\mathcal{F}(Q) := P_Q(\mathcal{F}(\mathbb{S}^d)) \qquad \text{for } Q\in\mathcal{Q}^d. \tag{6.3}$$

Theorem 6.1. *Let $\{P_Q\}_{Q\in\mathcal{Q}^d}$ be the family of the orthogonal projections as in Theorem 5.1. Then, there are constants $C = C(p,q,s,\mathcal{Q}^d)>0$ such that for $f\in\mathcal{F}(\mathbb{S}^d)$*

$$\|P_Q f\|_{\mathcal{F}(\mathbb{S}^d)} \le C\|f\|_{\mathcal{F}(\mathbb{S}^d)}.$$

Moreover, each $\mathcal{F}(Q)$ is a closed subspace of $\mathcal{F}(\mathbb{S}^d)$ and we have a direct sum decomposition

$$\mathcal{F}(\mathbb{S}^d) = \bigoplus_{Q\in\mathcal{Q}^d} \mathcal{F}(Q)$$

with the equivalence of norms

$$||f||_{\mathcal{F}(\mathbb{S}^d)} \asymp \sum_{Q\in\mathcal{Q}^d} ||P_Q f||_{\mathcal{F}(\mathbb{S}^d)}.$$

Proof. By Lemma 3.2 and Theorem 5.1 there exists a constant $C = C(r, p, \mathcal{Q}^d) > 0$ such that

$$\|P_Q f\|_{W_p^r(\mathbb{S}^d)} \le C\|f\|_{W_p^r(\mathbb{S}^d)} \qquad \text{for } f \in W_p^r(\mathbb{S}^d).$$

By the functorial property of interpolation spaces and (6.2) a similar property holds for Besov spaces. Since each P_Q is a projection, $\mathcal{F}(Q) = \ker(P_Q - \mathbf{I})$ is a closed subspace of $\mathcal{F}(\mathbb{S}^d)$. Moreover, the mapping $f \mapsto (P_Q f)_{Q\in\mathcal{Q}^d}$, $f \in \mathcal{F}(\mathbb{S}^d)$ is an isomorphism between $\mathcal{F}(\mathbb{S}^d)$ and $\bigoplus_{Q\in\mathcal{Q}^d} \mathcal{F}(Q)$ since

$$f = \sum_{Q\in\mathcal{Q}^d} P_Q f.$$

This completes the proof of Theorem 6.1. $\square$

6.2. **Localized Parseval frames on the sphere.** Ciesielski-Figiel decomposition of manifolds into cubes was used in the construction of wavelets on manifolds by Dahmen and Schneider [8] and by Kunoth and Sahner [16]. In addition, Narcovich, Petrushev, and Ward [19] have constructed well localized Parseval frames on the sphere, called needlets. In [20] they used these frames to characterize Besov and Triebel-Lizorkin spaces on the sphere. A generalization to compact homogeneous manifolds can be found in [10]. Here we shall give an alternative construction of wavelets on the sphere by transferring wavelets bases from $\mathbb{R}^d$ to patches Q via local diffeomorphisms.

Consider Daubechies multivariate wavelets. That is, for a fixed $N \ge 2$, let ${}_N\phi$ be a univariate, compactly supported scaling function with support $\operatorname{supp} {}_N\phi = [0, 2N-1]$ associated with the compactly supported, orthogonal univariate Daubechies wavelet ${}_N\psi$, see [9, Section 6.4]. For convenience, let $\psi^0 = {}_N\phi$ and $\psi^1 = {}_N\psi$. Let $E' = \{0,1\}^d$ be the vertices of the unit cube and let $E = E' \setminus \{0\}$ be the set of nonzero vertices. For each $\mathbf{e} = (e_1, \dots, e_d) \in E'$, define

$$\psi^{\mathbf{e}}(x) = \psi^{e_1}(x_1) \cdots \psi^{e_d}(x_d), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Observe that $\operatorname{supp} \psi^{\mathbf{e}} = [0, 2N-1]^d$.

Let $\mathcal{D}$ be the set of dyadic cubes in $\mathbb{R}^d$ of the form $I = 2^{-j}(k + [0,1]^d)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$. Denote the side length of I by $\ell(I) = 2^{-j}$. For any $\mathbf{e} \in E'$, define a wavelet scaled relative to I by

$$\psi_I^{\mathbf{e}}(x) = 2^{jd/2}\psi^{\mathbf{e}}(2^j x - k), \qquad x \in \mathbb{R}^d.$$

It is well-known that $\{\psi_I^{\mathbf{e}}(x) : I \in \mathcal{D}, \mathbf{e} \in E\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. Moreover, it is also an unconditional basis of $C^r(\mathbb{R}^d)$ and the Sobolev space $W_p^r(\mathbb{R}^d)$, $r = 0, 1, \dots$ and $1 \le p < \infty$ for sufficiently large choice of $N \ge N(r)$ depending on r. For our purposes it is convenient to consider wavelet systems localized to a cube.

