Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 32 (2013), 1–23
DOI: 10.4171/ZAA/2708

A Mean Characterization of Weighted Anisotropic Besov and Triebel-Lizorkin Spaces

Baode Li, Marcin Bownik, Dachun Yang and Wen Yuan

Abstract. In this article, the authors study weighted anisotropic Besov and Triebel-Lizorkin spaces associated with expansive dilations and A_∞ weights. The authors show that elements of these spaces are locally integrable when the smoothness parameter α is positive. The authors also characterize these spaces for small values of α in terms of a mean square function recently introduced in the context of Sobolev spaces in [Math. Ann. 354 (2012), 589-626] and isotropic Triebel-Lizorkin spaces in [Publ. Mat. 57 (2013), 57-82].

Keywords. anisotropic expansive dilation, Besov space, Triebel-Lizorkin space, Muckenhoupt weight, square function

Mathematics Subject Classification (2010). Primary 46A20, secondary 42B25, 42B35, 42C40, 46B10, 46E39

1. Introduction

Recently, Alabern, Mateu and Verdera [1] characterized the fractional Sobolev space $\dot{W}^{\alpha,p}(\mathbb{R}^n)$ for $\alpha \in (0,2)$ and $p \in (1,\infty)$ via a new square function

$$S_\alpha(f)(x) := \left\{ \int_0^\infty |f_{B(x,t)} - f(x)|^2 \frac{dt}{t^{1+2\alpha}} \right\}^{\frac{1}{2}}, \quad x \in \mathbb{R}^n,$$

B. Li: School of mathematics and statistics, Xinjiang University, Urumqi 830046, People's Republic of China; baodeli@xju.edu.cn
M. Bownik: Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA; mbownik@uoregon.edu
D. Yang (Corresponding author): School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China; dcyang@bun.edu.cn
W. Yuan: School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China; wenyuan@bun.edu.cn

where $f_{B(x,t)}$ denotes the *mean value* of f on the ball $B(x,t)$ with the center x and radius t. They showed that $f \in \dot{W}^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $\|S_\alpha(f)\|_{L^p(\mathbb{R}^n)} < \infty$. The key point for this result, as first observed by Wheeden [29] in the study of Lipschitz-type (Besov) spaces, and later independently by Alabern, Mateu and Verdera in [1], is that the square function S_α provides smoothness up to order 2 in the following sense: for all $f \in C^2(\mathbb{R}^n)$, $t \in (0,1)$ and $x \in \mathbb{R}^n$, $f_{B(x,t)} - f(x) = O(t^2)$, which follows from the second order Taylor expansion of f. Indeed, Wheeden in [29] obtained a general result which contains the above characterization for $\dot{W}^{\alpha,p}(\mathbb{R}^n)$ with $\alpha \in (0,2)$ and $p \in (1,\infty)$ as special cases. Via a similar observation, the corresponding characterization of higher order Sobolev spaces $\dot{W}^{\alpha,p}(\mathbb{R}^n)$ with $\alpha \in [2,\infty)$ and $p \in (1,\infty)$ was also obtained in [1]. Later, Yang, Yuan and Zhou [32] characterized the Triebel-Lizorkin space $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$, $p,\ q \in (1,\infty]$, via the square function

$$\mathcal{S}_{\alpha,q}(f)(x) := \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \left|f_{B(x,2^{-k})} - f(x)\right|^q\right\}^{\frac{1}{q}}, \quad x \in \mathbb{R}^n,$$

when the smoothness parameter $\alpha \in (0,2)$. Similar results for $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ with $\alpha \in (2,\infty) \setminus 2\mathbb{N}$ and $p,q \in (1,\infty]$, and for Besov spaces $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ with $\alpha \in (0,\infty) \setminus 2\mathbb{N}$, $p \in (1,\infty]$ and $q \in (0,\infty]$ were also obtained in [32] via some appropriately modified square function.

The main purpose of this paper is to extend the above characterizations of Besov and Triebel-Lizorkin spaces from the isotropic setting into the weighted anisotropic setting. These spaces are associated with general expansive dilations on $\mathbb{R}^n$ and Muckenhoupt A_∞ weights. The theory of function spaces in the weighted anisotropic setting, including Hardy spaces, Besov spaces and Triebel-Lizorkin spaces, has been proved to be a very general theory which includes the classical isotropic spaces, the parabolic spaces, and the weighted spaces as special cases. For more details about this theory, we refer to [2–14, 23–26] and their references. On the other hand, there has been a significant interest in providing alternative characterizations of function spaces (see, for example, [18, 20–22, 25, 27, 28, 30–33]). In particular, in this paper we extend results from [32] to the weighted anisotropic setting.

In order to formulate our results we begin with some necessary definitions.

Definition 1.1. A real $n \times n$ matrix A is called an *e*xpansive dilation, shortly a *d*ilation, if $\max_{\lambda\in\sigma(A)} |\lambda| > 1$, where $\sigma(A)$ is the set of all *e*igenvalues of A. A *quasi-norm* associated with expansive matrix A is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0,\infty)$, for simplicity, denoted as ρ, such that

(i) $\rho(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$;

(ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$, where $b := |\det A|$;

(iii) $\rho(x+y) \le H[\rho(x) + \rho(y)]$ for all $x,\ y \in \mathbb{R}^n$, where $H \in [1,\infty)$ is a constant.

It was pointed in [2, Lemma 2.2] that for any dilation A, there exist some constant $r \in (1,\infty)$ and a ellipsoid Δ, i. e. $\Delta := \{x \in \mathbb{R}^n : |Px| < 1\}$ for some nonnegative matrix P, such that

$$\Delta \subset r\Delta \subset A\Delta. \tag{1.1}$$

By a scaling we can additionally assume that the ellipsoid Δ in [2, Lemma 2.2] satisfies $|\Delta| = 1$. Let $B_k := A^k\Delta$ for all $k \in \mathbb{Z}$. By (1.1), we know that, for all $k \in \mathbb{Z}$, $B_k \subset B_{k+1}$ and $|B_k| = b^k$.

From [2, Lemma 2.4], we also deduce that any two homogeneous quasi-norms associated with a dilation A are equivalent. For our purposes it is enough to restrict to a quasi-norm $\rho := \rho_A$, as in [2, Definition 2.5], given by

$$\rho(x) := \begin{cases} b^k, & x \in B_{k+1} \setminus B_k; \\ 0, & x = 0. \end{cases}$$

For any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, we let $B_\rho(x, b^k) := \{y \in \mathbb{R}^n : \rho(y-x) < b^{k+1}\}$. These balls are convex sets and satisfy that $|B_\rho(x, b^k)| = b^k$ and $B_\rho(x, b^k) \subset B_\rho(x, b^j)$ with $j \geq k$.

We now recall the class of Muckenhoupt weights associated with A introduced in [6].

Definition 1.2. Let $p \in [1, \infty)$, A be a dilation and w a non-negative measurable function on $\mathbb{R}^n$. A function w is said to belong to the class $\mathcal{A}_p(A) := \mathcal{A}_p(\mathbb{R}^n; A)$ of Muckenhoupt weights, if there exists a positive constant C such that, when $p \in (1,\infty)$,

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}} \left\{ b^{-k} \int_{B_\rho(x,\, b^k)} w(y)\, dy \right\} \left\{ b^{-k} \int_{B_\rho(x,\, b^k)} [w(y)]^{\frac{-1}{p-1}}\, dy \right\}^{p-1} \leq C$$

and, when $p = 1$,

$$\sup_{x\in\mathbb{R}^n}\sup_{k\in\mathbb{Z}} \left\{ b^{-k} \int_{B_\rho(x,\, b^k)} w(y)\, dy \right\} \left\{ \operatorname*{esssup}_{y\in B_\rho(x,\, b^k)} [w(y)]^{-1} \right\} \leq C.$$

Define $\mathcal{A}_\infty(A) := \bigcup_{1\leq p<\infty} \mathcal{A}_p(A)$.

For any $w \in \mathcal{A}_\infty(A)$, define

$$q_w := \inf\{q \in [1, \infty) : w \in \mathcal{A}_q(A)\}. \tag{1.2}$$

Obviously, $q_w \in [1,\infty)$. If $q_w \in (1,\infty)$, by [7, p. 3072], it is easy to know that $w \notin \mathcal{A}_{q_w}(A)$. Moreover, even when $A = 2I_{n\times n}$, where $I_{n\times n}$ denotes the *unit matrix* of order $n \times n$, there exists a $w \in (\bigcap_{q>1} \mathcal{A}_q(A)) \setminus \mathcal{A}_1(A)$ such that $q_w = 1$; see Johnson and Neugebauer [19, p. 254, Remark].

Recall that $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type; see [6, Proposition 2.3]. For some basic properties of the above weights, we refer, for example, to [16, Chapter IV] and [6].

For all $p \in (0, \infty)$ and $w \in \mathcal{A}_\infty(A)$, the *w*eighted Lebesgue space $L^p_w(\mathbb{R}^n)$ is defined to be the space of all $w(x)\,dx$-measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \left\{\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right\}^{\frac{1}{p}} < \infty.$$

Denote by $\mathcal{S}(\mathbb{R}^n)$ the *s*et of all Schwartz functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ its *t*opological dual space. Let $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. Denote by $\mathcal{S}_\infty(\mathbb{R}^n)$ the *subspace of* $\mathcal{S}(\mathbb{R}^n)$ *given by*

$$\mathcal{S}_\infty(\mathbb{R}^n) = \left\{\phi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \phi(x) x^\gamma\,dx = 0 \textit{ for all } \gamma \in (\mathbb{Z}_+)^n\right\},$$

and $\mathcal{S}'_\infty(\mathbb{R}^n)$ its *t*opological dual space. It is not hard to show that any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\operatorname{supp}\widehat{\varphi}$ away from origin belongs to $\mathcal{S}_\infty(\mathbb{R}^n)$.

