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Spectra of Frame Operators with Prescribed Frame Norms

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Abstract. We study the set of possible finite spectra of self-adjoint operators with fixed diagonal. In the language of frame theory, this is equivalent to study of the set of finite spectra of frame operators with prescribed frame norms. We show several properties of such sets. We also give some numerical examples illustrating our results.

1. Introduction

The concept of frames in Hilbert spaces was originally introduced in the context of nonharmonic Fourier series by Duffin and Schaeffer [**12**] in 1950's. The advent of wavelet theory brought a renewed interest in frame theory as is attested by now classical books of Daubechies [**11**], Meyer [**26**], and Mallat [**24**]. For an introduction to frame theory we refer to the book by Christensen [**10**].

Definition 1.1. A sequence $\{f_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if there exist $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2 \qquad \text{for all } f \in \mathcal{H}. \tag{1.1}$$

The numbers A and B are called the *frame bounds*. The supremum over all A's and infimum over all B's which satisfy (1.1) are called the *optimal frame bounds*. If $A = B$, then $\{f_i\}$ is said to be a *tight frame*. In addition, if $A = B = 1$, then $\{f_i\}$ is called a *Parseval frame*. The frame operator is defined by $Sf = \sum_{i\in I}\langle f, f_i\rangle f_i$. It is well-known that S is a self-adjoint operator satisfying $A\mathbf{I} \le S \le B\mathbf{I}$.

The construction of frames with desired properties is a vast subject that is central to frame theory. Among the recently studied classes of frames with desired features are: Grassmanian frames, equiangular frames, equal norm tight frames, finite frames for sigma-delta quantization, fusion frames, frames for signal reconstruction without the phase, etc. In particular, the problem of constructing frames with prescribed norms and frame operator has been studied by many authors. Casazza and

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his collaborators [**7**–**9**] characterized norms of finite tight frames in terms of "fundamental frame inequality" using frame potential methods and gave an explicit and algorithmic construction of finite tight frames with prescribed norms. Kornelson and Larson [**22**] studied a similar problem for infinite dimensional Hilbert spaces using projection decomposition. Antezana, Massey, Ruiz, and Stojanoff [**1**] established the connection of this problem with the infinite dimensional Schur-Horn problem and gave refined necessary conditions and sufficient conditions. A beautifully simple and complete characterization of Parseval frame norms was given by Kadison [**18**, **19**], which easily extends to tight frames by scaling. The authors [**4**] have extended this result to the non-tight setting to characterize frame norms with prescribed optimal frame bounds. The second author [**17**] has characterized diagonals of self-adjoint operators with three points in the spectrum. This yields a characterization of frame norms whose frame operator has two point spectrum. Finally, the authors [**5**,**6**] have recently extended this result to operators with finite spectrum.

The above mentioned research was aimed primarily at characterizing diagonals of operators (or frame norms) with prescribed spectrum (or frame operator). However, it is equally interesting to consider a converse problem of characterizing spectra of operators with prescribed diagonal. In the language of frames, we are asking for possible spectra of frame operators for which the sequence of frame norms $\{||f_i||\}_{i\in I}$ is prescribed. That is, given $n \in \mathbb{N}$ and a sequence $\{d_i\}_{i\in I}$ in $[0,1]$ we consider the set

$$(1.2)\qquad \mathcal{A}_n = \mathcal{A}_n(\{d_i\}) = \big\{(A_1,\ldots,A_n)\in(0,1)^n : \forall_{j\neq k}\ A_j \neq A_k \\ \exists \text{ frame } \{f_i\}_{i\in I} \text{ such that } \forall_{i\in I}\ d_i = ||f_i||^2 \text{ and its} \\ \text{frame operator } S \text{ satisfies } \sigma(S) = \{A_1,\ldots,A_n,1\}\big\}.$$

In this work we shall always assume that there exists $\alpha \in (0,1)$ such that

$$\sum_{d_i<\alpha} d_i + \sum_{d_i\geq\alpha}(1-d_i) < \infty.$$

Otherwise, it can be shown by Theorems 2.3 and 2.7 that $\mathcal{A}_n(\{d_i\})$ is the set of all points in $(0,1)^n$ with distinct coordinates.

The second author [**17**, Theorem 7.1] has shown that the set $\mathcal{A}_1$ is always finite (possibly empty). In this paper we show some further properties such as a characterization of sequences $\{d_i\}$ for which $\mathcal{A}_1$ is nonempty. We also prove that the set $\mathcal{A}_2$ consists of a countable union of line segments. Moreover, one endpoint of each of these line segments must lie in the boundary of the unit square. Finally, we show that the sets $\mathcal{A}_n$ are nonempty for all $n \geq 2$ under the assumption that infinitely many d_i's satisfy $d_i \in (0,1)$. Moreover, we prove the optimality of this result.

2. Background results about Schur-Horn type theorems

The classical Schur-Horn theorem [**16**,**28**] characterizes diagonals of self-adjoint (Hermitian) matrices with given eigenvalues. It can be stated as follows, where $\mathcal{H}_N$ is N dimensional Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, i.e., $\mathcal{H}_N = \mathbb{R}^N$ or $\mathbb{C}^N$.

THEOREM 2.1 (Schur-Horn theorem). *Let $\{\lambda_i\}_{i=1}^N$ and $\{d_i\}_{i=1}^N$ be real sequences in nonincreasing order. There exists a self-adjoint operator $E : \mathcal{H}_N \to \mathcal{H}_N$ with* *eigenvalues* $\{\lambda_i\}$ *and diagonal* $\{d_i\}$ *if and only if*

$$\sum_{i=1}^{N} d_i = \sum_{i=1}^{N} \lambda_i \quad \text{and} \quad \sum_{i=1}^{n} d_i \le \sum_{i=1}^{n} \lambda_i \text{ for all } 1 \le n \le N. \tag{2.1}$$

The necessity of (2.1) is due to Schur [**28**] and the sufficiency of (2.1) is due to Horn [**16**]. It should be noted that (2.1) can equivalently be stated in terms of the convexity condition

$$(d_1, \dots, d_N) \in \operatorname{conv}\{(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(N)}) : \sigma \in S_N\}. \tag{2.2}$$

This characterization has attracted a significant interest and has been generalized in many remarkable ways. Some major milestones are the Kostant convexity theorem [**23**] and the convexity of moment mappings in symplectic geometry [**3**, **14**, **15**]. Moreover, the problem of extending Theorem 2.1 to an infinite dimensional dimensional Hilbert space $\mathcal{H}$ was also investigated. Neumann [**27**] gave an infinite dimensional version of the Schur-Horn theorem phrased in terms of ℓ^∞-closure of the convexity condition (2.2). Neumann's result can be considered an initial, albeit somewhat crude, solution of this problem. The first fully satisfactory progress was achieved by Kadison. In his influential work [**18**, **19**] Kadison discovered a characterization of diagonals of orthogonal projections acting on $\mathcal{H}$.

