Weighted Anisotropic Hardy Spaces and Their Applications in Boundedness of Sublinear Operators

MARCIN BOWNIK, BAODE LI, DACHUN YANG & YUAN ZHOU

ABSTRACT. In this paper we introduce and study weighted anisotropic Hardy spaces $H_w^p(\mathbb{R}^n; A)$ associated with general expansive dilations and A_∞ Muckenhoupt weights. This setting includes the classical isotropic Hardy space theory of Fefferman and Stein, the parabolic theory of Calderón and Torchinsky, and the weighted Hardy spaces of García-Cuerva, Strömberg, and Torchinsky.

We establish characterizations of these spaces via the grand maximal function and their atomic decompositions for $p \in (0, 1]$. Moreover, we prove the existence of finite atomic decompositions achieving the norm in dense subspaces of $H_w^p(\mathbb{R}^n; A)$. As an application, we prove that for a given admissible triplet $(p, q, s)_w$, if T is a sublinear operator and maps all $(p, q, s)_w$-atoms with $q < \infty$ (or all continuous $(p, q, s)_w$-atoms with $q = \infty$) into uniformly bounded elements of some quasi-Banach space $\mathcal{B}$, then T uniquely extends to a bounded sublinear operator from $H_w^p(\mathbb{R}^n; A)$ to $\mathcal{B}$. The last two results are new even for the classical weighted Hardy spaces on $\mathbb{R}^n$.

1. INTRODUCTION

The theory of Hardy spaces on the Euclidean space $\mathbb{R}^n$ plays an important role in various fields of analysis and partial differential equations; see, for examples, [14, 16, 27, 29–31]. One of the most important applications of Hardy spaces is that they are good substitutes of Lebesgue spaces when $p \in (0,1]$. For example, when $p \in (0,1]$, it is well-known that Riesz transforms are not bounded on $L^p(\mathbb{R}^n)$, however, they are bounded on Hardy spaces $H^p(\mathbb{R}^n)$.

On the other hand, there were several efforts of extending classical function spaces arising in harmonic analysis from Euclidean spaces to other domains and non-isotropic settings; see [2, 10–12, 17, 34, 36–38]. Calderón and Torchinsky initiated the study of Hardy spaces on $\mathbb{R}^n$ with anisotropic dilations [10–12]. The theory of Hardy spaces associated to expansive dilations was recently developed in [2, 5]. The other direction of extending classical function spaces is the study of weighted function spaces associated with general Muckenhoupt weights; see [4, 6–9, 18]. García-Cuerva [18] and Strömberg and Torchinsky [33] established a theory of weighted Hardy spaces on $\mathbb{R}^n$.

To establish the boundedness of operators in Hardy spaces on $\mathbb{R}^n$, one usually appeals to the atomic decomposition characterization (see [13, 23]) or the molecular characterization (see [35]) of Hardy spaces, which means that a function or distribution in Hardy spaces can be represented as a linear combination of functions of an elementary form, namely, atoms or molecules. Then, the boundedness of linear operators in Hardy spaces can be deduced from their behavior on atoms or molecules in principle.

However, Meyer [25, p. 513] (see also [3, 19]) gave an example of $f \in H^1(\mathbb{R}^n)$ whose norm cannot be achieved by its finite atomic decompositions via $(1,\infty,0)$-atoms. Based on this fact, a surprising example was constructed in [3, Theorem 2] that there exists a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$, which maps all $(1,\infty,0)$-atoms into bounded scalars, but yet cannot extend to a bounded linear functional on the whole $H^1(\mathbb{R}^n)$. This implies that the uniform boundedness in some quasi-Banach space $\mathcal{B}$ of a linear operator T on all (p,∞,s)-atoms does not generally guarantee the boundedness of T from $H^p(\mathbb{R}^n)$ to $\mathcal{B}$. This phenomenon has also essentially already been observed by Y. Meyer in [26, p. 19]. Recall that a function a on $\mathbb{R}^n$ is a (p,q,s)-atom, where $p \in (0,1]$, $p < q \in [1,\infty]$ and integer $s \geq \lfloor n(1/p-1) \rfloor$, if it satisfies the following three conditions:

- (support) $\operatorname{supp} a \subset B$ for some ball $B \subset \mathbb{R}^n$,
- (size) $\|a\|_{L^q(\mathbb{R}^n)} \leq |B|^{1/q-1/p}$,
- (vanishing moments) $\displaystyle\int_{\mathbb{R}^n} a(x)x^\alpha \, \mathrm{d}x = 0$ for all $|\alpha| \leq s$.

Here and in what follows, $\lfloor \alpha \rfloor$ for any $\alpha \in \mathbb{R}$ denotes the integer no more than α.

Motivated by this, Yabuta [40] gave some sufficient conditions for the boundedness of T from $H^p(\mathbb{R}^n)$ with $p \in (0,1]$ to $L^q(\mathbb{R}^n)$ with $q \geq 1$ or $H^q(\mathbb{R}^n)$ with $q \in [p,1]$. Yabuta's results were generalized to the setting of spaces of homogeneous type in [21]. However, these conditions are not necessary. In [41], a boundedness criterion was established using Lusin function characterizations of Hardy spaces as follows: a sublinear operator T uniquely extends to a bounded sublinear operator from $H^p(\mathbb{R}^n)$ with $p \in (0,1]$ to some quasi-Banach space

$\mathcal{B}$ if and only if T maps all $(p,2,s)$-atoms into uniformly bounded elements of $\mathcal{B}$. This result shows the structural difference between atomic characterization of $H^p(\mathbb{R}^n)$ via $(p,2,s)$-atoms and (p,∞,s)-atoms, which was also generalized to Hardy spaces H^p with p close to 1 on spaces of homogeneous type having the reverse doubling property in [42].