Denote by $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (or resp. $L^1_{\mathrm{loc}\,,w}(\mathbb{R}^n)$) the *space* of all locally integrable (or resp. $w(x)dx$-integrable) functions. In what follows, for any $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (or resp. $g \in L^1_{\mathrm{loc}\,,w}(\mathbb{R}^n)$), $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let

$$\fint_{B_\rho(x,\,b^k)} g(y)\,dy := \frac{1}{|B_\rho(x,\,b^k)|} \int_{B_\rho(x,\,b^k)} g(y)\,dy$$

(or resp.

$$\fint_{B_\rho(x,\,b^k)} g(y)w(y)\,dy := \frac{1}{w(B_\rho(x,\,b^k))} \int_{B_\rho(x,\,b^k)} g(y)\,w(y)dy,$$

where $w(B_\rho(x,\,b^k)) := \int_{B_\rho(x,\,b^k)} w(y)\,dy$). We say $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'_\infty(\mathbb{R}^n)$ means that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and the natural pairing $\langle f,\,\varphi\rangle$ given by the integral $\int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$ exists for all $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$ and induces an element of $\mathcal{S}'_\infty(\mathbb{R}^n)$.

We denote by $\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)$ the *w*eighted anisotropic Triebel-Lizorkin space; see Section 2 for its definition. Moreover, we introduce the following function spaces of Triebel-Lizorkin type via a variant of the square function $\mathcal{S}_{\alpha,q}$.

Definition 1.3. Let $\alpha \in \mathbb{R}$, $q \in (0, \infty]$ and $w \in \mathcal{A}_\infty(A)$.

(i) Let $p \in (0, \infty)$. The *space* $\mathbf{S}\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)$ is defined as the collection of all functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{\mathbf{S}\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)} := \|S_{\alpha,\,q}(f)\|_{L^p_w(\mathbb{R}^n)} < \infty$, where, for all $x \in \mathbb{R}^n$,

$$S_{\alpha,\,q}(f)(x) := \left\{\sum_{k\in\mathbb{Z}} b^{k\alpha q} \left|\fint_{B_\rho(x,\,b^{-k})} [f(y) - f(x)]\,dy\right|^q\right\}^{\frac{1}{q}}$$

with the usual modification made when $q = \infty$.

(ii) The space $\mathbf{S}\dot{\mathbf{F}}^{\alpha}_{\infty,\,q}(A;\,w)$ is defined as the collection of all functions $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathbf{S}\dot{\mathbf{F}}^{\alpha}_{\infty,\,q}(A;\,w)} := \sup_{z\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}}\left\{ \fint_{B_\rho(z,\,b^{-\ell})} \sum_{k\geq\ell} b^{k\alpha q} \times \left| \fint_{B_\rho(x,\,b^{-k})} [f(y)-f(x)]\,dy \right|^q w(x)\,dx \right\}^{\frac{1}{q}} < \infty$$

with the usual modification made when $q = \infty$.

Let λ_- and λ_+ be two positive numbers such that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+.$$

In the case when A is diagonalizable over $\mathbb{C}$, we can even take

$$\lambda_- := \min\{|\lambda| : \lambda \in \sigma(A)\} \ \text{ and } \ \lambda_+ := \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

Otherwise, we need to choose them sufficiently close to these equalities according to what we need in our arguments. Let $\zeta_\pm := \log_b \lambda_\pm$.

We first show that, in the sense of distributions, the elements in the weighted anisotropic Triebel-Lizorkin spaces $\dot{\mathbf{F}}^{\alpha}_{p,\,q}(A;\,w)$ are locally integrable.

Theorem 1.4. *Let $\alpha \in (0,\,\infty)$, $w \in \mathcal{A}_\infty(A)$, $p \in (q_w,\,\infty]$ with q_w as in* (1.2)*, and $q \in (0,\,\infty]$. Then $\dot{\mathbf{F}}^{\alpha}_{p,\,q}(A;\,w) \subset L^1_{\rm loc}(\mathbb{R}^n)$ in the sense of $\mathcal{S}'_\infty(\mathbb{R}^n)$.*

Using Theorem 1.4, we obtain the following characterization of $\dot{\mathbf{F}}^{\alpha}_{p,\,q}(A;\,w)$.

Theorem 1.5. *Let $\alpha \in (0,\,2\zeta_-)$, $w \in \mathcal{A}_\infty(\mathbb{R}^n)$, $p \in (q_w,\,\infty]$ with q_w as in* (1.2)*, and $q \in (1,\,\infty]$. Then $\dot{\mathbf{F}}^{\alpha}_{p,\,q}(A;\,w) = \mathbf{S}\dot{\mathbf{F}}^{\alpha}_{p,\,q}(A;\,w)$ with equivalent norms.*

The corresponding conclusions for Besov spaces are also true. Indeed, let $\alpha \in \mathbb{R}$ and $p,\,q \in (0,\,\infty]$. The space $\mathbf{S}\dot{\mathbf{B}}^{\alpha}_{p,\,q}(A;\,w)$ of Besov type is defined as the collection of all functions $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathbf{S}\dot{\mathbf{B}}^{\alpha}_{p,\,q}(A;\,w)} := \left\{ \sum_{k\in\mathbb{Z}} b^{k\alpha q} \left\| \fint_{B_\rho(\cdot,\,b^{-k})} [f(y)-f(\cdot)]\,dy \right\|^q_{L^p_w(\mathbb{R}^n)} \right\}^{\frac{1}{q}} < \infty.$$

Then we have analogous results to Theorems 1.4 and 1.5 for Besov spaces.

Theorem 1.6. *Let $\alpha \in (0,\,\infty)$, $w \in \mathcal{A}_\infty(A)$, $p \in (q_w,\,\infty]$ with q_w as in* (1.2)*, and $q \in (0,\,\infty]$. Then $\dot{\mathbf{B}}^{\alpha}_{p,\,q}(A;\,w) \subset L^1_{\rm loc}(\mathbb{R}^n)$ in the sense of $\mathcal{S}'_\infty(\mathbb{R}^n)$.*

Theorem 1.7. *Let $\alpha \in (0, 2\zeta_-)$, $p \in (q_w, \infty]$ with q_w as in (1.2), and $q \in (0, \infty]$. Then $\dot{\mathbf{B}}^\alpha_{p,q}(A; w) = \mathbf{S}\dot{\mathbf{B}}^\alpha_{p,q}(A; w)$ with equivalent norms.*

Remark 1.8. (i) Theorems 1.5 and 1.7 generalize the $S_{\alpha,q}$-function characterization of isotropic Triebel-Lizorkin and Besov spaces in [32, Theorems 1.2 and 4.1] and, in particular, the Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ in [1, Theorem 3], both in the case $\alpha \in (0, 2)$, to the anisotropic weighted cases. Indeed, as in [1, 32], let $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ and $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ denote, respectively, the classical Triebel-Lizorkin space and the classical Besov space (see [25]). Observe that the parameter α in $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ and $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ plays a different role from the parameter α used in $\dot{\mathbf{F}}^\alpha_{p,q}(A; w)$ and $\dot{\mathbf{B}}^\alpha_{p,q}(A; w)$ of the present paper, which is caused by the difference existing in the definitions of these function spaces in [1, 32] and the present article. To be precise, if $A = 2I_{n\times n}$, then $\zeta_- = 1/n$ and $b = 2^n$ and hence, when $A = 2I_{n\times n}$ and $w(x) := 1$ for all $x \in \mathbb{R}^n$, then $\dot{\mathbf{F}}^\alpha_{p,q}(A; w) = \dot{F}^{n\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{\mathbf{B}}^\alpha_{p,q}(A; w) = \dot{B}^{n\alpha}_{p,q}(\mathbb{R}^n)$. Thus, in this case, $\alpha \in (0, 2\zeta_-)$ if and only if $n\alpha \in (0, 2)$ and, therefore, Theorems 1.5 and 1.7 of the present paper coincide [32, Theorems 1.2 and 4.1] in the case $\alpha \in (0, 2)$.

(ii) In [32, Theorems 1.2 and 4.1], the corresponding characterizations of isotropic Triebel-Lizorkin and Besov spaces when $\alpha \in (2N, 2N + 2)$, $N \in \mathbb{N}$, were also obtained. However, it is still unknown whether the corresponding results of Theorems 1.5 and 1.7 are also true when $\alpha \geq 2\zeta_-$, due to the anisotropic structure of our spaces.

Remark 1.9. We point out that the inhomogeneous counterparts of Theorems 1.5 and 1.7 are also true. Indeed, by referring to the definitions of weighted inhomogeneous anisotropic Besov and Triebel-Lizorkin spaces (see [3, Definition 3.3] and [6, Definition 3.3]), we can also give the definitions of weighted inhomogeneous spaces $\mathbf{SB}^\alpha_{p,q}(A; w)$ and $\mathbf{SF}^\alpha_{p,q}(A; w)$. Then, using some arguments similar to those for the homogeneous case, we can obtain the desired inhomogeneous results. We omit the details.

In comparison with [32] this paper considers a very general setting of Muckenhoupt weights and anisotropic dilations. This necessitates a more complicated approach which uses some additional techniques adopted for the anisotropic setting in [2, 4–6]. A key role in our arguments is played by a special variant of the Calderón reproducing formula associated with anisotropic dilations; see Lemma 3.8 below.

The paper is organized as follows. In Section 2, we recall some basic notions and notation. The proofs of Theorems 1.4, 1.5 and 1.7 are given in Section 3.

2. Preliminaries

We begin with some basic notions and notation.

Denote by $\mathcal{S}(\mathbb{R}^n)$ the *s*pace of all Schwartz functions, whose topology is determined by a family of seminorms, $\{\|\cdot\|_{\mathcal{S}_{k,\,m}(\mathbb{R}^n)}\}_{k,\,m\in\mathbb{Z}_+}$, where, for all $k\in\mathbb{Z}_+$, $m\in(0,\,\infty)$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$,

$$\|\varphi\|_{\mathcal{S}_{k,\,m}(\mathbb{R}^n)} := \sup_{\alpha\in\mathbb{Z}_+^n,\ |\alpha|\le k}\ \sup_{x\in\mathbb{R}^n}(1+|x|)^m|\partial^\alpha\varphi(x)|.$$

The above norm can be also equivalently modified with $(1+|x|)^m$ replaced by $[1+\rho(x)]^m$. Here, for any $\alpha:=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}_+^n$, $|\alpha|:=\alpha_1+\cdots+\alpha_n$ and $\partial^\alpha:=(\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}$. It is known that $\mathcal{S}(\mathbb{R}^n)$ forms a locally convex topological vector space. Denote by $\mathcal{S}'(\mathbb{R}^n)$ the *t*opological dual space of $\mathcal{S}(\mathbb{R}^n)$ endowed with the weak $*$-topology. Denote by $\mathcal{P}(\mathbb{R}^n)$ the *c*ollection of all polynomials on $\mathbb{R}^n$. In what follows, for every $\varphi\in\mathcal{S}(\mathbb{R}^n)$, $k\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, let $\varphi_k(x):=b^k\varphi(A^kx)$.