THEOREM 2.2 (Kadison). *Let* $\{d_i\}_{i\in I}$ *be a sequence in* $[0,1]$ *and* $\alpha \in (0,1)$. *Define*

$$C(\alpha) = \sum_{d_i<\alpha} d_i, \qquad D(\alpha) = \sum_{d_i \ge \alpha} (1-d_i).$$

There exists an orthogonal projection on $\ell^2(I)$ *with diagonal* $\{d_i\}_{i\in I}$ *if and only if either:*

(i) $C(\alpha) = \infty$ *or* $D(\alpha) = \infty$, *or*
(ii) $C(\alpha) < \infty$ *and* $D(\alpha) < \infty$, *and*

$$C(\alpha) - D(\alpha) \in \mathbb{Z}. \tag{2.3}$$

The work by Gohberg and Markus [**13**] and Arveson and Kadison [**2**] extended the Schur-Horn Theorem 2.1 to positive trace class operators. This has been further extended to compact positive operators by Kaftal and Weiss [**21**]. These results are stated in terms of majorization inequalities as in (2.1), see also [**20**] for a detailed survey of recent progress on infinite Schur-Horn majorization theorems and their connections to operator ideals. Antezana, Massey, Ruiz, and Stojanoff [**1**] refined the results of Neumann [**27**]. Moreover, they showed the following connection between Schur-Horn type theorems and the existence of frames with prescribed norms and frame operators, see [**1**, Proposition 4.5] and [**4**, Proposition 2.3].

THEOREM 2.3. *Let* $\{d_i\}_{i\in I}$ *be a bounded sequence of positive numbers. Let* S *be a positive self-adjoint operator on a Hilbert space* $\mathcal{H}$. *Then the following are equivalent:*

(i) *there exists a frame* $\{f_i\}_{i\in I}$ *in* $\mathcal{H}$ *with the frame operator* S *such that* $d_i = ||f_i||^2$ *for all* $i \in I$,
(ii) *there exists a larger Hilbert space* $\mathcal{K} \supset \mathcal{H}$ *and a self-adjoint operator* E *acting on* $\ell^2(I)$, *which is unitarily equivalent with* $S \oplus \mathbf{0}$, *where* $\mathbf{0}$ *is the zero operator acting on* $\mathcal{K} \ominus \mathcal{H}$, *such that its diagonal* $\langle Ee_i, e_i\rangle = d_i$ *for all* $i \in I$.

The authors [**4**] have recently shown a variant of the Schur-Horn theorem for a class of locally invertible self-adjoint operators on $\mathcal{H}$.

THEOREM 2.4. *Let $0 < A < B < \infty$ and $\{d_i\}$ be a nonsummable sequence in $[0, B]$. Define the numbers*

$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i). \tag{2.4}$$

Then, there is a positive operator E on a Hilbert space $\mathcal{H}$ with $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$ and diagonal $\{d_i\}$ if and only if

$$\exists n \in \mathbb{N}_0 \qquad nA \leq C \leq A + B(n-1) + D. \tag{2.5}$$

As a corollary of Theorems 2.3 and 2.4 we obtain the characterization of sequences of frame norms.

COROLLARY 2.5. *Let $0 < A < B < \infty$ and $\{d_i\}$ be a nonsummable sequence in $[0, B]$. There exists a frame $\{f_i\}$ on a Hilbert space $\mathcal{H}$ with optimal frame bounds A and B, and $d_i = ||f_i||^2$, if and only if (2.5) holds.*

One should emphasize that the non-tight case is not a mere generalization of the tight case $A = B$ established by Kadison [**18, 19**]. Indeed, the non-tight case is qualitatively different from the tight case, since by setting $A = B$ in Theorem 2.4 we do not get the correct necessary and sufficient condition (2.3) previously discovered by Kadison. Another extension of Kadison's result [**18, 19**] was obtained by the second author [**17**] who characterized the set of diagonals of operators with three points in the spectrum.

THEOREM 2.6. *Let $0 < A < B < \infty$ and $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$ with $\sum d_i = \sum (B - d_i) = \infty$. Define C and D as in (2.4). There is a self-adjoint operator E with diagonal $\{d_i\}_{i \in I}$ and $\sigma(E) = \{0, A, B\}$ if and only if one following holds: (i) $C = \infty$, (ii) $D = \infty$, or (iii) $C, D < \infty$ and*

$$\exists N \in \mathbb{N} \; \exists k \in \mathbb{Z} \qquad C - D = NA + kB \quad \text{and} \quad C \geq (N + k)A.$$

Finally, the authors [**5**] showed the following characterization of diagonals of self-adjoint operators with finite spectrum. Theorem 2.7 becomes a foundation on which all subsequent results in this paper will be derived. We will often use this result under a convenient normalization that $B = 1$.

THEOREM 2.7. *Let $\{A_j\}_{j=0}^{n+1}$ be an increasing sequence of real numbers such that $A_0 = 0$ and $A_{n+1} = B$, $n \in \mathbb{N}$. Let $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$ with $\sum d_i = \sum (B - d_i) = \infty$. For each $\alpha \in (0, B)$, define*

$$C(\alpha) = \sum_{d_i < \alpha} d_i \quad \text{and} \quad D(\alpha) = \sum_{d_i \geq \alpha} (B - d_i). \tag{2.6}$$

Then, there exists a self-adjoint operator E with diagonal $\{d_i\}_{i \in I}$ and $\sigma(E) = \{A_0, A_1, \ldots, A_{n+1}\}$ if and only if either:

(i) *$C(B/2) = \infty$ or $D(B/2) = \infty$, or*
(ii) *$C(B/2) < \infty$ and $D(B/2) < \infty$, (and thus $C(\alpha), D(\alpha) < \infty$ for all $\alpha \in (0, B)$), and there exist $N_1, \ldots, N_n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that:*

$$C(B/2) - D(B/2) = \sum_{j=1}^{n} A_j N_j + kB, \tag{2.7}$$

and for all $r = 1, \dots, n$,

$$(B - A_r)C(A_r) + A_r D(A_r) \geq (B - A_r)\sum_{j=1}^{r} A_j N_j + A_r \sum_{j=r+1}^{n} (B - A_j)N_j. \tag{2.8}$$

3. Spectral set

In light of Theorem 2.3 we shall adopt the following definition for the set (1.2).