Recently, Meda, Sjögren and Vallarino independently obtained some similar results by grand maximal function characterizations of Hardy spaces on $\mathbb{R}^n$. In fact, let $p \in (0,1]$, $p < q \in [1,\infty]$ and integer $s \geq \lfloor n(1/p-1)\rfloor$. Let $H^{p,q,s}_{\text{fin}}(\mathbb{R}^n)$ be the set of all finite linear combinations of (p,q,s)-atoms. For any $f \in H^{p,q,s}_{\text{fin}}(\mathbb{R}^n)$, define

$$\begin{aligned}(1.1)\quad &\|f\|_{H^{p,q,s}_{\text{fin}}(\mathbb{R}^n)}\\ &= \inf\Big\{\Big[\sum_{j=1}^{k}|\lambda_j|^p\Big]^{1/p} : f = \sum_{j=1}^{k}\lambda_j a_j,\ k\in\mathbb{N},\ \{a_j\}_{j=1}^{k} \text{ are } (p,q,s)\text{-atoms}\Big\}.\end{aligned}$$

Meda, Sjögren and Vallarino in [24] proved the following result.

Theorem 1.1. *Let $p \in (0,1]$, $p < q \in [1,\infty]$ and integer $s \geq \lfloor n(1/p-1)\rfloor$. The quasi-norms $\|\cdot\|_{H^{p,q,s}_{\text{fin}}(\mathbb{R}^n)}$ and $\|\cdot\|_{H^p(\mathbb{R}^n)}$ are equivalent on $H^{p,q,s}_{\text{fin}}(\mathbb{R}^n)$ when $q < \infty$ and on $H^{p,q,s}_{\text{fin}}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ when $q = \infty$. Here, $C(\mathbb{R}^n)$ denotes the set of all continuous functions.*

From this, they further deduced that if T is a linear operator and maps all $(1,q,0)$-atoms with $q \in (1,\infty)$ or all continuous $(1,q,0)$-atoms with $q=\infty$ into uniformly bounded elements of some Banach space $\mathcal{B}$, then T uniquely extends to a bounded linear operator from $H^1(\mathbb{R}^n)$ to $\mathcal{B}$ which coincides with T on these $(1,q,0)$-atoms. These results were generalized in [20] to Hardy spaces H^p with p close to 1 on spaces of homogeneous type having the reverse doubling property.

The main purpose of this paper is twofold. The first goal is to introduce weighted anisotropic Hardy spaces $H^p_w(\mathbb{R}^n;A)$ associated with an expansive dilation A and $w \in \mathcal{A}_\infty(\mathbb{R}^n;A)$ (the weight class of Muckenhoupt). This setting includes the classical isotropic theory of Fefferman-Stein [16], the parabolic theory of Calderón-Torchinsky [11,12], the anisotropic Hardy spaces of Bownik [2], and the weighted Hardy spaces of García-Cuerva [18] and Strömberg-Torchinsky [33]. We introduce weighted anisotropic Hardy spaces $H^p_w(\mathbb{R}^n;A)$ via grand maximal functions and then establish their weighted atomic decomposition characterizations extending the results in [2].

The second goal is to generalize Theorem 1.1 to our setting. More precisely, assume that $(p,q,s)_w$ is an admissible triplet (see Definition 3.2 below). Let $H^{p,q,s}_{w,\text{fin}}(\mathbb{R}^n;A)$ be the set of all finite linear combinations of $(p,q,s)_w$-atoms, and for any $f \in H^{p,q,s}_{w,\text{fin}}(\mathbb{R}^n;A)$, define $\|f\|_{H^{p,q,s}_{w,\text{fin}}(\mathbb{R}^n;A)}$ as in (1.1) with (p,q,s)-atoms replaced by $(p,q,s)_w$-atoms. Then we show that Theorem 1.1 also holds for the more general quasi-norms $\|\cdot\|_{H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)}$ and $\|\cdot\|_{H^p_w(\mathbb{R}^n;A)}$. As an application, we then prove that if T is a sublinear operator and maps all $(p,q,s)_w$-atoms with $q<\infty$ (or all continuous $(p,q,s)_w$-atoms with $q=\infty$) into uniformly bounded elements of some quasi-Banach space $\mathcal{B}$, then T uniquely extends to a bounded sublinear operator from $H^p_w(\mathbb{R}^n;A)$ to $\mathcal{B}$ which coincides with T on these $(p,q,s)_w$-atoms. This extends both the results of Meda-Sjögren-Vallarino [24] and Yang-Zhou [41].

The paper is organized as follows. In Section 2, we first recall some notation and definitions concerning expansive dilations, Muckenhoupt weights, Schwartz functions and grand maximal functions; and we then obtain a basic approximation of the identity result (see Proposition 2.9 below) and the grand maximal function characterization (see Proposition 2.11 below) for $L^q_w(\mathbb{R}^n)$ with $q\in(q_w,\infty]$, where q_w is the critical index of w (see (2.8) below). In Section 3, we introduce weighted anisotropic Hardy spaces $H^p_{w,N}(\mathbb{R}^n;A)$ via grand maximal functions and weighted atomic anisotropic Hardy spaces $H^{p,q,s}_w(\mathbb{R}^n;A)$ for any admissible triplet $(p,q,s)_w$. Some basic properties of these spaces are also presented in this section. Section 4 is devoted to generalizing the Calderón-Zygmund decomposition associated to the grand maximal function on anisotropic $\mathbb{R}^n$ in [2] to the weighted setting. Applying this, in Section 5, we further prove that for any admissible triplet $(p,q,s)_w$, $H^p_{w,N}(\mathbb{R}^n;A)=H^{p,q,s}_w(\mathbb{R}^n;A)$ with equivalent norms; see Theorem 5.5 below. Moreover, in Section 6, we prove that $\|\cdot\|_{H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)}$ and $\|\cdot\|_{H^p_w(\mathbb{R}^n;A)}$ are equivalent quasi-norms on $H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$ when $q<\infty$ and on $H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)\cap C(\mathbb{R}^n)$ when $q=\infty$. Finally, in Section 7 we obtain criterions for boundedness of sublinear operators in $H^p_w(\mathbb{R}^n;A)$ (see Theorem 7.2 below). The results in Sections 6 and 7 are also new even for the classical weighted Hardy spaces on $\mathbb{R}^n$.