Now we recall the notion of weighted anisotropic Triebel-Lizorkin and Besov spaces; see [3, 4, 6]. In what follows, for any $\varphi\in L^1(\mathbb{R}^n)$, $\widehat{\varphi}$ denotes the *Fourier transform* of φ, namely, for all $\xi\in\mathbb{R}^n$,

$$\widehat{\varphi}(\xi):=\int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\varphi(x)\,dx.$$

Definition 2.1. Let $w\in\mathcal{A}_\infty(A)$, $\alpha\in\mathbb{R}$, $p,\ q\in(0,\,\infty]$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}\widehat{\varphi}\subset[-1/2,\,1/2]^n\setminus\{0\}\quad\text{and}\quad\sup_{j\in\mathbb{Z}}|\widehat{\varphi}((A^*)^j\xi)|>0\quad\text{for all}\quad\xi\in\mathbb{R}^n\setminus\{0\},\tag{2.1}$$

where A^* denotes the *adjoint* (*transpose*) of A.

The *w*eighted anisotropic homogeneous Triebel-Lizorkin space $\dot{\mathbf{F}}^\alpha_{p,q}(A;\,w)$ is defined as the collection of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)} := \left\|\left(\sum_{k\in\mathbb{Z}} b^{k\alpha q}|\varphi_k*f|^q\right)^{\frac1q}\right\|_{L^p_w(\mathbb{R}^n)}<\infty,\qquad p\in(0,\infty),$$

$$\|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)} := \sup_{x\in\mathbb{R}^n}\sup_{\ell\in\mathbb{Z}}\left\{\fint_{B_\rho(x,\,b^{-\ell})}\sum_{k\ge\ell} b^{k\alpha q}|\varphi_k*f(y)|^q w(y)\,dy\right\}^{\frac1q}<\infty,$$

with the usual modification made when $q=\infty$.

The *w*eighted anisotropic homogeneous Besov space $\dot{\mathbf{B}}^\alpha_{p,q}(A;\,w)$ is defined as the collection of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{\dot{\mathbf{B}}^\alpha_{p,\,q}(A;\,w)}<\infty$, where

$$\|f\|_{\dot{\mathbf{B}}^\alpha_{p,\,q}(A;\,w)} := \left\{\sum_{k\in\mathbb{Z}} b^{k\alpha q}\|\varphi_k*f\|^q_{L^p_w(\mathbb{R}^n)}\right\}^{\frac1q}$$

with the usual modifications made when $p=\infty$ or $q=\infty$.

Remark 2.2. The space $\dot{\mathbf{F}}^{\alpha}_{p,q}(A;\, w)$ was originally defined via dilated cubes in [4, Definition 3.2]. However, the estimate (3.14) in [4] indicates that the space $\dot{\mathbf{F}}^{\alpha}_{p,q}(A;\, w)$ can also be equivalently defined via dilated balls, as in the above definition.

Remark 2.3. (i) Notice that, if $\|f\|_{\dot{\mathbf{F}}^{\alpha}_{p,q}(A;\, w)} = 0$, then f is a polynomial. So the quotient space $\dot{\mathbf{F}}^{\alpha}_{p,q}(A;\, w)/\mathcal{P}(\mathbb{R}^n)$ is a quasi-Banach space. By abuse of the notation, the *space* $\dot{\mathbf{F}}^{\alpha}_{p,q}(A;\, w)/\mathcal{P}(\mathbb{R}^n)$ is also denoted by $\dot{\mathbf{F}}^{\alpha}_{p,q}(A;\, w)$ and its element $[f] := f + \mathcal{P}(\mathbb{R}^n)$ for simplicity by f. Similar observation applies to homogeneous Besov spaces.

(ii) By referring to [25, Section 5.1] or [6, p. 1479], we know that $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ can be identified with the space of all continuous functionals on the closed subspace $\mathcal{S}_\infty(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$.

Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $A \lesssim B$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, we then write $A \sim B$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its *characteristic function*. For any $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the *largest integer* not more than a.

3. Proofs of Main Results

In this section, we give the proofs of Theorems 1.4 and 1.5. The proofs of Theorems 1.6 and 1.7 are also sketched. To prove Theorem 1.4, we need the following three lemmas. The first lemma comes from [6, Lemmas 2.6 and 2.8].

Lemma 3.1. *Let $\varphi,\, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy* (2.1) *and*

$$\sum_{j\in\mathbb{Z}} \widehat{\varphi}((A^*)^j\xi)\widehat{\psi}((A^*)^j\xi) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{3.1}$$

Then, for any $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$,

$$f(\cdot) = \sum_{j\in\mathbb{Z}} \varphi_j * \psi_j * f(\cdot) = \sum_{j\in\mathbb{Z}} b^{-j} \sum_{k\in\mathbb{Z}^n} \varphi_j * f(A^{-j}k)\psi_j(\cdot - A^{-j}k)$$

in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$.

The next result follows from [4, (3.23)] and the fact that $w(x)\, dx$ is a ρ_A-doubling measure; see [4, Definition 2.5].

Lemma 3.2. *Let $\alpha \in \mathbb{R}$, $w \in \mathcal{A}_\infty(A)$, $p,\, q \in (0,\, \infty]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (2.1). Then there exist some positive constants C and $N \in \mathbb{Z}_+$ such that, for all $f \in \dot{\mathbf{F}}^\alpha_{p,q}(A;\, w)$ and integers $j \le 0$,*

$$\sup_{x\in\mathbb{R}^n} \frac{|\varphi_j * f(x)|}{[1+\rho(x)]^N} \le Cb^{-j\alpha}\|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)}.$$

To emphasize the dependence on φ of the norm in $\dot{\mathbf{F}}^\alpha_{p,q}(A;\, w)$, we let $\|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w,\varphi)}$ denote $\|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)}$. The following lemma shows that the space $\dot{\mathbf{F}}^\alpha_{p,q}(A;\, w)$ is independent of the choice of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (2.1); see [4, Corollary 3.6].

Lemma 3.3. *Let $\alpha \in \mathbb{R}$, $w \in \mathcal{A}_\infty(A)$, $p,\, q \in (0,\, \infty]$ and $\varphi^{(1)},\, \varphi^{(2)} \in \mathcal{S}(\mathbb{R}^n)$ satisfy (2.1). Then, for any $f \in \dot{\mathbf{F}}^\alpha_{p,q}(A;\, w)$, $\|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w,\varphi^{(1)})} \sim \|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w,\varphi^{(2)})}$ with the implicit positive constants independent of f.*

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $f \in \dot{\mathbf{F}}^\alpha_{p,q}(A;\, w)$ with $\alpha \in (0,\infty)$, $w \in \mathcal{A}_\infty(A)$, $p \in (q_w,\, \infty]$ with q_w as in (1.2) and $q \in (0,\, \infty]$. We only need to prove that there exists a function g such that $f = g$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ and $\int_{B_\rho(0,\,b^\ell)} |g(x)|\,dx < \infty$ for any $\ell \in \mathbb{Z}_+$.

Let $x \in B_\rho(0,\, b^\ell)$, $\varphi,\, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (2.1) and (3.1), and $L \in \mathbb{Z}_+$ be sufficiently large. Let

$$\mathrm{I}(x) := \sum_{j=-\infty}^{-1} b^{-j} \sum_{k\in\mathbb{Z}^n} \varphi_j * f(A^{-j}k)b^j \left[\psi(A^jx-k) - \sum_{|\gamma|\le L-1} \frac{\partial^\gamma\psi(-k)(A^jx)^\gamma}{\gamma!}\right]$$

and

$$\mathrm{II}(x) := \sum_{j=0}^{\infty} \varphi_j * \psi_j * f(x).$$

Then, by Lemma 3.1 and Remark 2.3(ii), we know that, for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$,

$$\langle f,\, \phi\rangle = \langle \mathrm{I},\, \phi\rangle + \langle \mathrm{II},\, \phi\rangle.$$

We claim that g, which is defined pointwise by $g(x) := \mathrm{I}(x) + \mathrm{II}(x)$, is the desired function.

By the Taylor remainder theorem, Lemma 3.2 and the fact that $|A^jx| \lesssim b^{j\zeta_-}|x|$ for any $j \le 0$ and $x \in B_\rho(0,\, b^\ell)$ (see [6, (2.8)]), we conclude that there exists some positive integer N such that

$$\mathrm{I}(x) \lesssim \|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)} \sum_{j=-\infty}^{-1} b^{j(L\zeta_- - \alpha)} \sum_{k\in\mathbb{Z}^n} \sup_{|\gamma|=L,\,\theta\in(0,\,1)} \frac{|x|^L[1+\rho(A^{-j}k)]^N}{[1+\rho(\theta A^jx-k)]^{N+2}}$$

$$\begin{aligned}&\lesssim C_{\ell,L}\|f\|_{\dot{\mathbf{F}}^{\alpha}_{p,q}(A;w)} b^{\ell L\zeta_-}\sum_{j=-\infty}^{-1} b^{j(L\zeta_--\alpha-N)}\\&\qquad\times\left\{\sum_{\rho(k)\le 2Hb^\ell}(2Hb^\ell)^N+\sum_{\rho(k)>2Hb^\ell}\frac{1}{[\rho(k)]^2}\right\},\end{aligned}$$

where $H\in[1,\infty)$ is as in Definition 1.1(iii) and L is chosen such that $L\zeta_->N+\alpha$. Let $Q_k:=k+[0,1)^n$. Notice that, for any $k\in\mathbb{Z}^n$ with $\rho(k)>2Hb^\ell$, from the estimate $1/\rho(k)\lesssim|Q_k|\inf_{y\in Q_k}1/\rho(y)$ and $\mathbb{R}^n=\cup_{k\in\mathbb{Z}^n}Q_k$, it follows that

$$\sum_{\rho(k)>2Hb^\ell}[\rho(k)]^{-2}\lesssim\sum_{\rho(k)>2Hb^\ell}\int_{Q_k}[\rho(y)]^{-2}\,dy\lesssim\int_{\rho(y)\gtrsim b^\ell}[\rho(y)]^{-2}\,dy\lesssim 1.$$

This, together with the previous arguments, implies that $\int_{B_\rho(0,b^\ell)}|\mathrm{I}(x)|\,dx<\infty$.