DEFINITION 3.1. Suppose that $\{d_i\}_{i \in I}$ is a sequence in $[0,1]$. For a given $n \in \mathbb{N}$ we define the *spectral set*

$$\mathcal{A}_n(\{d_i\}) = \big\{(A_1, \dots, A_n) \in (0,1)^n : \forall_{j \neq k}\ A_j \neq A_k,\ \exists \text{ self-adjoint operator } E \text{ on } \ell^2(I) \text{ with } \sigma(E) = \{0, A_1, \dots, A_n, 1\} \text{ and diagonal } \{d_i\}\big\}.$$

In order to apply Theorem 2.7 we shall assume that $\sum d_i = \sum(1 - d_i) = \infty$. This is not a true limitation. Indeed, the case when $\sum d_i < \infty$ or $\sum(1 - d_i) < \infty$ requires an application of a finite rank analogue of Theorem 2.7. This leads effectively to a finite dimensional case where the analysis is actually simpler, but also less interesting; see Remark 3.3. Furthermore, we will consider only sequences in the set

$$\mathcal{F} := \left\{\{d_i\} : 0 \leq d_i \leq 1 \text{ and } \exists\, \alpha \in (0,1) \text{ such that } \sum_{d_i < \alpha} d_i + \sum_{d_i \geq \alpha} (1 - d_i) < \infty\right\}.$$

Otherwise, Theorem 2.7(i) implies that $\mathcal{A}_n(\{d_i\})$ is the set of all points in $(0,1)^n$ with distinct coordinates, which is not interesting.

3.1. Three point spectrum. In this subsection we will look at properties of the set $\mathcal{A}_1(\{d_i\})$. In [**17**] it was shown that $\mathcal{A}_1(\{d_i\})$ is a finite (possibly empty) set for all $\{d_i\} \in \mathcal{F}$. Recall that we assume $\sum d_i = \sum(1 - d_i) = \infty$ so that we can apply Theorem 2.7.

DEFINITION 3.2. For a sequence $\{d_i\}_{i \in I} \in \mathcal{F}$ set

$$\eta = C(1/2) - D(1/2) - \lfloor C(1/2) - D(1/2) \rfloor,$$

and define the function $m : (0,1) \to \mathbb{Z}$ by

$$m(A) = C(A) - D(A) - \eta.$$

REMARK 3.1. Note that for $A < A'$

$$\begin{aligned} m(A') - m(A) &= C(A') - C(A) + D(A) - D(A') \\ &= \sum_{d_i \in [A, A')} d_i + \sum_{d_i \in [A, A')} (1 - d_i) = |\{i \colon d_i \in [A, A')\}|. \end{aligned} \tag{3.1}$$

Since $m(1/2) = \lfloor C(1/2) - D(1/2) \rfloor \in \mathbb{Z}$, from (3.1) we see that $m(A) \in \mathbb{Z}$ for all $A \in (0,1)$.

By Theorem 2.7 we have $A \in \mathcal{A}_1(\{d_i\})$ if and only if there exists $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that

$$C(A) - D(A) = AN + k \tag{3.2}$$

and

$$(1 - A)C(A) + AD(A) \geq (1 - A)AN. \tag{3.3}$$

In principle, to verify that $A \in \mathcal{A}_1$ one would need to check the inequality (3.3) for all N and k such that (3.2) holds. The next theorem shows that one needs only to check that (3.3) holds for a particular $N \in \mathbb{N}$.

THEOREM 3.3. *Let $\{d_i\}_{i\in I} \in \mathcal{F}$, and let η be as in Definition 3.2. Fix $A \in (0,1)$ and set*

$$q = \min\{j \in \mathbb{N} \colon jA - \eta \in \mathbb{Z}\}.$$

The number A is in $\mathcal{A}_1(\{d_i\})$ if and only if

$$(1-A)C(A) + AD(A) \geq (1-A)Aq. \tag{3.4}$$

REMARK 3.2. If there is no $j \in \mathbb{N}$ so that $jA - \eta \in \mathbb{Z}$, then we have $q = \min \varnothing = \infty$. In this case the inequality (3.4) does not hold and we conclude that $A \notin \mathcal{A}_1$.

PROOF. Assume that $A \in \mathcal{A}_1$. By Theorem 2.7 there exists $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that (3.2) and (3.3) hold. From (3.2) we see that $NA - \eta \in \mathbb{Z}$. Thus, $q \leq N$ and (3.4) holds.

Conversely, assume that (3.4) holds. By Theorem 2.7 it is enough to show that (3.2) and (3.3) hold with $N = q$ and $k = m(A) - qA + \eta$. It is clear that (3.3) holds, since this is exactly (3.4). Finally, we have

$$C(A) - D(A) = m(A) + \eta = qA + k,$$

which is exactly (3.2). □

Theorem 3.3 gives a finite algorithm to check whether a given number $A \in (0,1)$ is in $\mathcal{A}_1$. For each

$$1 \leq j \leq \left\lfloor \frac{(1-A)C(A) + AD(A)}{(1-A)A} \right\rfloor$$

we compute the quantity $jA - \eta$. Then, by Theorem 3.3 $A \in \mathcal{A}_1$ if and only if one of these numbers is an integer.