We finally make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts do not change through the whole paper. Denote by $\mathbb{N}$ the set $\{1,2,\dots\}$ and by $\mathbb{Z}_+$ the set $\mathbb{N}\cup\{0\}$. We use $f\lesssim g$ to denote $f\le Cg$, $f\gtrsim g$ to denote $f\ge Cg$, and if $f\lesssim g\lesssim f$, we then write $f\sim g$. For a set A, we denote by $\sharp A$ its cardinality.

2. PRELIMILARIES

We begin with recalling the following notions and properties concerning expansive dilations in [2, 6].

Definition 2.1. A real $n\times n$ matrix A is said to be an *expansive matrix*, sometimes shortly a *dilation*, if $\min_{\lambda\in\sigma(A)}|\lambda|>1$, where $\sigma(A)$ is the set of all eigenvalues of A.

Throughout the paper, we always let A be *a fixed dilation* and $b \equiv |\det A|$. Let λ_- and λ_+ be *positive numbers* such that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \leq \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+.$$

Furthermore, if A is diagonalizable over $\mathbb{C}$, then we take $\lambda_- = \min\{|\lambda| : \lambda \in \sigma(A)\}$ and $\lambda_+ = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

It was proved in [2, Lemma 2.2] that for a given dilation A, there exist an open ellipsoid Δ and $r > 1$ such that $\Delta \subset r\Delta \subset A\Delta$, and one can additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the n-dimensional Lebesgue measure of the set Δ. Set $B_k = A^k\Delta$ for $k \in \mathbb{Z}$. Then B_k is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$. Throughout the whole paper, let σ be the *minimum integer* such that $r^\sigma \geq 2$ and for any subset E of $\mathbb{R}^n$, let $E^\complement = \mathbb{R}^n \setminus E$. Then for all $k \in \mathbb{Z}$,

$$B_k + B_k \subset B_{k+\sigma}, \tag{2.1}$$

$$B_k + (B_{k+\sigma})^\complement \subset (B_k)^\complement, \tag{2.2}$$

where $E+F$ denotes the algebraic sums $\{x+y : x \in E,\ y \in F\}$ of sets $E, F \subset \mathbb{R}^n$.

Define the *step homogeneous quasi-norm* ρ associated to A and Δ by that for all $x \in \mathbb{R}^n$,

$$\rho(x) = \sum_{k\in\mathbb{Z}} b^{k-1}\chi_{B_k\setminus B_{k-1}}(x). \tag{2.3}$$

Obviously, for all $k \in \mathbb{Z}$, $B_k = \{x \in \mathbb{R}^n : \ \rho(x) < b^k\}$. From (2.1) and (2.2), it follows that for all $x, y \in \mathbb{R}^n$,

$$\rho(x+y) \leq b^\sigma \max\{\rho(x), \rho(y)\} \leq b^\sigma[\rho(x) + \rho(y)];$$

see [2, p. 8]. Moreover, $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [15], where dx is the n-dimensional Lebesgue measure.

Recall that the homogeneous quasi-norm associated with A was introduced in [2, Definition 2.3] as follows.

Definition 2.2. A *homogeneous quasi-norm* associated with an expansive matrix A is a measurable mapping $\rho : \mathbb{R}^n \to [0, \infty)$ satisfying

(i) $\rho(x) = 0$ if and only if $x = 0$;
(ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$;
(iii) $\rho(x+y) \leq H[\rho(x) + \rho(y)]$ for all $x, y \in \mathbb{R}^n$, where H is a constant no less than 1.

In the standard dyadic case $A = 2I_{n\times n}$, $\rho(x) = |x|^n$ is an example of homogeneous quasi-norms associated with A, where and in what follows, $I_{n\times n}$ denotes the $n \times n$ *unit matrix* and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. It was proved that all homogeneous quasi-norms associated to a given dilation A are equivalent; see [2, Lemma 2.4]. Therefore, for a given expansive dilation A, in what follows, for the convenience, we always use the step homogeneous quasi-norm ρ as in (2.3).

The following inequalities concerning A, ρ and the Euclidean norm $|\cdot|$ established in [2, Section 2] are used in this paper: There exists a positive constant C such that

$$C^{-1}[\rho(x)]^{\ln(\lambda_-)/\ln b} \le |x| \le C[\rho(x)]^{\ln(\lambda_+)/\ln b} \quad \text{for all } \rho(x) \ge 1, \tag{2.4}$$

$$C^{-1}[\rho(x)]^{\ln(\lambda_+)/\ln b} \le |x| \le C[\rho(x)]^{\ln(\lambda_-)/\ln b} \quad \text{for all } \rho(x) < 1, \tag{2.5}$$

and

$$C^{-1}(\lambda_-)^j|x| \le |A^jx| \le C(\lambda_+)^j|x| \quad \text{for all } j \ge 0, \tag{2.6}$$

$$C^{-1}(\lambda_+)^j|x| \le |A^jx| \le C(\lambda_-)^j|x| \quad \text{for all } j < 0. \tag{2.7}$$

We also need the following slight variant of the Whitney covering lemma, which generalizes Lemma 2.7 of [2]. Here we borrow some ideas from Lemma 2 in [30, p. 15].