On the other hand, notice that, for any $w\in\mathcal{A}_\infty(A)$ and $p\in(q_w,\infty]$, we have $w\in\mathcal{A}_p(A)$ and $w^{-p'/p}\in\mathcal{A}_{p'}(A)$ with p' satisfying $1/p+1/p'=1$. By Hölder's inequality, $\alpha\in(0,\infty)$, $q\in(0,\infty]$ and Lemma 3.3 with $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$ satisfying (2.1) and (3.1), we conclude that

$$\begin{aligned}\int_{B_\rho(0,b^\ell)}|\mathrm{II}(x)|\,dx&\le\sum_{j=0}^\infty\int_{B_\rho(0,b^\ell)}|\varphi_j*\psi_j*f(x)|\,dx\\&\lesssim\|f\|_{\dot{\mathbf{F}}^{\alpha}_{p,q}(A;w,\varphi*\psi)}\sum_{j=0}b^{-j\alpha}\int_{B_\rho(0,b^\ell)}[w(x)]^{-\frac{p'}{p}}\,dx\\&\lesssim C_{w,\ell}\|f\|_{\dot{\mathbf{F}}^{\alpha}_{p,q}(A;w)}.\end{aligned}$$

Combining the above estimates on $\mathrm{I}(x)$ and $\mathrm{II}(x)$, we see that g is locally integrable, which completes the proof of Theorem 1.4. $\square$

Theorem 1.5 follows immediately as a consequence of the following Theorems 3.4 and 3.7. Thus, to finish the proof of Theorem 1.5, it suffices to prove the following Theorems 3.4 and 3.7.

Theorem 3.4. *Let $\alpha\in(0,2\zeta_-)$, $w\in\mathcal{A}_\infty(A)$, $p\in(q_w,\infty]$ and $q\in(1,\infty]$. If $f\in\dot{\mathbf{F}}^{\alpha}_{p,q}(A;w)$, then there exists a polynomial P_f such that $f+P_f\in\mathbf{S}\dot{\mathbf{F}}^{\alpha}_{p,q}(A;w)$. Moreover, $\|f+P_f\|_{\mathbf{S}\dot{\mathbf{F}}^{\alpha}_{p,q}(A;w)}\le C\|f\|_{\dot{\mathbf{F}}^{\alpha}_{p,q}(A;w)}$, where C is a positive constant independent of f.*

To prove Theorem 3.4, we need the following lemma; see [6, Lemma 3.6] and [4, Proposition 3.15].

Lemma 3.5. *For any $\varphi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$ satisfying* (2.1)*, there exists $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^n)$ satisfying* (2.1) *such that $\varphi^{(1)}$ and $\psi^{(1)}$ satisfy* (3.1)*. Moreover, for any $\alpha \in \mathbb{R}$, $w \in \mathcal{A}_\infty(A)$, $p,\, q \in (0,\, \infty]$ and $f \in \dot{\mathbf{F}}^\alpha_{p,\,q}(A;\, w)$, there exist polynomials $\{P_j^{(1)}\}_{j\in\mathbb{Z}}$ and $P_f^{(1)}$ with degrees not more than $\lfloor \alpha/\zeta_- \rfloor$ such that*

$$f + P_f^{(1)} = \lim_{i\to-\infty} \left\{ \sum_{j=i}^{\infty} \varphi_j^{(1)} * \psi_j^{(1)} * f + P_i^{(1)} \right\} \tag{3.2}$$

in $\mathcal{S}'(\mathbb{R}^n)$. Let $(\varphi^{(2)},\, \psi^{(2)})$ be another pair of Schwartz functions also satisfying (2.1) *and* (3.1)*. Then, the corresponding polynomial $P_f^{(2)}$ satisfies*

$$\deg(P_f^{(1)} - P_f^{(2)}) \le \lfloor \alpha/\zeta_- \rfloor.$$

By [5, Corollary 3.7], we have the following embedding result.

Lemma 3.6. *Let $w \in \mathcal{A}_\infty(A)$, $\alpha \in \mathbb{R}$, $p \in (0,\, \infty]$ and $q_1,\, q_2 \in (0,\, \infty]$ with $q_1 \le q_2$. Then $\dot{\mathbf{F}}^\alpha_{p,\,q_1}(A;\, w) \hookrightarrow \dot{\mathbf{F}}^\alpha_{p,\,q_2}(A;\, w)$.*

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. By Theorem 1.4, we know that $\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\, w) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ in the sense of distributions. So, for any $f \in \dot{\mathbf{F}}^\alpha_{p,\,q}(A;\, w)$, we only need to prove that there exists a polynomial P_f such that $\|f + P_f\|_{\mathbf{S}\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)} \le C\|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)}$.

Let φ and ψ be, respectively, as in $\varphi^{(1)}$ and $\psi^{(1)}$ of Lemma 3.5. In this case, we denote the corresponding $P_f^{(1)}$ and $\{P_i^{(1)}\}_{i\in\mathbb{Z}}$ in (3.2), respectively, by P_f and $\{P_i\}_{i\in\mathbb{Z}}$. Then (3.2) holds for f and the degrees of the polynomials $\{P_i\}_{i\in\mathbb{Z}}$ in (3.2) are not more than $\lfloor \alpha/\zeta_- \rfloor$. Since $\alpha \in (0,\, 2\zeta_-)$, each P_i has degree at most 1, and thus $P_i(x) := \sum_{j=1}^n a_{i,\,j} x_j + b_i$, $x := (x_1,\, \dots,\, x_n)$, for some constants $a_{i,\,j}$ and b_i. Furthermore, notice that, for any $k \in \mathbb{Z}$, $B_\rho(0,\, b^{-k}) = A^{-k}\Delta$ is symmetric at origin due to (1.1) and $\int_{B_\rho(0,\,b^{-k})} z_i\, dz = 0$ for all $i \in \{1, \dots, n\}$. Then, for any $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, we have

$$P_i(x) - \fint_{B_\rho(x,\,b^{-k})} P_i(z)\, dz = P_i(x) - \fint_{B_\rho(0,\,b^{-k})} P_i(x+z)\, dz = 0. \tag{3.3}$$

Moreover, if $\varphi^{(i)}$ and $\psi^{(i)}$, $i \in \{1, 2\}$, satisfy (2.1) and

$$\sum_{k\in\mathbb{Z}} \widehat{\varphi^{(i)}}((A^*)^{-k}\xi)\widehat{\psi^{(i)}}((A^*)^{-k}\xi) = 1 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\},$$

then, by Lemma 3.5, $P_f^{(1)} - P_f^{(2)}$ is a polynomial of degree not more than $\lfloor \alpha/\zeta_- \rfloor \le 1$, where $P_f^{(i)}$ is as in (3.2) corresponding to $\varphi^{(i)}$ and $\psi^{(i)}$ for $i \in \{1, 2\}$. Then, by (3.3), for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, we see that

$$P_f^{(1)}(x) - P_f^{(2)}(x) - \fint_{B_\rho(x,\,b^{-k})} [P_f^{(1)}(z) - P_f^{(2)}(z)]\,dz = 0. \tag{3.4}$$

Let $\widetilde{f} := f + P_f$. By (3.2), we have

$$\widetilde{f} - \widetilde{f}_{B(\cdot,b^{-k})} = \sum_{j\in\mathbb{Z}} (\varphi_j - \chi_k * \varphi_j) * \psi_j * f \tag{3.5}$$

in $\mathcal{S}'(\mathbb{R}^n)$, where $\widetilde{f}_{B(\cdot,b^{-k})} := \fint_{B(\cdot,\,b^{-k})} \widetilde{f}(y)\,dy$, $\chi := \frac{\chi_{B_\rho(0,1)}}{|B_\rho(0,1)|}$ and $\chi_k := b^k \chi(A^k\cdot)$. From (3.4), it follows that $\widetilde{f} - \widetilde{f}_{B(\cdot,b^{-k})}$ is independent of the choices of φ and ψ satisfying (2.1) and (3.1). Then, to prove Theorem 3.4, it suffices to show that, when $p \in (q_w, \infty)$ and $q \in (1, \infty]$,

$$\begin{aligned}&\left\{\int_{\mathbb{R}^n} \left[\sum_{k\in\mathbb{Z}} b^{k\alpha q} \left(\sum_{j\in\mathbb{Z}} |[\varphi_j - \chi_k * \varphi_j] * \psi_j * f(x)|\right)^q\right]^{\frac{p}{q}} w(x)dx\right\}^{\frac{1}{p}}\\ &\quad\lesssim \|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)}.\end{aligned} \tag{3.6}$$

When $p = \infty$ and $q \in (1, \infty]$ we need to show that, for all $x \in \mathbb{R}^n$ and $\ell \in \mathbb{Z}$,

$$\begin{aligned}&\left\{\fint_{B_\rho(x,\,b^{-\ell})} \sum_{k\geq\ell} b^{k\alpha q} \left(\sum_{j\in\mathbb{Z}} |(\varphi_j - \chi_k * \varphi_j) * \psi_j * f(y)|\right)^q w(y)\,dy\right\}^{\frac{1}{q}}\\ &\quad\lesssim \|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)}.\end{aligned} \tag{3.7}$$

Indeed, if (3.6) holds, then, for each $k \in \mathbb{Z}$, we have

$$\int_{\mathbb{R}^n} \left[\sum_{j\in\mathbb{Z}} |(\varphi_j - \chi_k * \varphi_j) * \psi_j * f(x)|\right]^p w(x)\,dx < \infty,$$

which implies that (3.5) holds in $L^p_w(\mathbb{R}^n)$ and hence almost everywhere. Therefore, for every $k \in \mathbb{Z}$,

$$|\widetilde{f} - \widetilde{f}_{B(\cdot,b^{-k})}| \leq \sum_{j\in\mathbb{Z}} |(\varphi_j - \chi_k * \varphi_j) * \psi_j * f|$$

almost everywhere, and hence $\|\widetilde{f}\|_{\mathbf{S}\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)}$ is dominated by the left hand side of (3.6), which implies that $\|\widetilde{f}\|_{\mathbf{S}\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)} \lesssim \|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)}$. Similarly, if (3.7) holds, then (3.5) holds in $L^1_{\mathrm{loc}\,,\,w}(\mathbb{R}^n)$ and hence almost everywhere and, therefore, an argument similar to the above leads to $\|\widetilde{f}\|_{\mathbf{S}\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)} \lesssim \|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)}$.