Definition 6.5. Suppose that $J = \prod_{i=1}^d [a_i, b_i]$ is a cube in $\mathbb{R}^d$ and $\epsilon > 0$. Define its ϵ enlargement by $J_\epsilon = \prod_{i=1}^d [a_i - \epsilon, b_i + \epsilon]$. Let $j_0 \in \mathbb{Z}$ be the smallest integer such that

$$(2N-1)2^{-j_0} < \epsilon/2. \tag{6.4}$$

For any $j \ge j_0$, consider families of dyadic cubes

$$\mathcal{D}_j = \{I \in \mathcal{D} : \ell(I) = 2^{-j} \text{ and } \operatorname{supp} \psi_I^{\mathbf{e}} \subset J_\epsilon\} \qquad \text{and} \qquad \mathcal{D}_{j_0}^+ = \bigcup_{j=j_0}^{\infty} \mathcal{D}_j.$$

Define a localized wavelet system relative to the cube J and $\epsilon > 0$ by

$$S(J,\epsilon) := \{\psi_I^{\mathbf{e}} : \mathbf{e} \in E, I \in \mathcal{D}_{j_0}^+\} \cup \{\psi_I^0 : I \in \mathcal{D}_{j_0}\}.$$

Lemma 6.1. *The localized wavelet system $S(J,\epsilon)$ has following properties:*

- *$S(J,\epsilon)$ is an orthonormal sequence in $L^2(J_\epsilon)$,*
- *for every $f \in L^2(J_\epsilon)$ with $\operatorname{supp} f \subset J_{\epsilon/2}$ we have*

$$\|f\|_{L^2}^2 = \sum_{\mathbf{e}\in E} \sum_{I \in \mathcal{D}_{j_0}^+} |\langle f, \psi_I^{\mathbf{e}}\rangle|^2 + \sum_{I\in\mathcal{D}_{j_0}} |\langle f, \psi_I^0\rangle|^2. \tag{6.5}$$

- *magnitudes of coefficients $\{|\langle f, g\rangle|\}_{g\in S(J,\epsilon)}$ characterize functions $f \in \mathcal{F}(\mathbb{R}^d)$ satisfying $\operatorname{supp} f \subset J_{\epsilon/2}$, where $\mathcal{F}$ is either the Sobolev space W_p^r or the Besov space $B_{p,q}^s$, $0 < s < r$, $1 < p, q < \infty$.*

Proof. This follows from from the properties of the corresponding (global) orthonormal wavelet basis

$$\{\psi_I^{\mathbf{e}} : \mathbf{e} \in E, I \in \mathcal{D}, \ell(I) \le 2^{-j_0}\} \cup \{\psi_I^0 : I \in \mathcal{D}, \ell(I) = 2^{-j_0}\} \tag{6.6}$$

and the following elementary observation that is a consequence of (6.4). For any $f \in L^2(J_\epsilon)$ satisfying $\operatorname{supp} f \subset J_{\epsilon/2}$, and for any $I \in \mathcal{D}$ with $\ell(I) \le 2^{-j_0}$ and $\mathbf{e} \in E' = E \cup \{0\}$

$$\langle f, \psi_I^{\mathbf{e}}\rangle \ne 0 \implies I \in \mathcal{D}_{j_0}^+.$$

Thus, the wavelet coefficients of such f with respect to the system (6.6) must vanish for dyadic cubes that do not belong to $\mathcal{D}_{j_0}^+$. Hence, (6.5) holds. The same argument applies to any other function space $\mathcal{F}$, which can be characterized by wavelet coefficients of the system (6.6), such as Sobolev space W_p^r, $1 < p < \infty$, see [17, Section 6.2]. In particular, for any function $f \in W_p^r(\mathbb{R}^d)$ with $\operatorname{supp} f \subset J_{\epsilon/2}$ we have

$$||f||_{W_p^r} \asymp \left\| \left(\sum_{(\mathbf{e},I)\in(E\times\mathcal{D}_{j_0}^+)\cup(\{0\}\times\mathcal{D}_{j_0})} |\langle f, \psi_I^{\mathbf{e}}\rangle|^2 |I|^{-1-2r} \chi_I \right)^{1/2} \right\|_{L^p} < \infty.$$