Lemma 2.3. *Let Ω be an open proper subset of $\mathbb{R}^n$. Then for each integer $d \ge 0$, there exist a positive constant L depending only on d, a sequence $\{x_j\}_j \subset \Omega$ and a sequence $\{\ell_j\}_j$ of integers such that*

(i) $\Omega = \bigcup_j (x_j + B_{\ell_j})$,

(ii) $(x_i + B_{\ell_i - 2\sigma}) \cap (x_j + B_{\ell_j - 2\sigma}) = \emptyset$ *for all i, j with $i \ne j$,*

(iii) $(x_j + B_{\ell_j + d}) \cap \Omega^\complement = \emptyset$ *and* $(x_j + B_{\ell_j + d + 1}) \cap \Omega^\complement \ne \emptyset$ *for all j,*

(iv) $(x_i + B_{\ell_i + d - 2\sigma}) \cap (x_j + B_{\ell_j + d - 2\sigma}) \ne \emptyset$ *implies that* $|\ell_i - \ell_j| \le \sigma$,

(v) $\sharp\{j : (x_i + B_{\ell_i + d - 2\sigma}) \cap (x_j + B_{\ell_j + d - 2\sigma}) \ne \emptyset\} \le L$ *for all i.*

Proof. For any $x \in \Omega$, let $\ell(x) = \max\{\ell \in \mathbb{Z} : x + B_\ell \subset \Omega\}$. We first claim that $\ell(x) \in \mathbb{Z}$. To see this, since $\Omega^\complement \ne \emptyset$, we let $z \in \Omega^\complement$; then for any $x \in \mathbb{R}^n$, we have $b^{\ell(x)} \le \rho(x - z) < \infty$ and thus $\ell(x) < \ln[\rho(x - z)]/\ln b < \infty$. On the other hand, for any given $x \in \Omega$, the fact that Ω is open implies that there exists a $\delta \in (0, 1)$ such that $\{y \in \mathbb{R}^n : |x - y| < \delta\} \subset \Omega$. By (2.1) and (2.5), for any $z \in B_k$ with $k = \lfloor \ln(\delta/C)/\ln(\lambda_-) \rfloor - 1 < 0$ and C as in (2.5), we have $|z| < \delta$, which implies that $x + B_k \subset \{y \in \mathbb{R}^n : |x - y| < \delta\}$ and therefore $\ell(x) \ge k > -\infty$. Thus, the claim holds.

Obviously, the collection $\{x + B_{\ell(x) - d - 2\sigma}\}_{x \in \Omega}$ forms a cover of Ω. Now, let $\{x_j + B_{\ell(x_j) - d - 2\sigma}\}_j$ be a maximal disjoint subcollection of Ω, namely, for any i, j with $i \ne j$, $(x_i + B_{\ell(x_i) - d - 2\sigma}) \cap (x_j + B_{\ell(x_j) - d - 2\sigma}) = \emptyset$, and for any $x \in \mathbb{R}^n$, there exists k such that $(x + B_{\ell(x) - d - 2\sigma}) \cap (x_k + B_{\ell(x_k) - d - 2\sigma}) \ne \emptyset$.

For all j, set $\ell_j = \ell(x_j) - d$. Obviously, (ii) and (iii) hold.

To prove (i), for any $x \in \Omega$, there exists i such that $(x + B_{\ell(x) - d - 2\sigma}) \cap (x_i + B_{\ell(x_i) - d - 2\sigma}) \ne \emptyset$. We claim that $|\ell(x) - \ell(x_i)| \le \sigma$. If this is true, then by (2.1),

$$x - x_i \subset B_{\ell(x)-d-2\sigma} + B_{\ell(x_i)-d-2\sigma} \\ \subset B_{\ell(x_i)-d-\sigma} + B_{\ell(x_i)-d-2\sigma} \subset B_{\ell(x_i)-d},$$

which implies that $x \in x_i + B_{\ell(x_i)-d} = x_i + B_{\ell_i}$ and thus gives (i). To prove the claim, if $\ell(x) \geq \ell(x_i)+\sigma+1$, then by (2.1), $x_i - x \subset B_{\ell(x_i)-d-2\sigma} + B_{\ell(x)-d-2\sigma} \subset B_{\ell(x)-d-\sigma}$, which together with (2.2) implies that

$$x_i + B_{\ell(x_i)+1} \subset x + (x_i - x) + B_{\ell(x_i)+1} \\ \subset x + B_{\ell(x)-d-\sigma} + B_{\ell(x)-\sigma} \subset x + B_{\ell(x)} \subset \Omega.$$

This implies that $\ell(x_i) + 1 \leq \ell(x_i)$ by the definition of $\ell(x_i)$, which is a contradiction. Thus $\ell(x) \leq \ell(x_i) + \sigma$. By interchanging the roles of x and x_i, we also have $\ell(x_i) \leq \ell(x) + \sigma$, which verifies the claim and hence, (i).

The proofs for (iv) and (v) are, respectively, as in Lemma 2.7 (iv) and (v) of [2]. We omit the details. This completes the proof of Lemma 2.3. $\square$

Remark 2.4. Lemma 2.3 when $|\Omega| < \infty$ is just Lemma 2.7 of [2] except that Lemma 2.3 (ii) is replaced by $(x_i + B_{\ell_i - \sigma}) \cap (x_j + B_{\ell_j - \sigma}) = \emptyset$ for all i, j with $i \neq j$.

For any locally integrable function f, the *Hardy-Littlewood maximal function* $M(f)$ is defined by

$$M(f)(x) \equiv \sup_{k \in \mathbb{Z}} \sup_{y \in x + B_k} \frac{1}{|B_k|} \int_{y+B_k} |f(z)|\, dz, \quad x \in \mathbb{R}^n.$$

Bownik in [2, Theorem 3.6] proved that M is bounded on $L^p(\mathbb{R}^n)$ with $p \in (1, \infty]$ and bounded from $L^1(\mathbb{R}^n)$ to weak-$L^1(\mathbb{R}^n)$.

Recall that the weight class of Muckenhoupt associated to A was introduced in [6].