To prove (3.6), we consider $\sum_{j\le k}$ and $\sum_{j>k}$ separately. Notice that, for any smooth function Φ on $\mathbb{R}$,

$$\Phi(1)=\Phi(0)+\int_0^1\Phi'(s)\,ds=\Phi(0)+\Phi'(0)+\int_0^1(1-s)\Phi''(s)\,ds. \tag{3.8}$$

Let $\Phi(s):=\varphi(A^jx+sz)$ for $s\in[0,1]$ and $x,\,z\in\mathbb{R}^n$. Then (3.8) can be written as

$$\varphi(A^jx+z)=\varphi(A^jx)+(\nabla\varphi)(A^jx)z^t+\int_0^1(1-s)z(\nabla^2\varphi)(A^jx+sz)z^t\,ds,$$

where z^t denotes the *transpose* of z.

Notice that, for any $x\in\mathbb{R}^n$, $z\in B_\rho(0,\,b^{j-k})$ with $j\le k$ and $s\in[0,\,1]$,

$$1+\rho(A^jx+sz)\ge 1+\rho(A^jx)/H-\rho(sz)\ge\rho(A^jx),$$

and hence

$$1+\rho(A^jx-sz)\gtrsim 1+\rho(A^jx).$$

Therefore, when $j\le k$, by the fact that $|x|\le C[\rho(x)]^{\zeta_-}$ for all $\rho(x)\le 1$ (see [2, Section 2]), we see that, for all $x\in\mathbb{R}^n$,

$$\begin{aligned}|\chi_k*\varphi_j(x)-\varphi_j(x)|&=\left|\fint_{B_\rho(0,1)}b^j\left[\varphi(A^jx+A^{j-k}z)-\varphi(A^jx)\right]\,dz\right|\\&=\left|\fint_{B_\rho(0,b^{j-k})}b^j\left[\varphi(A^jx+z)-\varphi(A^jx)\right]\,dz\right|\\&=\left|\fint_{B_\rho(0,b^{j-k})}b^j\int_0^1(1-s)z(\nabla^2\varphi)(A^jx+sz)\,z^t\,ds\,dz\right|\\&\lesssim b^{2(j-k)\zeta_-}\frac{b^j}{[1+\rho(A^jx)]^L},\end{aligned}\tag{3.9}$$

where $L\in(1,\,\infty)$. Hence,

$$\begin{aligned}|(\chi_k*\varphi_j-\varphi_j)*\psi_j*f(x)|&\lesssim b^{2(j-k)\zeta_-}\int_{\mathbb{R}^n}\frac{b^j}{[1+\rho(A^jy)]^L}|\psi_j*f(x-y)|\,dy\\&\lesssim b^{2(j-k)\zeta_-}M_\rho(\psi_j*f)(x).\end{aligned}\tag{3.10}$$

Here M_ρ is the *Hardy-Littlewood maximal operator* defined, for all locally integrable functions f on $\mathbb{R}^n$, by

$$M_\rho f(x):=\sup_{\substack{r>0\\ y\in B_\rho(x,\,r)}}\fint_{B_\rho(y,\,r)}|f(z)|\,dz,\quad x\in\mathbb{R}^n.$$

Then, by choosing any $\delta \in (0, 2\zeta_- - \alpha)$, together with Hölder's inequality, we conclude that

$$\begin{aligned}
\mathrm{I}_1 &:= \left\{ \int_{\mathbb{R}^n} \left[\sum_{k\in\mathbb{Z}} b^{k\alpha q} \left(\sum_{j\le k} |(\varphi_j - \chi_k * \varphi_j) * \psi_j * f(x)| \right)^q \right]^{p/q} w(x)\,dx \right\}^{1/p} \\
&\lesssim \left\{ \int_{\mathbb{R}^n} \left\{ \sum_{k\in\mathbb{Z}} b^{k\alpha q} \sum_{j\le k} b^{(j-k)(2\zeta_- - \delta)q} [M_\rho(\psi_j * f)(x)]^q \right\}^{p/q} w(x)\,dx \right\}^{1/p} \\
&\lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{j\in\mathbb{Z}} b^{j\alpha q} \, [M_\rho(\psi_j * f)(x)]^q \right]^{p/q} w(x)\,dx \right\}^{1/p}. \qquad (3.11)
\end{aligned}$$

This, together with the Fefferman-Stein vector-valued maximal inequality (see [6, Theorem 2.5]), implies that

$$\mathrm{I}_1 \lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{j\in\mathbb{Z}} b^{j\alpha q} |\psi_j * f(x)|^q \right]^{p/q} w(x)\,dx \right\}^{1/p} \lesssim \|f\|_{\dot{\mathbf{F}}^{\alpha}_{p,\,q}(A;\,w)}.$$

On the other hand, notice that, when $j > k$, for all $x \in \mathbb{R}^n$, it holds that

$$\begin{aligned}
|(\chi_k * \varphi_j - \varphi_j) * \psi_j * f(x)| &\le |\chi_k * \varphi_j * \psi_j * f(x)| + |\varphi_j * \psi_j * f(x)| \\
&\le \chi_k * [M_\rho(\psi_j * f)](x) + M_\rho(\psi_j * f)(x) \\
&\lesssim M_\rho \circ M_\rho(\psi_j * f)(x),
\end{aligned}$$

where $M_\rho \circ M_\rho$ denotes the *composition* of M_ρ and M_ρ. Then, by taking $\delta \in (0, \alpha)$ and Hölder's inequality, we see that

$$\begin{aligned}
\mathrm{I}_2 &:= \left\{ \int_{\mathbb{R}^n} \left[\sum_{k\in\mathbb{Z}} b^{k\alpha q} \left(\sum_{j> k} |(\varphi_j - \chi_k * \varphi_j) * \psi_j * f(x)| \right)^q \right]^{p/q} w(x)\,dx \right\}^{1/p} \\
&\lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{k\in\mathbb{Z}} b^{(\alpha-\delta)qk} \sum_{j> k} b^{j\delta q} [M_\rho \circ M_\rho(\psi_j * f)(x)]^q \right]^{p/q} w(x)\,dx \right\}^{1/p} \\
&\lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{j\in\mathbb{Z}} b^{j\alpha q} \, [M_\rho \circ M_\rho(|\psi_j * f|)(x)]^q \right]^{p/q} w(x)\,dx \right\}^{1/p}. \qquad (3.12)
\end{aligned}$$

This, together with the Fefferman-Stein vector-valued maximal inequality, implies that $\mathrm{I}_2 \lesssim \|f\|_{\dot{\mathbf{F}}^{\alpha}_{p,\,q}(A;\,w)}$. This proves (3.6).

To prove (3.7), we split the sum over $j \in \mathbb{Z}$ into three parts: $j < \ell \le k$, $\ell \le j \le k$, and $j > k \ge \ell$. If $j \le \ell \le k$, then, from (3.9) and Hölder's inequality, we deduce that, for all $y \in \mathbb{R}^n$,

$$|(\chi_k * \varphi_j - \varphi_j) * \psi_j * f(y)| \lesssim b^{2(j-k)\zeta_-} \int_{\mathbb{R}^n} \frac{b^j}{[1+\rho(A^j z)]^L} |\psi_j * f(y-z)|\,dz$$
$$\lesssim b^{2(j-k)\zeta_-} \sum_{i=0}^{\infty} b^{i(1-L)} ⨍_{B_\rho(y,\, b^{i-j})} |\psi_j * f(z)|\,dz.$$

Moreover, for any $y \in \mathbb{R}^n$, $i \ge 0$ and $r_0 > \max\{q_w,\, q\}$ with $q \in (1,\, \infty]$ and q_w as in (1.2), by Hölder's inequality with r_0, the definition of $\mathcal{A}_{r_0}(A)$ and Lemma 3.6 with $q_1 = q$ and $q_2 = r_0$, we have

$$⨍_{B_\rho(y,\, b^{i-j})} |\psi_j * f(z)|\,dz \lesssim \left\{ ⨍_{B_\rho(y,\, b^{i-j})} |\psi_j * f(z)|^{r_0} w(z)\,dz \right\}^{\frac{1}{r_0}}$$
$$\lesssim b^{-j\alpha} \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\, r_0}(A;\, w)} \lesssim b^{-j\alpha} \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\, q}(A;\, w)}.$$

From this and $L \in (1,\, \infty)$, it follows that

$$|(\chi_k * \varphi_j - \varphi_j) * \psi_j * f(y)| \lesssim b^{2(j-k)\zeta_-} b^{-j\alpha} \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\, q}(A;\, w)}.$$

This, together with $\alpha \in (0,\, 2\zeta_-)$, implies that

$$\left\{ ⨍_{B_\rho(x,\, b^{-\ell})} \sum_{k \ge \ell} b^{k\alpha q} \left[\sum_{j \le \ell} |(\varphi_j - \chi_k * \varphi_j) * \psi_j * f(y)| \right]^q w(y)\,dy \right\}^{1/q}$$
$$\lesssim \left\{ ⨍_{B_\rho(x,\, b^{-\ell})} \sum_{k \ge \ell} b^{k\alpha q} \left[\sum_{j \le \ell} b^{2(j-k)\zeta_-} b^{-j\alpha} \right]^q w(y)\,dy \right\}^{1/q} \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\, q}(A;\, w)}$$
$$\lesssim \|f\|_{\dot{\mathbf{F}}^{\alpha}_{\infty,\, q}(A;\, w)}.$$