THEOREM 3.4. *Let $\{d_i\}_{i\in I} \in \mathcal{F}$, and let η be as in Definition 3.2. If $\eta \neq 0$, then $\eta \in \mathcal{A}_1(\{d_i\})$. In particular, if $\{d_i\}$ is not the diagonal of a projection, then $\mathcal{A}_1(\{d_i\}) \neq \varnothing$.*

PROOF. By Theorem 3.3 it is enough to show that (3.4) holds with $A = \eta$. Note that

$$q = \min\{j \in \mathbb{N} \colon j\eta - \eta \in \mathbb{Z}\} = 1.$$

Thus, it is enough to verify that

$$(1-\eta)C(\eta) + \eta D(\eta) \geq (1-\eta)\eta. \tag{3.5}$$

If $m(\eta) \geq 0$, then

$$C(\eta) \geq C(\eta) - D(\eta) = m(\eta) + \eta \geq \eta.$$

If $m(\eta) \leq -1$, then

$$D(\eta) \geq D(\eta) - C(\eta) = -m(\eta) - \eta \geq 1 - \eta.$$

In either case we conclude that (3.5) holds. □

The next theorem characterizes the sequences $\{d_i\}$ such that $\mathcal{A}_1(\{d_i\}) = \varnothing$.

THEOREM 3.5. *Let $\{d_i\}_{i\in I} \in \mathcal{F}$, and let η be as in Definition 3.2. The set $\mathcal{A}_1 = \varnothing$ if and only if $\eta = 0$ and*

$$C(1/2) + D(1/2) < 1. \tag{3.6}$$

That is, $\mathcal{A}_1$ is empty if and only if $\{d_i\}$ is the diagonal of a projection and $1/2 \notin \mathcal{A}_1$.

PROOF. We begin by assuming that $\mathcal{A}_1 = \varnothing$. By Theorem 3.4 we must have $\eta = 0$, otherwise $\eta \in \mathcal{A}_1$ and thus $\mathcal{A}_1 \neq \varnothing$. Thus, we may assume $\eta = 0$. Since $1/2 \notin \mathcal{A}_1$, Theorem 3.3 shows (3.6).

Conversely, assume that $\eta = 0$ and (3.6) holds. Since $C(1/2) - D(1/2) \in \mathbb{Z}$ we conclude that $C(1/2) = D(1/2) < 1/2$ and $m(1/2) = 0$. For the sake of contradiction assume that there is some $A \in \mathcal{A}_1 \cap (0, 1/2]$. From (3.2) we see that $A \in \mathbb{Q}$. That is, $A = p/q$ for some $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$. From (3.2) we have

$$-m(A) = m(1/2) - m(A) = |\{i : A \leq d_i < 1/2\}| \tag{3.7}$$

Using (3.7) and the definition of $m(A)$ we have

$$\begin{aligned}(1 - A)C(A) + AD(A) &= -Am(A) + C(A) = -Am(A) + \sum_{d_i<A} d_i \\ &= -Am(A) + C(1/2) - \sum_{A\leq d_i<1/2} d_i \\ &\leq -Am(A) + C(1/2) - A|\{i : A \leq d_i < 1/2\}| = C(1/2) \\ &< \frac{1}{2} \leq 1 - A \leq (1 - A)p = (1 - A)Aq.\end{aligned}$$

This implies that (3.4) does not hold, and by Theorem 3.3 $A \notin \mathcal{A}_1$. Consequently, $\mathcal{A}_1 \cap (0, 1/2] = \varnothing$. A similar argument shows that $\mathcal{A}_1 \cap (1/2, 1) = \varnothing$, and thus $\mathcal{A}_1 = \varnothing$. □

REMARK 3.3. Recall that in this section it is assumed that $\sum d_i = \sum(1-d_i) = \infty$ so that Theorem 2.7 applies. However, all of the results in this subsection hold under the weaker assumption that both $\sum d_i \geq 1$ and $\sum(1 - d_i) \geq 1$. In the case that $\{d_i\}$ or $\{1 - d_i\}$ is summable the proofs are similar to those above. The difference being that the applications of Theorem 2.7 are replaced by a finite rank version of the Schur-Horn theorem [**21**].

3.2. Finite point spectrum. In this subsection we will look at properties of the set $\mathcal{A}_n(\{d_i\})$, where $n \geq 2$.

DEFINITION 3.6. Let $\{d_i\}_{i\in I} \in \mathcal{F}$ and let η be as in Definition 3.2. By Theorem 2.7 a point $(A_1, \dots, A_n)$ with $0 < A_1 < \dots < A_n < 1$ belongs to $\mathcal{A}_n(\{d_i\})$ if and only there exist $N_1, \dots, N_n \in \mathbb{N}$ such that

$$\sum_{j=1}^{n} A_j N_j \equiv \eta \mod 1 \tag{3.8}$$

and for all $r = 1, \dots, n$,

$$(1 - A_r)C(A_r) + A_r D(A_r) \geq (1 - A_r)\sum_{j=1}^{r} A_j N_j + A_r \sum_{j=r+1}^{n} (1 - A_j)N_j. \tag{3.9}$$

Given $(N_1, \ldots, N_n) \in \mathbb{N}^n$ define the set

(3.10)
$$\mathcal{A}_n^{N_1,\ldots,N_n}(\{d_i\}) = \Big\{(A_1, \ldots, A_n) : 0 < A_1 < \ldots < A_n < 1, \text{ (3.8) and (3.9) hold}\Big\}.$$

As an immediate consequence of Theorem 2.7 we have that

$$\mathcal{A}_n(\{d_i\}) = \bigcup_{(N_1,\ldots,N_n)\in\mathbb{N}^n} \bigcup_{\sigma\in\Sigma_n} \sigma \circ \mathcal{A}_n^{N_1,\ldots,N_n}(\{d_i\}), \tag{3.11}$$

where $\circ$ denotes the action of the permutation group Σ_n on $\mathbb{R}^n$.