Definition 2.5. Let $p \in (1, \infty)$ and w be a nonnegative measurable function on $\mathbb{R}^n$. The function w is said to belong to the *weight class of Muckenhoupt* $\mathcal{A}_p \equiv \mathcal{A}_p(\mathbb{R}^n; A)$, if there exists a positive constant C such that

$$\sup_{x \in \mathbb{R}^n} \sup_{k \in \mathbb{Z}} \left\{ \frac{1}{|B_k|} \int_{x+B_k} w(y)\, dy \right\} \left\{ \frac{1}{|B_k|} \int_{x+B_k} [w(y)]^{-1/(p-1)}\, dy \right\}^{p-1} \leq C.$$

The function w is said to belong to the weight class of Muckenhoupt $\mathcal{A}_1 \equiv \mathcal{A}_1(\mathbb{R}^n; A)$, if there exists a positive constant C such that

$$\sup_{x \in \mathbb{R}^n} \sup_{k \in \mathbb{Z}} \left\{ \frac{1}{|B_k|} \int_{x+B_k} w(y)\, dy \right\} \left\{ \sup_{y \in x+B_k} [w(y)]^{-1} \right\} \leq C.$$

Define $\mathcal{A}_\infty \equiv \bigcup_{1 \leq p < \infty} \mathcal{A}_p$.

Recall that $(\mathbb{R}^n, \rho, dx)$ is a space of homogeneous type. For some basic properties of $\mathcal{A}_p$ weights, we refer the reader to [19, Chapter IV], [30, Chapter V], [33], and [39]. Here we only state some properties that will be used later. In fact, it is easy to see that $\mathcal{A}_p \subset \mathcal{A}_q$ for $1 \le p < q \le \infty$. If $w \in \mathcal{A}_p$ with $p \in (1, \infty)$, then there exists an $\varepsilon \in (0, p-1]$ such that $w \in \mathcal{A}_{p-\varepsilon}$ by the reverse Hölder inequality. For any given $w \in \mathcal{A}_\infty$, define the *critical index* of w by

$$q_w \equiv \inf\{p \in [1, \infty) : w \in \mathcal{A}_p\}. \tag{2.8}$$

Obviously, $q_w \in [1, \infty)$. If $q_w \in (1, \infty)$, then $w \notin \mathcal{A}_{q_w}$; and if $q_w = 1$, Johnson and Neugebauer [22, p. 254] gave an example of $w \notin \mathcal{A}_1$ such that $q_w = 1$.

In what follows, for any $w \in \mathcal{A}_\infty$ and any Lebesgue measurable set E, let $w(E) = \int_E w(x)\,dx$. For any $w \in \mathcal{A}_\infty$, $L^p_w(\mathbb{R}^n)$ with $p \in (0, \infty)$ denotes the *set of all measurable functions* f such that

$$\|f\|_{L^p_w(\mathbb{R}^n)} \equiv \left\{ \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \right\}^{1/p} < \infty,$$

and $L^\infty_w(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. The space weak-$L^1_w(\mathbb{R}^n)$ denotes the *set of all measurable functions* f such that

$$\|f\|_{\text{weak}-L^1_w(\mathbb{R}^n)} \equiv \sup_{\lambda > 0} \lambda \cdot w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) < \infty.$$

Moreover, we have the following conclusions.

Proposition 2.6.

(i) *Let $p \in [1, \infty)$ and $w \in \mathcal{A}_p$. Then there exists a positive constant C such that for all $x \in \mathbb{R}^n$ and $k, m \in \mathbb{Z}$ with $k \le m$,*

$$C^{-1} b^{(m-k)/p} w(x + B_k) \le w(x + B_m) \le C b^{(m-k)p} w(x + B_k).$$

(ii) *Let $w \in \mathcal{A}_\infty$. Then the Hardy-Littlewood maximal operator M is bounded on $L^p_w(\mathbb{R}^n)$ if and only if $w \in \mathcal{A}_p$ with $p \in (1, \infty)$; and M is bounded from $L^1_w(\mathbb{R}^n)$ to weak-$L^1_w(\mathbb{R}^n)$ if and only if $w \in \mathcal{A}_1$.*

Proof. Proposition 2.6 (ii) is just Theorem 2.4 of [6].

To see (i), recall that if $w \in \mathcal{A}_p$, then for any measurable sets $E \subset B$,

$$\left(\frac{|B|}{|E|}\right)^{1/p} \lesssim \frac{w(B)}{w(E)} \lesssim \left(\frac{|B|}{|E|}\right)^p$$

(see [33, pp. 7–8]). For any $x \in \mathbb{R}^n$ and $k, m \in \mathbb{Z}$ with $k \le m$, if we set $B = x + B_m$ and $E = x + B_k$, then

$$w(x + B_m) \lesssim \left(\frac{|B_m|}{|B_k|}\right)^p w(x + B_k) \lesssim b^{(m-k)p} w(x + B_k),$$

and

$$w(x + B_m) \gtrsim \left(\frac{|B_m|}{|B_k|}\right)^{1/p} w(x + B_k) \gtrsim b^{(m-k)/p} w(x + B_k).$$

This completes the proof of Proposition 2.6. □

We remark that Proposition 2.6 (i) implies that the measure $w(x)\,\mathrm{d}x$ is doubling and thus $(\mathbb{R}^n, \rho, w(x)\,\mathrm{d}x)$ is also a space of homogenous type.

Now we recall the space of Schwartz functions and its dual space in [2].

Definition 2.7. A complex valued function φ on $\mathbb{R}^n$ is said to belong to the *Schwartz class* $S(\mathbb{R}^n)$, if φ is infinitely differentiable and for every $\alpha \in (\mathbb{Z}_+)^n$ and $m \in \mathbb{Z}_+$,

$$\|\varphi\|_{\alpha,m} \equiv \sup_{x\in\mathbb{R}^n} [\rho(x)]^m\, |\partial^\alpha \varphi(x)| < \infty,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$.

The space $S(\mathbb{R}^n)$ endowed with pseudonorms $\{\|\cdot\|_{\alpha,m}\}_{\alpha\in(\mathbb{Z}_+)^n, m\in\mathbb{Z}_+}$ becomes a complete locally convex topological vector space. Moreover, from (2.4) and (2.5), it follows that $S(\mathbb{R}^n)$ coincides with the classical space of Schwartz functions. The *dual space* of $S(\mathbb{R}^n)$, i.e., the *space of tempered distributions on* $\mathbb{R}^n$, is denoted by $S'(\mathbb{R}^n)$.