If $\ell < j \le k$, then, for any $x \in \mathbb{R}^n$ and $y \in B_\rho(x,\, b^{-\ell})$, using (3.10), we know that

$$|(\chi_k * \varphi_j - \varphi_j) * \psi_j * f(y)| \lesssim b^{2(j-k)\zeta_-} \left[\int_{\mathbb{R}^n} \frac{b^j |\psi_j * f(z)| \chi_{B_\rho(x, 2Hb^{-\ell})}(z)}{[1+b^j \rho(y-z)]^L}\,dz \right.$$
$$\left. + \int_{\mathbb{R}^n} \frac{b^j |\psi_j * f(z)| \chi_{\mathbb{R}^n \setminus B_\rho(x, 2Hb^{-\ell})}(z)}{[1+b^j \rho(y-z)]^L}\,dz \right].$$

Since $L \in (1,\, \infty)$, we have

$$\int_{\mathbb{R}^n} \frac{b^j |\psi_j * f(z)| \chi_{B_\rho(x, 2Hb^{-\ell})}(z)}{[1+b^j \rho(y-z)]^L}\,dz \lesssim M_\rho \left(|\psi_j * f| \chi_{B_\rho(x, 2Hb^{-\ell})} \right)(y). \tag{3.13}$$

On the other hand, notice that, if $y \in B_\rho(x, b^{-\ell})$ and $z \in \mathbb{R}^n \setminus B_\rho(x, 2Hb^{-\ell})$, then $z \in \mathbb{R}^n \setminus B_\rho(y, b^{-\ell})$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{b^j|\psi_j * f(z)|\chi_{\mathbb{R}^n\setminus B_\rho(x,2Hb^{-\ell})}(z)}{[1+b^j\rho(y-z)]^L}\,dz &\lesssim \int_{\mathbb{R}^n\setminus B_\rho(y,b^{-\ell})} \frac{b^j|\psi_j * f(z)|}{[b^j\rho(y-z)]^L}\,dz\\ &\lesssim \sum_{i\ge j-\ell} b^{i(1-L)} \fint_{B_\rho(y,\, b^{i-j+1})} |\psi_j * f(z)|\,dz\\ &\lesssim b^{(j-\ell)(1-L)} b^{-j\alpha} \|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)}.\end{aligned}$$

Thus, for all $y \in B_\rho(x, b^{-\ell})$, we see that

$$\begin{aligned}|(\chi_k * \varphi_j - \varphi_j) * \psi_j * f(y)| &\lesssim b^{2(j-k)\zeta_-} M_\rho(\psi_j * f\chi_{B_\rho(x,\,2Hb^{-\ell})})(y)\\ &\quad + b^{2(j-k)\zeta_-} b^{(j-\ell)(1-L)} b^{-j\alpha} \|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)}.\end{aligned}\tag{3.14}$$

Using (3.13), (3.14), Hölder's inequality, and the Fefferman-Stein vector-valued maximal inequality, by an estimate similar to (3.11), we conclude that

$$\begin{aligned}&\left\{\fint_{B_\rho(x,\,b^{-\ell})} \sum_{k\ge\ell} b^{k\alpha q}\left[\sum_{\ell<j\le k} |(\varphi_j - \chi_k*\varphi_j)*\psi_j*f(y)|\right]^q w(y)\,dy\right\}^{\frac1q}\\ &\quad\lesssim \left\{\fint_{B_\rho(x,\,b^{-\ell})} \sum_{k\ge\ell}\left[\sum_{\ell<j\le k} b^{(j-k)(2\zeta_- -\alpha)} b^{j\alpha} M_\rho(\psi_j * f\chi_{B_\rho(x,\,2Hb^{-\ell})})(y)\right]^q w(y)\,dy\right\}^{\frac1q}\\ &\qquad + \left\{\fint_{B_\rho(x,\,b^{-\ell})} \sum_{k\ge\ell} b^{k\alpha q}\left[\sum_{\ell<j\le k} b^{2(j-k)\zeta_-} b^{(j-\ell)(1-L)} b^{-j\alpha}\right]^q w(y)\,dy\right\}^{\frac1q} \|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)}\\ &\quad\lesssim \left\{\fint_{B_\rho(x,\,b^{-\ell})} \sum_{\ell<j} \left[b^{j\alpha} M_\rho(\psi_j * f\chi_{B_\rho(x,2Hb^{-\ell})})(y)\right]^q w(y)\,dy\right\}^{\frac1q} + \|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)}\\ &\quad\lesssim \|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)}.\end{aligned}$$

Similarly, if $j > k \ge \ell$, then we see that, for all $y \in B_\rho(x, b^{-\ell})$,

$$\begin{aligned}&|(\chi_k * \varphi_j - \varphi_j) * \psi_j * f(y)|\\ &\quad\lesssim M_\rho\left(\chi_{B_\rho(x,2Hb^{-\ell})} M_\rho\left(\psi_j * f\chi_{B_\rho(x,4H^2b^{-\ell})}\right)\right)(y) + b^{-j\alpha} b^{(j-\ell)(1-L)} \|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)}.\end{aligned}$$

This, together with an estimate similar to (3.12), implies that

$$\left\{\fint_{B_\rho(x,\,b^{-\ell})} \sum_{k\ge\ell} b^{k\alpha q}\left(\sum_{j>k} |(\varphi_j - \chi_k*\varphi_j)*\psi_j*f(y)|\right)^q dy\right\}^{1/q} \lesssim \|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)}.$$

Combining the estimates in the above three cases, we then obtain (3.7) and hence complete the proof of Theorem 3.4. $\square$

The following theorem shows another part of Theorem 1.5.

Theorem 3.7. *Let* $\alpha \in (0, 2\zeta_-)$, $w \in \mathcal{A}_\infty(A)$, $p \in (q_w, \infty]$ *and* $q \in (1, \infty]$. *If* $f \in \mathbf{S}\dot{\mathbf{F}}^\alpha_{p,q}(A;\, w)$, *then* $f \in \dot{\mathbf{F}}^\alpha_{p,q}(A;\, w)$ *and there exists a positive constant* C, *independent of* f, *such that* $\|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)} \le C\|f\|_{\mathbf{S}\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)}$.

The proof of Theorem 3.7 relies on the following variant of the Calderón reproducing formula.

Lemma 3.8. *Let* $\chi := \frac{\chi_{B_\rho(0,1)}}{|B_\rho(0,1)|}$, $L \in \mathbb{Z}_+ \cup \{-1\}$ *and* $N \in \mathbb{N}$. *Then there exist* $j_0 := j_0(A;\, n) \in \mathbb{Z}_+$ *and* $\phi,\, \psi \in \mathcal{S}(\mathbb{R}^n)$ *satisfying that* $\operatorname{supp}\phi \subset B_\rho(0,1)$,

$$\int_{\mathbb{R}^n} \phi(x) x^\gamma \, dx = 0$$

for all $|\gamma| \le L$, *and* $\operatorname{supp}\widehat{\psi} \subset [-1/2,\, 1/2]^n \setminus \{0\}$ *such that, for all* $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}_j(\xi)\widehat{\phi}_j(\xi)[\widehat{\chi}_j(\xi) - \widehat{\chi}_{j-j_0}(\xi)] = 1. \tag{3.15}$$

Moreover, for every $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$, *there exist polynomials* $\{P_j\}_{j\in\mathbb{Z}}$ *and* P_f *such that*

$$f + P_f = \lim_{i\to-\infty} \left\{ \sum_{j=i}^{\infty} \phi_j * \psi_j * (f_{B_\rho(\cdot,\, b^j)} - f_{B_\rho(\cdot,\, b^{j-j_0})}) + P_i \right\} \tag{3.16}$$

in $\mathcal{S}'(\mathbb{R}^n)$.

Proof. It suffices to show (3.15). The proof of (3.16) follows from (3.15) and an argument similar to that used in the proof of [6, Lemma 2.6].

We shall construct $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\psi} \subset \{\xi \in \mathbb{R}^n :\ C_1 \le |\xi| \le C_2\}$ for two positive constants $C_1 < C_2$ to be chosen later. First we will show that there exist $j_0 \in \mathbb{Z}_+$ and a positive constant C_0 such that

$$|\widehat{\chi}(\xi) - \widehat{\chi_{-j_0}}(\xi)| \ge C_0 > 0 \qquad \text{for all } C_1 \le |\xi| \le C_2. \tag{3.17}$$

Notice that $B_\rho(0,\, 1) = \Delta = \{x \in \mathbb{R}^n : |Px| < 1\}$ for some nonnegative matrix P. Moreover, by [17, p. 429], we know that $\widehat{\chi_{B(0,1)}}(\xi) = J_{n/2}(2\pi|\xi|)/|\xi|^{n/2}$ for all $\xi \in \mathbb{R}^n$, where the Bessel function is given by

$$J_v(t) := \frac{(\frac{t}{2})^v}{\Gamma(v+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{its}(1-s^2)^{v-\frac{1}{2}}\, ds, \quad t \in \mathbb{R}.$$

Since $\chi(x)=\chi_{B(0,1)}(Px)$ for all $x\in\mathbb{R}^n$, it follows that, for all $\xi\in\mathbb{R}^n$,

$$\widehat{\chi}(\xi)=|\det(P^{-1})|\widehat{\chi_{B(0,1)}}((P^{-1})^*\xi)=\frac{|\det(P^{-1})|J_{n/2}(2\pi|(P^{-1})^*\xi|)}{|(P^{-1})^*\xi|^{n/2}},$$

where $(P^{-1})^*$ denotes the *transpose* of P^{-1}. By $\widehat{\chi_{-j_0}}(\xi)=\widehat{\chi}((A^*)^{j_0}\xi)$, we then see that, for all $\xi\in\mathbb{R}^n$,

$$\widehat{\chi_{-j_0}}(\xi)=\frac{|\det(P^{-1})|J_{\frac{n}{2}}(2\pi|(P^{-1})^*(A^*)^{j_0}\xi|)}{|(P^{-1})^*(A^*)^{j_0}\xi|^{\frac{n}{2}}}.$$