THEOREM 3.7. *Let $\{d_i\}_{i\in I} \in \mathcal{F}$ be such that*

$$\{i \in I : d_i \in (0, 1)\} \text{ is infinite,} \tag{3.12}$$

and let η be as in Definition 3.2. Then, the set $\mathcal{A}_n^{N_1,\ldots,N_n}(\{d_i\})$ is nonempty if one of the following holds:

(i) *the sequence $\{d_i\}$ is a diagonal of a projection, i.e., $\eta = 0$, and both of the sets $\{i \in I : d_i \in (0, 1/2)\}$ and $\{i \in I : d_i \in (1/2, 1)\}$ are infinite,*
(ii) *$\eta > 0$, $N_1 = 1$, and $\{i \in I : d_i \in (1/2, 1)\}$ is infinite,*
(iii) *$\eta > 0$, $N_n = 1$, and $\{i \in I : d_i \in (0, 1/2)\}$ is infinite,*

PROOF. By our hypothesis (3.12) and $C(1/2), D(1/2) < \infty$, we have that $\{i : d_i \in (0, \varepsilon)\}$ or $\{i : d_i \in (1 - \varepsilon, 1)\}$ are infinite for every $\varepsilon > 0$. This implies that at least one of the following holds:

$$\forall_{\alpha\in(0,1)}\ C(\alpha) > 0 \qquad \text{and} \qquad \lim_{\alpha\to0^+} D(\alpha) = \infty, \tag{3.13}$$

$$\forall_{\alpha\in(0,1)}\ D(\alpha) > 0 \qquad \text{and} \qquad \lim_{\alpha\to1^-} C(\alpha) = \infty. \tag{3.14}$$

First we shall prove that (i) or (ii) imply the existence of $0 < \varepsilon < 1 - \eta$ such that for any $A_1 \in (\eta, \eta + \varepsilon)$ and $A_2 \in (1 - \varepsilon, 1)$, there exist $A_3 < \ldots < A_n$ such that (3.9) holds.

CASE (i). Suppose that $\{d_i\}$ is a diagonal of a projection, i.e., $\eta = 0$, and both (3.13) and (3.14) hold. Thus, there exists $\varepsilon > 0$ such that

$$D(A_1) \geq \sum_{j=1}^{n} N_j \qquad \text{for all } A_1 \in (0, \varepsilon), \tag{3.15}$$

$$C(A_2) \geq N_1 + N_2 \qquad \text{for all } A_2 \in (1 - \varepsilon, 1). \tag{3.16}$$

Once A_1 and A_2 satisfying (3.15) and (3.16) are chosen we will show inductively that there exist $A_3 < \ldots < A_n$ such that (3.17) holds:

$$\forall_{r=2,\ldots,n}\ C(A_r) \geq \sum_{j=1}^{r} N_j \qquad \text{and} \qquad D(A_r) \geq \sum_{j=r+1}^{n} (1 - A_j)N_j. \tag{3.17}$$

Indeed, once $A_1, \ldots, A_{r-1}$, $r \geq 3$, are defined, by (3.14) we can choose A_r sufficiently close to 1 such that

$$C(A_r) \geq \sum_{j=1}^{r} N_j \qquad \text{and} \qquad (1 - A_r)\sum_{j=r}^{n} N_j \leq D(A_{r-1}).$$

Since

$$\sum_{j=r}^{n}(1-A_j)N_j \le (1-A_r)\sum_{j=r}^{n} N_j$$

this inductive process yields $(A_1,\dots,A_n)$ such that (3.15) and (3.17) are satisfied. Thus, (3.9) holds.

CASE (ii). We assume that $\eta > 0$, $N_1 = 1$, and (3.14) holds. There are two subcases to consider. Suppose that $C(\eta) > \eta$. By (3.14) this implies that there exists $1-\eta > \varepsilon > 0$ such that (3.16) holds and

$$C(A_1) \ge A_1 + \varepsilon \sum_{j=2}^{n} N_j \qquad \text{for all } A_1 \in (\eta, \eta+\varepsilon). \tag{3.18}$$

Hence, for any $A_1 \in (\eta,\eta+\varepsilon)$ and $A_2 \in (1-\varepsilon, 1)$ we have

$$(1-A_1)C(A_1)+A_1D(A_1) \ge (1-A_1)\Big(A_1 + (1-A_2)\sum_{j=2}^{n} N_j\Big) \ge A_1 \sum_{j=1}^{n}(1-A_j)N_j.$$

Suppose that $C(\eta) \le \eta$. Since $C(\alpha)-D(\alpha) \equiv \eta \mod 1$ for all α, by (3.14) we have $D(\eta) > 1-\eta$. Then again by (3.14) we can choose $1-\eta > \varepsilon > 0$ such that (3.16) holds and

$$D(A_1) \ge 1 - A_1 + \varepsilon \sum_{j=2}^{n} N_j \qquad \text{for } A_1 \in (\eta, \eta+\varepsilon). \tag{3.19}$$

Hence, for any $A_1 \in (\eta,\eta+\varepsilon)$ and $A_2 \in (1-\varepsilon, 1)$ we have

$$(1-A_1)C(A_1)+A_1D(A_1) \ge A_1\Big(1 - A_1 + (1-A_2)\sum_{j=2}^{n} N_j\Big) \ge A_1 \sum_{j=1}^{n}(1-A_j)N_j.$$

In either case, by an inductive argument as in case (i) one can show that there exist $A_3 < \ldots < A_n$ such that (3.17) holds. Thus, (3.9) holds.

It remains to prove that that we can find a solution to (3.9) which, in addition, satisfies (3.8). Choose $A_2 \in (0,1)$ close enough to 1 such that $1-A_2 < \varepsilon/\sum_{j=2}^{n} N_j$. Then,

$$\sum_{j=2}^{n} A_j N_j \equiv -\sum_{j=2}^{n}(1-A_j)N_j \equiv -x \mod 1,$$

for some $0 < x < \varepsilon$. Thus, by choosing $A_1 = \eta + x/N_1$ we have (3.8), and thus $(A_1,\dots,A_n) \in \mathcal{A}_n^{N_1,\dots,N_n}(\{d_i\})$. This completes the proof of Theorem 3.7 under assumptions (i) and (ii). Finally, case (iii) follows by symmetry from (ii). □

THEOREM 3.8. *Let $\{d_i\} \in \mathcal{F}$. The set $\mathcal{A}_n(\{d_i\})$ is nonempty for each $n \ge 2$ if and only if $\{i\colon d_i \in (0,1)\}$ is infinite.*

PROOF. Assume that $n \ge 2$ and (3.12) holds. Theorem 3.7 and the identity (3.11) shows that $\mathcal{A}_n \ne \varnothing$ unless we are in the special case when $\{d_i\}$ is a diagonal of a projection, $\eta = 0$, and only one of the sets $\{i \in I : d_i \in (0, 1/2)\}$ or $\{i \in I : d_i \in (1/2, 1)\}$ is infinite. Without loss of generality we can assume that $\{i \in I : d_i \in (1/2, 1)\}$ is finite since the other case is done by symmetry. This implies that

$$k_0 = \sum_{i\in I_0} d_i < \infty, \qquad \text{where } I_0 = \{i \in I : d_i \in [0,1)\}.$$

Moreover $\sum d_i = \infty$ implies that $I_1 = \{i \in I : d_i = 1\}$ is infinite.