Lemma 2.8. *Let $w \in \mathcal{A}_\infty$, q_w be as in (2.8), and $p \in (q_w, \infty]$. Then*

(i) *if $1/p + 1/p' = 1$, then $S(\mathbb{R}^n) \subset L^{p'}_{w^{-1/(p-1)}}(\mathbb{R}^n)$;*
(ii) *$L^p_w(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ and the inclusion is continuous.*

Proof. We only prove the case $p < \infty$. The proof for the case $p = \infty$ is easier and we omit the details. Since $p \in (q_w, \infty)$, then $w \in \mathcal{A}_p$. Therefore, by the definition of $\mathcal{A}_p$, for all $k \in \mathbb{Z}$, we have

$$\int_{B_k} [w(x)]^{-1/(p-1)}\,\mathrm{d}x \lesssim [w(B_k)]^{-1/(p-1)}\, |B_k|^{p'}, \tag{2.9}$$

where $1/p + 1/p' = 1$. Then, by (2.9), for any $\varphi \in S(\mathbb{R}^n)$, we have

$$\begin{aligned} &\|\varphi\|_{L^{p'}_{w^{-1/(p-1)}}(\mathbb{R}^n)} \\ &\quad\lesssim \left(\|\varphi\|_{0,0} + \|\varphi\|_{0,2}\right)\left\{1 + \sum_{k=1}^{\infty} \int_{B_k\setminus B_{k-1}} \frac{1}{[\rho(x)]^{2p'}} [w(x)]^{-1/(p-1)}\,\mathrm{d}x\right\}^{1/p'} \\ &\quad\lesssim \left(\|\varphi\|_{0,0} + \|\varphi\|_{0,2}\right) < \infty. \end{aligned} \tag{2.10}$$

Thus (i) holds.

To see (ii), for any $f \in L^p_w(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, by the Hölder inequality and (2.10), we have

$$\begin{aligned}|\langle f, \varphi\rangle| &\le \|f\|_{L^p_w(\mathbb{R}^n)} \left\{\int_{\mathbb{R}^n} |\varphi(x)|^{p'} [w(x)]^{-1/(p-1)}\,dx\right\}^{1/p'} \\ &\lesssim \|f\|_{L^p_w(\mathbb{R}^n)} (\|\varphi\|_{0,0} + \|\varphi\|_{0,2}),\end{aligned}$$

which implies the desired conclusions of (ii) and hence, completes the proof of Lemma 2.8. ◻

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{Z}$, set

$$\varphi_k(x) \equiv b^{-k}\varphi(A^{-k}x) \tag{2.11}$$

for all $x \in \mathbb{R}^n$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. Then $\{\varphi_k\}_{k\in\mathbb{Z}}$ forms *an approximation of the identity*. Precisely, we have the following conclusions.

Proposition 2.9. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$.*

(i) *For any $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, $\psi * \varphi_k \to \psi$ in $\mathcal{S}(\mathbb{R}^n)$ as $k \to -\infty$ and $f * \varphi_k \to f$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \to -\infty$.*
(ii) *Let $w \in \mathcal{A}_\infty$ and q_w be as in (2.8). If $q \in (q_w, \infty)$, then for any $f \in L^q_w(\mathbb{R}^n)$, $f * \varphi_k \to f$ in $L^q_w(\mathbb{R}^n)$ as $k \to -\infty$.*

Proof. Proposition 2.9 (i) is just Proposition 3.8 of [2].

To prove (ii), denote by $L^\infty_c(\mathbb{R}^n)$ the set of all bounded functions with compact support. Obviously, $L^\infty_c(\mathbb{R}^n)$ is dense in $L^q_w(\mathbb{R}^n)$. For any $g \in L^\infty_c(\mathbb{R}^n)$, since $L^\infty_c(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, by Theorem 1.25 in [32, p. 13], we know that $g * \varphi_k \to g$ almost everywhere as $k \to -\infty$. Recall that $q_w \in [1, \infty)$. By $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have $|\varphi_k * g(x)| \lesssim M(g)(x)$ for all $x \in \mathbb{R}^n$, which together with Proposition 2.6 (ii) and the Lebesgue dominated convergence theorem implies that $g * \varphi_k \to g$ in $L^q_w(\mathbb{R}^n)$ with $q \in (q_w, \infty)$ as $k \to -\infty$. From this and the density of $L^\infty_c(\mathbb{R}^n)$ in $L^q_w(\mathbb{R}^n)$, it further follows the desired conclusion (ii), which completes the proof of Proposition 2.9. ◻

Let $N \in \mathbb{Z}_+$ and

$$\mathcal{S}_N(\mathbb{R}^n) \equiv \Big\{\varphi \in \mathcal{S}(\mathbb{R}^n) : \|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)} \equiv \sup_{x\in\mathbb{R}^n} \sup_{|\alpha|\le N} |\partial^\alpha \varphi(x)|\,[1+\rho(x)]^N \le 1\Big\}.$$

Definition 2.10. Let $N \in \mathbb{Z}_+$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, the *nontangential grand maximal function* $M_N(f)$ of f is defined for all $x \in \mathbb{R}^n$ by

$$M_N(f)(x) \equiv \sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)} \sup_{k\in\mathbb{Z}} \sup_{y\in x+B_k} |f * \varphi_k(y)|,$$

and the *radial grand maximal function* $M_N^0(f)$ of f is defined for all $x \in \mathbb{R}^n$ by

$$M_N^0(f)(x) \equiv \sup_{\varphi \in S_N(\mathbb{R}^n)} \sup_{k \in \mathbb{Z}} |f * \varphi_k(x)|,$$

where φ_k for $k \in \mathbb{Z}$ is as in (2.11).

For every $N \in \mathbb{Z}_+$, there exists a positive constant C such that for all $f \in S'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M_N^0(f)(x) \le M_N(f)(x) \le CM_N^0(f)(x); \tag{2.12}$$

see [2, Proposition 3.10]. Moreover, we have the following conclusions.