Therefore, for all $\xi\in\mathbb{R}^n$, we know that

$$\begin{aligned}
&|\widehat{\chi}(\xi)-\widehat{\chi_{-j_0}}(\xi)|\\
&\quad=\frac{\pi^{n/2}}{\Gamma(n/2+1/2)\Gamma(1/2)}\\
&\quad\quad\times\left|\left\{\int_{-1}^{1}\left[e^{2\pi i|(P^{-1})^*\xi|s}-e^{2\pi i|(P^{-1})^*(A^*)^{j_0}\xi|s}\right](1-s^2)^{\frac{n}{2}-\frac12}\,ds\right\}\right|\\
&\quad\gtrsim\left|\int_0^1\left[\cos(2\pi|(P^{-1})^*\xi|s)-\cos(2\pi|(P^{-1})^*(A^*)^{j_0}\xi|s)\right](1-s^2)^{\frac{n}{2}-\frac12}\,ds\right|. \qquad (3.18)
\end{aligned}$$

Since the spectrum $\sigma(A^*)=\sigma(A)$, by [2, (2.1)], we conclude that there exists a positive constant $C_3:=C_3(A;\,n)$ such that, for any $\xi\in\mathbb{R}^n$ and $j\in\mathbb{Z}_+$,

$$1/C_3(\lambda_-)^j|\xi|\le|(A^*)^j\xi|\le C_3(\lambda_+)^j|\xi|.$$

Thus, we can pick an integer $j_0:=j_0(A;\,n)\in\mathbb{N}$ large enough such that, for any $\xi\in\mathbb{R}^n\setminus\{0\}$,

$$2|(P^{-1})^*\xi|<|(P^{-1})^*(A^*)^{j_0}\xi|. \qquad (3.19)$$

Choose a positive constant C_2 sufficiently close to zero and a positive constant $C_1:=C_2/(8\|A\|)$ such that, for any $\xi\in\mathbb{R}^n$ with $C_1\le|\xi|\le C_2$,

$$|(P^{-1})^*(A^*)^{j_0}\xi|<1/8, \qquad (3.20)$$

where $\|A\|:=\{\sum_{i,j=1}^n|a_{i,j}|^2\}^{1/2}$ for $A:=(a_{i,j})_{1\le i,j\le n}$. By (3.19) and (3.20), we see that there exists a positive constant C_4 such that

$$0<C_4\le\pi[|(P^{-1})^*(A^*)^{j_0}\xi|\pm|(P^{-1})^*\xi|]<\pi/4 \qquad \text{for } C_1\le|\xi|\le C_2.$$

Consequently, using the fact that $\sin t\ge t/2$ for $t\in[0,\,\frac{\pi}{4})$, we conclude that, for $s\in(0,1)$,

$$\cos 2\pi s|(P^{-1})^*\xi|-\cos 2\pi s|(P^{-1})^*(A^*)^{j_0}\xi|$$

$$= 2\sin(\pi s(|(P^{-1})^*(A^*)^{j_0}\xi| - |(P^{-1})^*\xi|))\sin(\pi s(|(P^{-1})^*(A^*)^{j_0}\xi| + |(P^{-1})^*\xi|))$$
$$\geq 2\frac{\pi s(|(P^{-1})^*(A^*)^{j_0}\xi| - |(P^{-1})^*\xi|)}{2}\frac{\pi s(|(P^{-1})^*(A^*)^{j_0}\xi| + |(P^{-1})^*\xi|)}{2} \gtrsim s^2.$$

Combining this inequality with (3.18), we obtain (3.17), namely, for all $C_1 \leq |\xi| \leq C_2$,

$$|\widehat{\chi}(\xi) - \widehat{\chi_{-j_0}}(\xi)| \gtrsim \int_0^1 s^2(1-s^2)^{n/2-1/2}\,ds > 0.$$

For any fixed $L \in \mathbb{Z}_+ \cup \{-1\}$, select a positive smooth function ϕ on $\mathbb{R}^n$ such that $\operatorname{supp}\phi \subset B_\rho(0,1)$, $\int_{\mathbb{R}^n}\phi(x)x^\gamma\,dx = 0$ for all $|\gamma| \leq L$, particularly when $L = -1$, the above vanishing condition is void, and $|\widehat{\phi}(\xi)| \geq C > 0$ for all $C_1 \leq |\xi| \leq C_2$, where C is a positive constant. For a construction of such ϕ, see [15, Theorem 2.6]. Then $\phi * (\chi - \chi_{-j_0}) \in C_c^\infty(\mathbb{R}^n)$, has vanishing moments till order L and satisfies that

$$\left|\widehat{\phi}(\xi)[\widehat{\chi}(\xi) - \widehat{\chi_{-j_0}}(\xi)]\right| \gtrsim 1 \tag{3.21}$$

for all $C_1 \leq |\xi| \leq C_2$.

Choose $g \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{g}$ is nonnegative, $\operatorname{supp} g \subset \{\xi \in \mathbb{R}^n :\ C_1 \leq |\xi| \leq C_2\}$ and $g(\xi) \geq C > 0$ if $2C_1 \leq |\xi| \leq C_2/2$, where C is a positive constant. We claim that, for any $\xi \in \mathbb{R}^n \setminus \{0\}$, there exists some $j \in \mathbb{Z}$ such that $|(A^*)^j\xi| \in [2C_1, C_2/2]$. Since $2C_1\|A\| = C_2/2$, the smallest $j \in \mathbb{Z}$ such that $|(A^*)^j\xi| \geq 2C_1$ does this job. Let $F := \sum_{j\in\mathbb{Z}} g((A^*)^{-j}\cdot)$. Then, F is a bounded smooth function and $F((A^*)^j\cdot) \equiv F$ for all $j \in \mathbb{Z}$ and $F(\xi) \gtrsim 1$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Define $h := g/F$. Then, $h \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp} h \subset \{\xi \in \mathbb{R}^n :\ C_1 \leq |\xi| \leq C_2\}$ and, for all $\xi \neq 0$, $\sum_{j\in\mathbb{Z}} h((A^*)^{-j}\xi) = 1$. By (3.21), we can define a Schwartz function ψ by setting $\widehat{\psi} := h\{\widehat{\phi}[\widehat{\chi} - \widehat{\chi_{-j_0}}]\}^{-1}$. Then

$$\operatorname{supp}\widehat{\psi} \subset \{\xi \in \mathbb{R}^n :\ C_1 \leq |\xi| \leq C_2\}$$

and, for all $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}_j(\xi)\widehat{\phi}_j(\xi)[\widehat{\chi}_j(\xi) - \widehat{\chi}_{j-j_0}(\xi)] = \sum_{j\in\mathbb{Z}} \widehat{h}((A^*)^{-j}\xi) = 1.$$

This finishes the proof of Lemma 3.8. □

Proof of Theorem 3.7. Let $\alpha \in (0, 2\zeta_-)$, $f \in \mathbf{S\dot{F}}^\alpha_{p,q}(A; w)$ and j_0 be as in Lemma 3.8. By Lemma 3.8 and $f \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$, we conclude that

$$f = \sum_{j\in\mathbb{Z}} \phi_j * \psi_j * (\chi_j - \chi_{j-j_0}) * f = \sum_{j\in\mathbb{Z}} \phi_j * \psi_j * (f_{B_\rho(\cdot, b^{-j})} - f_{B_\rho(\cdot, b^{j_0-j})}),$$

which, modulo polynomials, holds in $\mathcal{S}'(\mathbb{R}^n)$. Here, ϕ and ψ are as in Lemma 3.8. Let φ be as in (2.1). For any $k \in \mathbb{Z}$, we have

$$\varphi_k * f = \sum_{j\in\mathbb{Z}} \varphi_k * \phi_j * \psi_j * (f_{B_\rho(\cdot,\, b^{-j})} - f_{B_\rho(\cdot,\, b^{j_0-j})}).$$

Notice that, for all k, $j \in \mathbb{Z}$, and $x \in \mathbb{R}^n$,

$$|\varphi_k * \phi_j * \psi_j(x)| = |\varphi_k * (\phi * \psi)_j(x)| \lesssim b^{-(s+1)|j-k|\zeta_-} \frac{b^{\min\{j,\,k\}}}{[1 + b^{\min\{j,\,k\}}\rho(x)]^L}, \tag{3.22}$$

where s, L can be chosen sufficiently large; see [8, Lemma 5.4]. Thus,

$$|\varphi_k * \phi_j * \psi_j * g| = |\varphi_k * (\phi * \psi)_j * g| \lesssim b^{-2|j-k|\zeta_-} M_\rho(g).$$

Therefore, when $p \in (q_w, \infty)$, from Definition 2.1, Hölder's inequality and the Fefferman-Stein vector-valued maximal inequality, we infer that

$$\begin{aligned}
\|f\|_{\dot{\mathbf{F}}^\alpha_{p,\,q}(A;\,w)} &\lesssim \left\| \left\{ \sum_{k\in\mathbb{Z}} b^{k\alpha q} \left| \sum_{j\in\mathbb{Z}} b^{-2|k-j|\zeta_-} M_\rho \left(f_{B_\rho(\cdot,\, b^{-j})} - f_{B_\rho(\cdot,\, b^{j_0-j})} \right) \right|^q \right\}^{1/q} \right\|_{L^p_w(\mathbb{R}^n)} \\
&\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} b^{j\alpha q} \left[M_\rho \left(f_{B_\rho(\cdot,\, b^{-j})} - f_{B_\rho(\cdot,\, b^{j_0-j})} \right) \right]^q \right\}^{1/q} \right\|_{L^p_w(\mathbb{R}^n)} \\
&\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} b^{j\alpha q} \left| [f_{B_\rho(\cdot,\, b^{-j})} - f] - [f_{B_\rho(\cdot,\, b^{j_0-j})} - f] \right|^q \right\}^{1/q} \right\|_{L^p_w(\mathbb{R}^n)} \\
&\lesssim \left\| \left\{ \sum_{j\in\mathbb{Z}} b^{j\alpha q} \left| f_{B_\rho(\cdot,\, b^{-j})} - f \right|^q \right\}^{1/q} \right\|_{L^p_w(\mathbb{R}^n)} \lesssim \|S_{\alpha,\,q}(f)\|_{L^p_w(\mathbb{R}^n)}.
\end{aligned}$$

When $p = \infty$, we need to show that

$$\left\{ \fint_{B_\rho(x,\, b^{-\ell})} \sum_{k\ge\ell} b^{k\alpha q} \left(\sum_{j\in\mathbb{Z}} \left| \varphi_k * \phi_j * \psi_j * \left[f_{B_\rho(\cdot,\, b^{-j})} - f_{B_\rho(\cdot,\, b^{j_0-j})} \right](y) \right| \right)^q w(y)\, dy \right\}^{1/q}$$

is controlled by $\|f\|_{\mathbf{S}\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)}$ uniformly in $x \in \mathbb{R}^n$ and $\ell \in \mathbb{Z}$. The proof of this is similar to that of (3.7), where we need to split $\sum_{j\in\mathbb{Z}}$ into three parts. Using (3.22) one can show that $\|f\|_{\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)} \lesssim \|f\|_{\mathbf{S}\dot{\mathbf{F}}^\alpha_{\infty,\,q}(A;\,w)}$. We leave the details to the reader. This finishes the proof of Theorem 3.7. $\square$

Finally, we sketch the details needed to deal with Besov spaces.