□

The localized wavelet system $S(J,\epsilon)$ can be transferred to the sphere $\mathbb{S}^d$ via the spherical coordinates Ψ_d. Consider the change of variables operator

$$\mathbf{T}_d : L^2([0,\pi]^{d-1} \times [0, 2\pi]) \to L^2(\mathbb{S}^d)$$

given by

$$\mathbf{T}_d(\psi)(u) = \frac{\psi({\Psi_d}^{-1}(u))}{\sqrt{J_d(\Psi_d^{-1}(u))}}, \qquad u \in \mathbb{S}^d,$$

where J_d is the Jacobian of Ψ_d given by

$$J_d(\theta_d, \theta_{d-1}, \dots, \theta_1) = |\sin^{d-1}\theta_d \sin^{d-2}\theta_{d-1} \cdots \sin\theta_2|.$$

Since the set where Ψ_d is not 1-1 has measure zero, by the change of variables formula, $\mathbf{T}_d$ is an isometric isomorphism.

Definition 6.6. Let $\mathcal{Q}^d$ be a partition of $\mathbb{S}^d$ as in Definition 5.2. Fix an interior patch $Q \in \mathcal{Q}^d$ of the form $Q = \Psi_d(J)$, where

$$J = [\vartheta^d_{i_d-1}, \vartheta^d_{i_d}] \times \cdots \times [\vartheta^1_{i_1-1}, \vartheta^1_{i_1}], \qquad 1 \le i_1 \le n_1 \text{ and } 2 \le i_k \le n_k - 1 \text{ for } 2 \le k \le d.$$

Let $\delta > 0$ be such that (3.9) holds. Let $\epsilon = 2\delta$ so that we have

$$J_{\epsilon/2} = [\vartheta^d_{i_d-1} - \delta, \vartheta^d_{i_d} + \delta] \times \cdots \times [\vartheta^1_{i_1-1} - \delta, \vartheta^1_{i_1} + \delta]. \tag{6.7}$$

Then, we can map the orthogonal wavelet system $S(J,\epsilon)$ on $\mathbb{R}^d$ into the sphere $\mathbb{S}^d$ using the operator $\mathbf{T}_d$. That is, we define the localized wavelet system relative to the interior patch Q as

$$\mathcal{S}(Q) = \mathbf{T}_d(S(J,\epsilon)) = \{\mathbf{T}_d(\psi^{\mathbf{e}}_I) : \mathbf{e} \in E, I \in \mathcal{D}^+_{j_0}\} \cup \{\mathbf{T}_d(\psi^0_I) : I \in \mathcal{D}_{j_0}\}. \tag{6.8}$$

In the case $Q \in \mathcal{Q}^d$ is a boundary patch the above definition needs to be modified due to singularities of the Jacobian J_d.

Definition 6.7. Fix a boundary patch of the form $Q = \Psi_d(J)$, where

$$J = [\vartheta^d_{i_d-1}, \vartheta^d_{i_d}] \times \cdots \times [\vartheta^{k+1}_{i_{k+1}-1}, \vartheta^{k+1}_{i_{k+1}}] \times [0, \vartheta^k_1] \times \Psi^{-1}_{k-1}(\mathbb{S}^{k-1}),$$

where $2 \le k \le d$. The case when $[0,\vartheta^k_1]$ is replaced by $[\vartheta^k_{n_k-1}, \pi])$ is dealt in a similar way. Let

$$J'_\delta = [\vartheta^d_{i_d-1} - \delta, \vartheta^d_{i_d} + \delta] \times \cdots \times [\vartheta^{k+1}_{i_{k+1}-1} - \delta, \vartheta^{k+1}_{i_{k+1}} + \delta] \times [0, \vartheta^k_1 + \delta] \times \Psi^{-1}_{k-1}(\mathbb{S}^{k-1}), \tag{6.9}$$

When $\vartheta^k_1 + \delta$ is sufficiently small ($< \pi/4$) one can find an orthogonal linear map $O : \mathbb{R}^d \to \mathbb{R}^d$ such that $\Psi_d(J'_\delta) \subset O(\Psi_d(\tilde{J}))$, where

$$\tilde{J} = [a_d, b_d] \times \ldots \times [a_2, b_2] \times [a_1, b_1],$$

$0 < a_1 < b_1 < 2\pi$, and $0 < a_i < b_i < \pi$ for $i = 2, \ldots, d$. Once we choose $0 < \epsilon < \min\{a_i, \pi - b_i : i = 2, \ldots, d\}$, we define the localized wavelet system relative to the boundary patch Q as

$$\mathcal{S}(Q) = D_O(\mathbf{T}_d(S(\tilde{J}, \epsilon))), \tag{6.10}$$

where $D_O f(x) = f(Ox)$ is a rotation operator by O.