Proposition 2.11. *Let A be an expansive dilation and $N \ge 2$.*

(i) *There exists a positive constant C such that for all $f \in (L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap S'(\mathbb{R}^n))$ and almost everywhere $x \in \mathbb{R}^n$, $|f(x)| \le M_N^0(f)(x) \le CM(f)(x)$.*

(ii) *If $w \in \mathcal{A}_p$ with $p \in (1, \infty]$, then $f \in L^p_w(\mathbb{R}^n)$ if and only if $f \in S'(\mathbb{R}^n)$ and $M_N^0(f) \in L^p_w(\mathbb{R}^n)$; moreover, $\|f\|_{L^p_w(\mathbb{R}^n)} \sim \|M_N^0(f)\|_{L^p_w(\mathbb{R}^n)}$.*

(iii) *If $w \in \mathcal{A}_1$, then M_N^0 is bounded from $L^1_w(\mathbb{R}^n)$ to weak-$L^1_w(\mathbb{R}^n)$.*

Proof. Let $\varphi \in S_N(\mathbb{R}^n)$ have compact support. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, write $f = \sum_{q \in \mathcal{J}} (f\chi_Q)$, where $\mathcal{J}$ denotes the set of all classical dyadic cubes of $\mathbb{R}^n$ with side length 1 and χ_Q denotes the characteristic function of Q. Then obviously, for all $x \in \mathbb{R}^n$,

$$\varphi_k * f(x) = \sum_{Q \in \mathcal{J}} \varphi_k * (f\chi_Q)(x).$$

Since $f\chi_Q \in L^1(\mathbb{R}^n)$, by [32, Theorem 1.25], we have that for almost everywhere $x \in \mathbb{R}^n$, $\varphi_k * (f\chi_Q)(x) \to f(x)\chi_Q(x)$, which further implies $\varphi_k * f(x) \to f(x)$. Thus, for almost everywhere $x \in \mathbb{R}^n$, $|f(x)| \le M_N^0(f)(x)$.

On the other hand, for any $\varphi \in S_N(\mathbb{R}^n)$, since $N \ge 2$, we have that for all $k \in \mathbb{Z}$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$|(\varphi_k * f)(x)| \le \int_{\mathbb{R}^n} \frac{b^{-k}}{[1 + b^{-k}\rho(x-y)]^2} |f(y)|\,\mathrm{d}y \lesssim M(f)(x),$$

which implies that for all $x \in \mathbb{R}^n$, $M_N^0(f)(x) \lesssim M(f)(x)$. This verifies (i).

By (i) and Proposition 2.6 (ii), we have that if $w \in \mathcal{A}_1$, then M_N^0 are bounded from $L^1_w(\mathbb{R}^n)$ to weak-$L^1_w(\mathbb{R}^n)$, which gives (iii).

To see (ii), if $f \in S'(\mathbb{R}^n)$ and $M_N^0(f) \in L^p_w(\mathbb{R}^n)$, obviously, $\{f * \varphi_k : k \in \mathbb{Z}\}$ is bounded in $L^p_w(\mathbb{R}^n)$. By the Alaoglu theorem there exists a subsequence $\{k_j\}_{j\in\mathbb{N}}$ with $k_j \to -\infty$ such that $\{f * \varphi_{k_j}\}_{j\in\mathbb{N}}$ converges weak-$*$ in $L^p_w(\mathbb{R}^n)$. Notice that $(L^p_w(\mathbb{R}^n))^* = L^{p'}_{w^{-1/(p-1)}}(\mathbb{R}^n)$. By Lemma 2.8 (i), we know that $\{f * \varphi_{k_j}\}_{j\in\mathbb{N}}$ converges also in $\mathcal{S}'(\mathbb{R}^n)$. By Proposition 2.9 (i), this limit is just f and thus $f \in L^p_w(\mathbb{R}^n)$, which together with (i) implies that $\|f\|_{L^p_w(\mathbb{R}^n)} \lesssim \|M^0_N(f)\|_{L^p_w(\mathbb{R}^n)}$. Conversely, if $f \in L^p_w(\mathbb{R}^n)$, by $w \in \mathcal{A}_p$ and Lemma 2.8 (ii), we see that $f \in (L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n))$. Thus, by (i) and Proposition 2.6 (ii), we have $M^0_N(f) \in L^p_w(\mathbb{R}^n)$ and $\|M^0_N(f)\|_{L^p_w(\mathbb{R}^n)} \lesssim \|f\|_{L^p_w(\mathbb{R}^n)}$, which gives (ii) and hence completes the proof of Proposition 2.11. ∎

We remark that by (2.12), Proposition 2.11 still holds with M^0_N replaced by M_N.

3. The Grand Maximal Function Definition of Hardy Spaces

In this section, we introduce weighted anisotropic Hardy spaces via grand maximal functions and weighted anisotropic atomic Hardy spaces. Some basic properties of these spaces are also presented.

Definition 3.1. Let $p \in (0,\infty]$, A be an expansive dilation, $w \in \mathcal{A}_\infty$, and q_w be as in (2.8). Set

$$N_{p,w} \equiv \begin{cases} \left\lfloor \left(\dfrac{q_w}{p} - 1\right) \dfrac{\ln b}{\ln(\lambda_-)} \right\rfloor + 2, & p \le q_w; \\ 2, & p > q_w. \end{cases}$$

For each $N \ge N_{p,w}$, the *weighted anisotropic Hardy space* associated with the dilation A is defined by

$$H^p_{w,N}(\mathbb{R}^n; A) \equiv \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : M_N(f) \in L^p_w(\mathbb{R}^n) \right\}.$$

Moreover, we define $\|f\|_{H^p_{w,N}(\mathbb{R}^n;A)} \equiv \|M_N(f)\|_{L^p_w(\mathbb{R}^n)}$.