Using the finite rank Schur-Horn theorem [**4**, Theorem 3.2] one can show that there exists a self-adjoint operator E_0 with diagonal $\{d_i\}_{i\in I_0}$ and spectrum $\sigma(E_0) = \{0, A_1, \ldots, A_n\}$ for some $0 < A_1 < \ldots < A_n$. This can be proved by an induction argument on $n \geq 2$. For the base case $n = 2$ we consider an eigenvalue sequence which consists of $A_1 = k_0(1 - A_2)$ and k_0 copies of A_2. It is easy to verify that this sequence fulfills majorization condition of [**4**, Theorem 3.2] when A_2 is sufficiently close to 1. For the inductive step suppose we have a finite rank operator with required diagonal and positive eigenvalues $A_1 < \ldots < A_n$, where eigenvalue A_1 has multiplicity 1. We split the smallest eigenvalue A_1 into two eigenvalues δ and $A_1 - \delta$, $\delta > 0$. Then one can show that the resulting eigenvalue sequence satisfies the assumptions of [**4**, Theorem 3.2] for sufficiently small $\delta > 0$. Observe that the operator $E = E_0 \oplus \mathbf{I}$, where $\mathbf{I}$ is the identity on $\ell^2(I_1)$, has spectrum $\sigma(E) = \sigma(E_0) \cup \{1\}$ and diagonal $\{d_i\}_{i\in I}$. Applying Theorem 2.7 implies that $(A_1, \ldots, A_n) \in \mathcal{A}_n$. Thus, $\mathcal{A}_n$ is nonempty.

Conversely, assume that $\mathcal{A}_n$ is nonempty for all $n \geq 2$. On the contrary, suppose that $I_2 = \{i \in I : d_i \in (0,1)\}$ is finite and has n elements. Since $\mathcal{A}_{n+1} \neq \varnothing$ there exists an operator E with spectrum $\sigma(E) = \{0, A_1, \ldots, A_{n+1}, 1\}$ and diagonal $\{d_i\}_{i\in I}$. Then, E can be decomposed as $E = E' \oplus P$, where E' acts on $\ell^2(I_2)$ and P is a projection, see [**4**, Proof of Theorem 5.1]. Consequently, E' acts on n dimensional space, but yet has at least $n + 1$ points in the spectrum. This contradiction finishes the proof of Theorem 3.8. □

In order to study more subtle properties of the set $\mathcal{A}_n(\{d_i\})$ it is useful to prove the following lemma.

LEMMA 3.9. *Let $\{d_i\}_{i\in I} \in \mathcal{F}$ and let η be as in Definition 3.2. Then the function $f : (0,1) \to (0,\infty)$ defined by $f(\alpha) = (1-\alpha)C(\alpha) + \alpha D(\alpha)$ is piecewise linear, continuous, concave, and it satisfies*

$$\lim_{\alpha\to 0^+} f(\alpha) = \lim_{\alpha\to 1^-} f(\alpha) = 0. \tag{3.20}$$

Moreover, $f'(\alpha) \equiv -\eta \pmod 1$ for every $\alpha \in (0,1) \setminus \{d_i : i \in I\}$.

PROOF. The continuity of f at each $\alpha \in (0,1) \setminus \{d_i : i \in I\}$ is clear from the definition. For $\alpha = d_{i_0}$ we see that

$$\lim_{\alpha\to d_{i_0}^-} f(\alpha) = (1 - d_{i_0}) \sum_{d_i < d_{i_0}} d_i + d_{i_0} \sum_{d_i \geq d_{i_0}} (1 - d_i) = f(d_{i_0}),$$

and

$$\begin{aligned}
\lim_{\alpha\to d_{i_0}^+} f(\alpha) &= (1 - d_{i_0}) \sum_{d_i \leq d_{i_0}} d_i + d_{i_0} \sum_{d_i > d_{i_0}} (1 - d_i) \\
&= (1 - d_{i_0}) \sum_{d_i < d_{i_0}} d_i + d_{i_0} \sum_{d_i > d_{i_0}} (1 - d_i) + (1 - d_{i_0}) d_{i_0} |\{i \in I : d_i = d_{i_0}\}| \\
&= (1 - d_{i_0}) \sum_{d_i < d_{i_0}} d_i + d_{i_0} \sum_{d_i \geq d_{i_0}} (1 - d_i) = f(d_{i_0}).
\end{aligned}$$

This shows that f is continuous on $(0,1)$.

The set $(0,1) \setminus \{d_i : i \in I\}$ is a countable collection of open intervals. On each of these intervals both $C(\alpha)$ and $D(\alpha)$ are constant, and thus $f'(\alpha) = D(\alpha) - C(\alpha) \equiv$

$-\eta \pmod 1$ for $\alpha \in (0,1)\setminus\{d_i \colon i \in I\}$. We deduce that f is linear on any subinterval $I \subset (0,1)$, which does not contain any of d_i's. Moreover, $f'(\alpha)$ is decreasing on the set where it is defined. Consequently, f is concave.