Lemma 6.2. *The localized wavelet system $\mathcal{S}(Q)$ has the following properties:*

- *$\mathcal{S}(Q)$ is an orthogonal sequence in $L^2(\mathbb{S}^d)$ for any $Q \in \mathcal{Q}^d$,*
- *$P_Q(\mathcal{S}(Q))$ is a Parseval frame for $P_Q(L^2(\mathbb{S}^d))$. That is, for any $f \in P_Q(L^2(\mathbb{S}^d))$*

$$\|f\|^2_{L^2} = \sum_{g \in \mathcal{S}(Q)} |\langle f, P_Q(g)\rangle|^2. \tag{6.11}$$

- *magnitudes of coefficients $\{|\langle f, g\rangle|\}_{g \in \mathcal{P}_Q(S(Q))}$ characterize functions $f \in \mathcal{F}(Q)$, where $\mathcal{F}$ is either the Sobolev space W^r_p or the Besov space $B^s_{p,q}$, $0 < s < r$, $1 < p, q < \infty$, and $\mathcal{F}(Q)$ is given by (6.3).*

Proof. $\mathcal{S}(Q)$ is an orthogonal sequence in $L^2(\mathbb{S}^d)$ as an immediate consequence of Lemma 6.1, Definition 6.6, and the fact that $\mathbf{T}_d$ is an isometric isomorphism. Since $\mathcal{S}(Q)$ is not a basis of $L^2(\mathbb{S}^d)$, one can not automatically deduce that $P_Q(\mathcal{S}(Q))$ is a Parseval frame for $P_Q(L^2(\mathbb{S}^d))$. Instead, we need to use the following argument.

First, suppose that $Q \in \mathcal{Q}^d$ is an interior patch as in Definition 6.6. Take any $f \in P_Q(L^2(\mathbb{S}^d))$. Then, by (5.4) we have

$$\operatorname{supp} \mathbf{T}_d^{-1} f \subset J_{\epsilon/2},$$

where $J_\epsilon/2$ is given by (6.7). By Lemma 6.1, (6.5), and (6.8) we have

$$||\mathbf{T}_d^{-1} f||^2 = \sum_{g \in S(J,\epsilon)} |\langle \mathbf{T}_d^{-1} f, g\rangle|^2 = \sum_{g \in S(J,\epsilon)} |\langle f, \mathbf{T}_d g\rangle|^2 = \sum_{g \in \mathcal{S}(Q)} |\langle P_Q f, g\rangle|^2 = \sum_{g \in \mathcal{S}(Q)} |\langle f, P_Q g\rangle|^2.$$

Likewise, suppose that Q is a boundary patch as in Definition 6.7. Then, for any $f \in P_Q(L^2(\mathbb{S}^d))$ we have

$$\operatorname{supp} \mathbf{T}_d^{-1} D_O^{-1} f \subset \tilde{J},$$

where $\tilde{J}$ is given by (6.9). Using a similar calculation as above and (6.10) yields

$$||\mathbf{T}_d^{-1} D_O^{-1} f||^2 = \sum_{g \in S(\tilde{J},\epsilon)} |\langle \mathbf{T}_d^{-1} D_O^{-1} f, g\rangle|^2 = \sum_{g \in \mathcal{S}(Q)} |\langle P_Q f, g\rangle|^2 = \sum_{g \in \mathcal{S}(Q)} |\langle f, P_Q g\rangle|^2.$$

This shows (6.11) and thus the Parseval frame property of $P_Q(\mathcal{S}(Q))$. Finally, the last conclusion follows along the same lines from the last part of Lemma 6.1. $\square$

Now that Lemma 6.2 is established, we define the wavelet system $\mathcal{S}(\mathcal{Q}^d)$ with respect to the partition $\mathcal{Q}^d$ of the sphere $\mathbb{S}^d$ as

$$\mathcal{S}(\mathcal{Q}^d) := \bigcup_{Q \in \mathcal{Q}^d} P_Q(\mathcal{S}(Q)).$$

Then we have the following result.

Theorem 6.2. *The collection $\mathcal{S}(\mathcal{Q}^d)$ is a Parseval frame in $L^2(\mathbb{S}^d)$. Moreover, magnitudes of coefficients $\{|\langle f, g\rangle|\}_{g \in \mathcal{S}(\mathcal{Q}^d)}$ characterize functions f in $\mathcal{F}(\mathbb{S}^d)$, where $\mathcal{F}$ is either the Sobolev space W_p^r or the Besov space $B_{p,q}^s$, $0 < s < r$, $1 < p, q < \infty$.*

Proof. The first part is an immediate consequence of Theorem 5.1 and Lemma 6.2. The moreover part follows from Theorem 6.1 and Lemma 6.2. $\square$

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