Proof of Theorem 1.7. The proof of this theorem is similar to that of Theorem 1.5. Thus, we only prove an analogue of the estimate of I_1 in the proof of Theorem 3.4 for Besov spaces. The other estimates are left to the reader. For any $p > q_w$ and $q \in (0, \infty)$, by (3.10), Minkowski's inequality with $p \in (q_w, \infty)$ and the $L^p_w(\mathbb{R}^n)$-boundedness of M_ρ, we have

$$\begin{aligned}
\mathrm{I}_1 &:= \left\{\sum_{k\in\mathbb{Z}} b^{kq\alpha}\left[\int_{\mathbb{R}^n}\left(\sum_{j\le k}|(\varphi_j-\chi_k*\varphi_j)*\psi_j*f(x)|\right)^p w(x)\,dx\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&\lesssim \left\{\sum_{k\in\mathbb{Z}}\left[\int_{\mathbb{R}^n}\left(\sum_{j\le k} b^{(j-k)(2\zeta_--\alpha)+j\alpha}M_\rho(\psi_j*f)(x)\right)^p w(x)\,dx\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&\lesssim \left\{\sum_{k\in\mathbb{Z}}\left[\sum_{j\le k} b^{(j-k)(2\zeta_--\alpha)+j\alpha}\|\psi_j*f\|_{L^p_w(\mathbb{R}^n)}\right]^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

When $q=\infty$, it is easy to obtain $\mathrm{I}_1 \lesssim \|f\|_{\dot{\mathbf{B}}^\alpha_{p,\,q}(A;\,w)}$. When $q\in(0,\,1]$, we use the q-triangle inequality to deduce the same conclusion. Finally, when $q\in(1,\,\infty)$, we choose $\delta\in(0, 2\zeta_- -\alpha)$. Then, by Hölder's inequality, we see that

$$\begin{aligned}
\mathrm{I}_1 &\lesssim \left\{\sum_{k\in\mathbb{Z}}\left[\sum_{j\le k} b^{(j-k)(2\zeta_--\alpha-\delta)+j\alpha}\|\psi_j*f\|_{L^p_w(\mathbb{R}^n)} b^{(j-k)\delta}\right]^q\right\}^{\frac{1}{q}}\\
&\lesssim \left\{\sum_{k\in\mathbb{Z}}\sum_{j\le k} b^{q(j-k)(2\zeta_--\alpha-\delta)+qj\alpha}\|\psi_j*f\|^q_{L^p_w(\mathbb{R}^n)}\right\}^{\frac{1}{q}} \lesssim \|f\|_{\dot{\mathbf{B}}^\alpha_{p,\,q}(A;\,w)}.
\end{aligned}$$

Combining the estimates above, we conclude that $\mathrm{I}_1 \lesssim \|f\|_{\dot{\mathbf{B}}^\alpha_{p,\,q}(A;\,w)}$ for $q\in(0,\,\infty]$ and $p\in(q_w,\,\infty)$. The same estimate also holds for $p=\infty$ with usual modifications, which completes the proof of Theorem 1.7. □

Acknowledgement. This project is partially supported by the National Natural Science Foundation of China (Grant Nos. 11001234, 11161044, 11171027, 11361020 and 11101038), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120003110003) and the Fundamental Research Funds for Central Universities of China (Grant No. 2012LYB26). Marcin Bownik is partially supported by a grant from the Simons Foundation (Grant No. 244422).

References

[1] Alabern, R., Mateu, J. and Verdera, J., A new characterization of Sobolev spaces on $\mathbb{R}^n$. *Math. Ann.* 354 (2012), 589 – 626.

[2] Bownik, M., *Anisotropic Hardy Spaces and Wavelets.* Mem. Amer. Math. Soc. 164 (2003), No. 781.

[3] Bownik, M., Atomic and molecular decompositions of anisotropic Besov spaces. *Math. Z.* 250 (2005), 539 – 571.

[4] Bownik, M., Anisotropic Triebel-Lizorkin spaces with doubling measures. *J. Geom. Anal.* 17 (2007), 387 – 424.

[5] Bownik, M., Duality and interpolation of anisotropic Triebel-Lizorkin spaces. *Math. Z.* 259 (2008), 131 – 169.

[6] Bownik, M. and Ho, K.-P., Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces. *Trans. Amer. Math. Soc.* 358 (2006), 1469 – 1510.

[7] Bownik, M., Li, B., Yang, D. and Zhou, Y., Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators. *Indiana Univ. Math. J.* 57 (2008), 3065 – 3100.

[8] Bownik, M., Li, B., Yang, D. and Zhou, Y., Weighted anisotropic product Hardy spaces and boundedness of sublinear operators. *Math. Nachr.* 283 (2010), 392 – 442.

[9] Calderón, A.-P., An atomic decomposition of distributions in parabolic H^p spaces. *Adv. Math.* 25 (1977), 216 – 225.

[10] Calderón, A.-P. and Torchinsky, A., Parabolic maximal functions associated with a distribution. *Adv. Math.* 16 (1975), 1 – 64.

[11] Calderón, A.-P. and Torchinsky, A., Parabolic maximal functions associated with a distribution. II. *Adv. Math.* 24 (1977), 101 – 171.

[12] Fabes, E. B. and Rivière, N. M., Singular intervals with mixed homogeneity. *Studia Math.* 27 (1966), 19 – 38.

[13] Fabes, E. B. and Rivière, N. M., Symbolic calculus of kernels with mixed homogeneity, In: *Singular Integrals.* (Proc. Sympos. Pure Math., Chicago, Ill., 1966), Providence, R. I.: Amer. Math. Soc. 1967, pp. 106 – 127.

[14] Folland, G. B. and Stein, E. M., *Hardy Spaces on Homogeneous Group.* Mathematical Notes 28. Princeton, N. J.: Princeton University Press 1982.

[15] Frazier, M. and Jawerth, B., Decomposition of Besov spaces. *Indiana U. Math. J.* 34 (1985), 777 – 799.

[16] García-Cuerva, J. and Rubio de Francia, J. L., *Weighted Norm Inequalities and Related Topics.* North-Holland Mathematics Studies 116. Notas de Matemática [Mathematical Notes] 104. Amsterdam: North-Holland Publishing Co. 1985.

[17] Grafakos, L., *Classical Fourier Analysis. Second edition.* Graduate Texts in Mathematics 249/250. New York: Springer 2008.

[18] Haroske, D., and Triebel, H., Embeddings of function spaces: a criterion in terms of differences. *Complex Var. Elliptic Equ.* 56 (2011), 931 – 944.

[19] Johnson, R. and Neugebauer, C. J., Homeomorphisms preserving A_p. *Rev. Mat. Iberoamericana* 3 (1987), 249 – 273.

[20] Koskela, P., Yang, D. and Zhou, Y., A characterization of Hajłasz-Sobolev and Triebel-Lizorkin spaces via grand Littlewood-Paley functions. *J. Funct. Anal.* 258 (2010), 2637 – 2661.

[21] Koskela, P., Yang, D. and Zhou, Y., Pointwise characterizations of Besov and Triebel-Lizorkin spaces and quasiconformal mappings. *Adv. Math.* 226 (2011), 3579 – 3621.

[22] Müller, D. and Yang, D., A difference characterization of Besov and Triebel-Lizorkin spaces on RD-spaces. *Forum Math.* 21 (2009), 259 – 298.

[23] Rivière, N. M., Singular integrals and multiplier operators. *Ark. Mat.* 9 (1971), 243 – 278.

[24] Schmeisser, H.-J. and Triebel, H., *Topics in Fourier Analysis and Function Spaces*. Chichester: John Wiley & Sons Ltd. 1987.

[25] Triebel, H., *Theory of Function Spaces*. Basel: Birkhäuser Verlag 1983.

[26] Triebel, H., *Theory of Function Spaces III*. Basel: Birkhäuser Verlag 2006.

[27] Triebel, H., Sobolev-Besov spaces of measurable functions. *Studia Math.* 201 (2010), 69 – 86.

[28] Triebel, H., Limits of Besov norms. *Arch. Math.* 96 (2011), 169 – 175.

[29] Wheeden, R. L., Lebesgue and Lipschitz spaces and integrals of the Marcinkiewicz type. *Studia Math.* 32 (1969), 73 – 93.

[30] Yang, D., New characterizations of Hajłasz-Sobolev spaces on metric spaces. *Sci. China Ser. A* 46 (2003), 675 – 689.

[31] Yang, D. and Yuan, W., A new class of function spaces connecting Triebel-Lizorkin spaces and Q spaces. *J. Funct. Anal.* 255 (2008), 2760 – 2809.

[32] Yang, D., Yuan, W. and Zhou, Y., A new characterization of Triebel-Lizorkin spaces on $\mathbb{R}^n$. *Publ. Mat.* 57 (2013), 57 – 82.

[33] Yuan, W., Sickel, W. and Yang, D., *Morrey and Campanato Meet Besov, Lizorkin and Triebel*. Lecture Notes in Mathematics 2005. Berlin: Springer-Verlag 2010.

Received January 24, 2013; revised November 15, 2013