Finally, to prove (3.20) observe that for all $0 < \alpha < \beta < 1$

$$\begin{aligned}(1-\beta)C(\beta) &= (1-\beta)C(\alpha) + (1-\beta)\sum_{\alpha \le d_i < \beta} d_i \\ &\le (1-\beta)C(\alpha) + (1-\beta)\beta|\{i\colon \alpha \le d_i < \beta\}| \\ &\le (1-\beta)C(\alpha) + \beta \sum_{\alpha \le d_i < \beta}(1-d_i) = (1-\beta)C(\alpha) + \beta D(\alpha) - \beta D(\beta).\end{aligned}$$

Rearranging gives

$$f(\beta) = (1-\beta)C(\beta) + \beta D(\beta) \le (1-\beta)C(\alpha) + \beta D(\alpha). \tag{3.21}$$

It can similarly be shown that (3.21) holds for $0 < \beta < \alpha < 1$. From this we deduce that for any $\alpha \in (0,1)$

$$\limsup_{\beta \to 0^+} f(\beta) \le C(\alpha) \quad \text{and} \quad \limsup_{\beta \to 1^-} f(\beta) \le D(\alpha).$$

Since $C(\alpha) \to 0$ as $\alpha \to 0^+$, $D(\alpha) \to 0$ as $\alpha \to 1^-$, and $f(\beta) \ge 0$ for all $\beta \in (0,1)$ we have

$$\lim_{\beta \to 0^+} f(\beta) = \lim_{\beta \to 1^-} f(\beta) = 0.$$

□

REMARK 3.4. Suppose that $0 = A_0 < A_1 < \ldots < A_n < A_{n+1} = 1$, and $N_j \in \mathbb{N}$, $j = 1, \ldots, \mathbb{N}$. Observe that Lemma 3.9 can be also applied to a finite sequence $\{\widetilde{d}_j\}_{j=1}^{\sigma}$, where $\sigma = N_1 + \ldots + N_n$, is given by

$$(\widetilde{d}_1, \ldots, \widetilde{d}_\sigma) = (\underbrace{A_1, \ldots, A_1}_{N_1}, \underbrace{A_2, \ldots, A_2}_{N_2}, \ldots, \underbrace{A_n, \ldots, A_n}_{N_n}).$$

Let $g(\alpha) = (1-\alpha)\widetilde{C}(\alpha) + \alpha\widetilde{D}(\alpha)$, where $\widetilde{C}(\alpha)$ and $\widetilde{D}(\alpha)$ are defined by (2.6) for the sequence $\{\widetilde{d}_j\}_{j=1}^{\sigma}$. Then, the majorization inequalities (2.8) can be restated as

$$f(\alpha) \ge g(\alpha) \qquad \text{for } \alpha = A_r,\ r = 1, \ldots, n. \tag{3.22}$$

Since f is a concave function and g is a piecewise linear function with knots at $A_1, \ldots, A_n$, (3.22) is equivalent to

$$f(\alpha) \ge g(\alpha) \qquad \text{for all } \alpha \in (0,1). \tag{3.23}$$

Thus, the majorization inequality (2.8) is equivalent to an inequality (3.23) between two functions defined as above out of two sequences $\{d_i\}_{i\in I}$ and $\{\widetilde{d}_j\}_{j=1}^{\sigma}$, resp.

This observation will play a key role in the proof of the following lemma.

LEMMA 3.10. *Let $\{d_i\}_{i\in I} \in \mathcal{F}$ be a sequence in $[0,1]$. Let $N_j \in \mathbb{N}$, $j = 1, \ldots, n$. Suppose that $(A_1, \ldots, A_n) \in \mathcal{A}_n^{N_1,\ldots,N_n}(\{d_i\})$. Then, for any choice of $r, s = 1, \ldots, n$, $r < s$, and any $0 < A'_1 < \ldots < A'_n < 1$ satisfying:*

$$A_{r-1} < A'_r < A_r \qquad \text{and} \qquad A_s < A'_s < A_{s+1}, \tag{3.24}$$

$$A'_j = A_j, \qquad \text{for } j \ne r, s, \tag{3.25}$$

$$A'_r N_r + A'_s N_s = A_r N_r + A_s N_s, \tag{3.26}$$

we have $(A_1', \dots, A_n') \in \mathcal{A}_n^{N_1,\dots,N_n}(\{d_i\})$.

Proof. Let h be the function defined as in Lemma 3.9 corresponding to the sequence

$$(\underbrace{A_1', \dots, A_1'}_{N_1}, \underbrace{A_2', \dots, A_2'}_{N_2}, \dots, \underbrace{A_n', \dots, A_n'}_{N_n}).$$

By (3.26) we have that (3.8) holds for $\{(A_j', N_j)\}_{j=1}^n$. Moreover, in order to establish (3.9) for $\{(A_j', N_j)\}_{j=1}^n$, by Remark 3.4 it suffices to show that $g(\alpha) \ge h(\alpha)$ for all $\alpha \in (0,1)$. This follows immediately from the following identity

$$h(\alpha) - g(\alpha) = \begin{cases} 0 & \alpha \in (0, A_r') \cup (A_s', 1), \\ N_r(A_r' - \alpha) & \alpha \in (A_r', A_r), \\ N_r(A_r' - A_r) = N_s(A_s - A_s') & \alpha \in (A_r, A_s), \\ N_s(\alpha - A_s') & \alpha \in (A_s, A_s'). \end{cases} \tag{3.27}$$

The proof of (3.27) is a calculation which we present only for $\alpha \in (A_r', A_r)$. By comparing common terms appearing in g and h we see by (3.24) and (3.25) that they cancel out except when $j = r, s$. That is, by (3.26) we have

$$\begin{aligned} h(\alpha) - g(\alpha) &= (1-\alpha)N_r A_r' + \alpha N_s(1 - A_s') - \alpha N_r(1 - A_r) - \alpha N_s(1 - A_s) \\ &= N_r A_r' + \alpha N_s - \alpha A_r - \alpha N_s = N_r(A_r' - \alpha). \end{aligned}$$

This shows that (3.9) holds for $(A_1', \dots, A_n')$ and completes the proof of Lemma 3.10. ☐

As a consequence of Lemma 3.10 and some standard results in majorization theory [**25**] we have the following result which bears a close resemblance to the Schur-Horn Theorem 2.1.

Theorem 3.11. *Let* $\{d_i\}_{i \in I} \in \mathcal{F}$ *be a sequence in* $[0,1]$. *Let* $N_j \in \mathbb{N}$, $j = 1, \dots, n$. *Suppose that* $(A_1, \dots, A_n) \in \mathcal{A}_n^{N_1,\dots,N_n}(\{d_i\})$. *Then, for any* $0 < A_1' < \dots < A_n' < 1$ *satisfying*

$$\sum_{j=1}^n A_j' N_j = \sum_{j=1}^n A_j N_j \qquad \text{and} \qquad \sum_{j=1}^m A_j' N_j \le \sum_{j=1}^m A_j N_j \qquad \text{for } 1 \le m \le n,$$

we have $(A_1', \dots, A_n') \in \mathcal{A}_n^{N_1,\dots,N_n}(\{d_i\})$.

Proof. For simplicity we shall sketch the proof only in the special case $N_1 = \dots = N_n = 1$, where results from the majorization theory are readily applicable. By [**25**, Lemma 2.B.1] sequence $(A_1', \dots, A_n')$ can be derived from $(A_1, \dots, A_n)$ by a successive finite application of T-transforms, also known as convex moves. These operations correspond to successive applications of Lemma 3.10. Also by analyzing the proof of [**25**, Lemma 2.B.1] it is apparent that these T-transforms preserve strict monotonicity of $(A_1, \dots, A_n)$ at each step. ☐

3.3. Four point spectrum. As a consequence of Lemma 3.10 we have the following result about $\mathcal{A}_2^{N_1,N_2}(\{d_i\})$.

Lemma 3.12. *Let* $\{d_i\}_{i \in I} \in \mathcal{F}$ *be a sequence in* $[0,1]$ *and* $N_1, N_2 \in \mathbb{N}$. *Then, the set* $\mathcal{A}_2^{N_1,N_2}(\{d_i\})$ *consists of a finite number (possibly zero) of line segments with slopes* $-N_1/N_2$. *One endpoint of each of these line segments must lie in the boundary of the unit square* $(0,1)^2$.

Proof. By (2.7) any $(A_1, A_2) \in \mathcal{A}_2^{N_1,N_2}(\{d_i\})$ must satisfy for some $k \in \mathbb{Z}$

$$C(1/2) - D(1/2) = N_1 A_1 + N_2 A_2 + k.$$

The above equation defines a line with slope $-N_1/N_2$ in (A_1, A_2) plane which can intersect the unit square $(0,1)^2$ only for finitely many values of $k \in \mathbb{Z}$. By Lemma 3.10 for any point (A_1', A_2') with $0 < A_1' < A_1$ and $A_2 < A_2' < 1$, which lies on the same line as (A_1, A_2), we have $(A_1', A_2') \in A_2^{N_1,N_2}(\{d_i\})$. This together with Theorem 3.7 completes the proof. □

As a corollary of Theorem 3.8 and Lemma 3.12, we obtain the following description of the set $\mathcal{A}_2(\{d_i\})$.

Corollary 3.13. *Let $\{d_i\}_{i\in I} \in \mathcal{F}$ be a sequence in $[0,1]$ satisfying (3.12). Then, the set $\mathcal{A}_2(\{d_i\})$ is nonempty and it consists of a countable union of line segments. Moreover, one endpoint of each of these line segments must lie in the boundary of the unit square.*

We shall illustrate Corollary 3.13 for the symmetric geometric sequence already studied in [**17**].

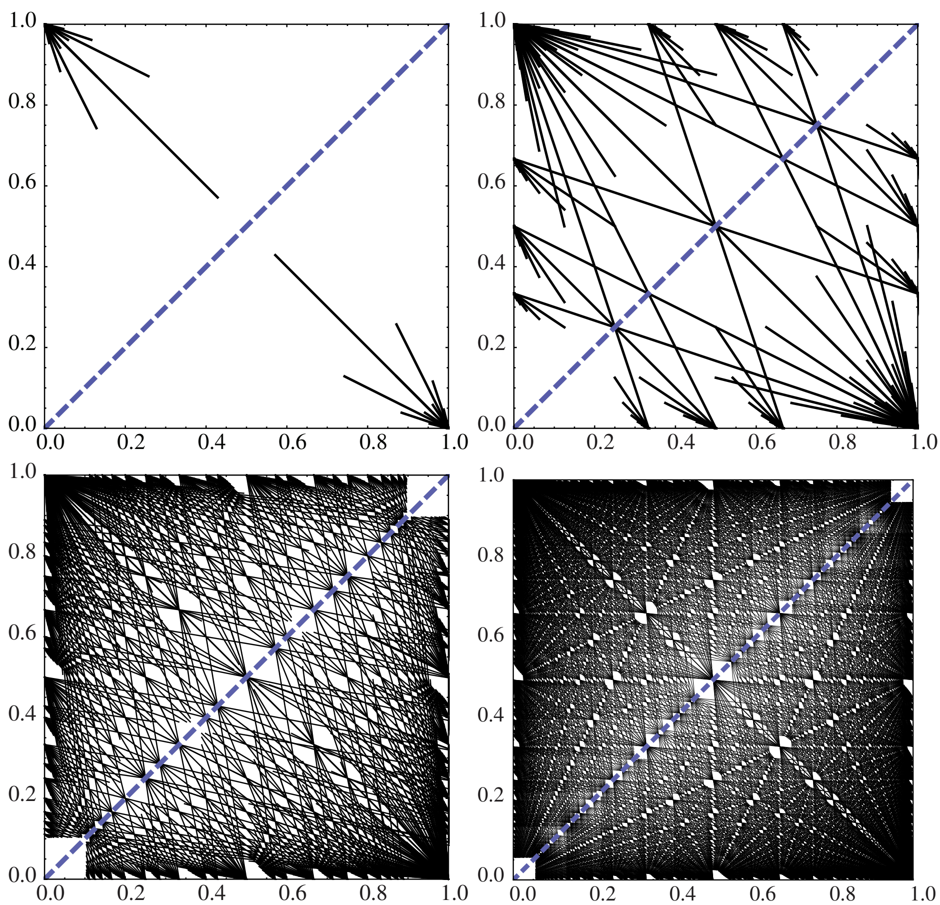

Figure 1. The sets $\mathcal{A}_2(\{d_i\})$ for $\beta = 0.3$, 0.5, 0.7, 0.8, resp.

Example 3.14. Let $\beta \in (0,1)$ and define the sequence $\{d_i\}_{i\in\mathbb{Z}\setminus\{0\}}$ by

$$d_i = \begin{cases} 1-\beta^i & i>0 \\ \beta^{-i} & i<0 \end{cases}.$$

Using the characterization from Theorem 2.7 and numerical calculations performed with *Mathematica*, Figure 1 depicts the set $\mathcal{A}_2(\{d_i\})$ for different values of the parameter β.

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