For any integers N, $\tilde{N}$ with $N_{p,w} \le N \le \tilde{N}$, since the facts that $\mathcal{S}_{\tilde{N}}(\mathbb{R}^n) \subset \mathcal{S}_N(\mathbb{R}^n) \subset \mathcal{S}_{N_{p,w}}(\mathbb{R}^n)$ imply that $M_{\tilde{N}}(f)(x) \le M_N(f)(x) \le M_{N_{p,w}}(f)(x)$ for all $x \in \mathbb{R}^n$, we have

$$H^p_{w,N_{p,w}}(\mathbb{R}^n; A) \subset H^p_{w,N}(\mathbb{R}^n; A) \subset H^p_{w,\tilde{N}}(\mathbb{R}^n; A) \tag{3.1}$$

and the inclusions are continuous.

Notice that if $p \in (q_w, \infty]$ and $N \ge N_{p,w} = 2$, then by Proposition 2.11 (ii), we have $H^p_{w,N}(\mathbb{R}^n; A) = L^p_w(\mathbb{R}^n)$ with equivalent norms. However, if $p \in (1, q_w)$, the element of $H^p_{w,N}(\mathbb{R}^n; A)$ may be a distribution, and hence, $H^p_{w,N}(\mathbb{R}^n; A) \neq L^p_w(\mathbb{R}^n)$ (see [33, p. 86]); but, by Proposition 2.11 (i), we have $(H^p_{w,N}(\mathbb{R}^n; A) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)) \subset L^p_w(\mathbb{R}^n)$. For applications considered in this paper, we concentrate only on $H^p_{w,N}(\mathbb{R}^n; A)$ with $p \in (0, 1]$.

We remark that if $w \equiv 1$, then $H^p_{w,N}(\mathbb{R}^n; A)$ is just the Hardy space $H^p_A(\mathbb{R}^n)$ in [2] when $p \in (0,1]$ and $L^p(\mathbb{R}^n)$ when $p \in (1,\infty]$ (see [2, p. 17]), and the index $N_{p,w}$ just coincides with the N_p therein (see [2, p. 17]).

We introduce the following weighted anisotropic atoms.

Definition 3.2. Let A be an expansive dilation, $w \in \mathcal{A}_\infty$ and q_w be as in (2.8). A triplet $(p,q,s)_w$ is called to be *admissible*, if $p \in (0,1]$, $q \in (q_w, \infty]$ and $s \in \mathbb{N}$ with $s \geq \lfloor (q_w/p - 1)\ln b/\ln(\lambda_-)\rfloor$. A function a on $\mathbb{R}^n$ is said to be a $(p,q,s)_w$*-atom* if

(i) $\operatorname{supp} a \subset x_0 + B_j$ for some $j \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^n$,

(ii) $\|a\|_{L^q_w(\mathbb{R}^n)} \leq [w(x_0 + B_j)]^{1/q - 1/p}$,

(iii) $\displaystyle\int_{\mathbb{R}^n} a(x)x^\alpha\,dx = 0$ for $\alpha \in (\mathbb{Z}_+)^n$ with $|\alpha| \leq s$.

When $w \equiv 1$, we write (p,q,s)-atom instead of $(p,q,s)_w$-atom.

We remark that if $A = 2I_{n\times n}$, $w \in \mathcal{A}_\infty$ and $p \in (0,1]$, $H^p_{w,N}(\mathbb{R}^n; A)$ is just the weighted Hardy space in [18, 33] and the least vanishing moment of atoms, $\lfloor (q_w/p - 1)\ln b/\ln \lambda_-\rfloor$, in this case becomes $\lfloor (q_w/p - 1)n\rfloor$ which coincides with the index in [18, 33].

Definition 3.3. Let A be an expansive dilation, $w \in \mathcal{A}_\infty$ and $(p,q,s)_w$ be an admissible triplet. The *weighted atomic anisotropic Hardy space* $H^{p,q,s}_w(\mathbb{R}^n; A)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that $f = \sum_{i=1}^\infty \lambda_i a_i$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$, $\sum_{i=1}^\infty |\lambda_i|^p < \infty$, and $\{a_i\}_{i\in\mathbb{N}}$ are $(p,q,s)_w$-atoms. Moreover, the *quasi-norm* of $f \in H^{p,q,s}_w(\mathbb{R}^n; A)$ is defined by

$$\|f\|_{H^{p,q,s}_w(\mathbb{R}^n;A)} \equiv \inf\Big\{\Big[\sum_{i=1}^\infty |\lambda_i|^p\Big]^{1/p}\Big\},$$

where the infimum is taken over all the decompositions of f as above.

It is easy to see that if the triplets $(p,q,s)_w$ and $(p,\tilde{q},\tilde{s})_w$ are admissible and satisfy $\tilde{q} \leq q$ and $\tilde{s} \leq s$, then $(p,q,s)_w$-atoms are $(p,\tilde{q},\tilde{s})_w$-atoms, which further implies that $H^{p,q,s}_w(\mathbb{R}^n; A) \subset H^{p,\tilde{q},\tilde{s}}_w(\mathbb{R}^n; A)$ and the inclusion is continuous.

Though $(\mathbb{R}^n, \rho, w(x)\,dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [15], the atoms in Definition 3.2 are different from those in [15] since the vanishing moments for the weighted atoms are with respect to the measure dx, not to $w(x)\,dx$, and thus the Coifman-Weiss atomic Hardy spaces on $(\mathbb{R}^n, \rho, w(x)\,dx)$ are different from the weighted atomic anisotropic Hardy spaces $H^{p,q,s}_w(\mathbb{R}^n; A)$.

We give some basic properties concerning $H^p_{w,N}(\mathbb{R}^n; A)$ and $H^{p,q,s}_w(\mathbb{R}^n; A)$.

Proposition 3.4. *Let A be an expansive dilation and $w \in \mathcal{A}_\infty$. If $p \in (0,1]$ and $N \geq N_{p,w}$, then the inclusion $H^p_{w,N}(\mathbb{R}^n; A) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is continuous.*

Proof. Let $f \in H^p_{w,N}(\mathbb{R}^n; A)$. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}|\langle f, \varphi\rangle| = |f * \tilde{\varphi}(0)| &\le \|\tilde{\varphi}\|_{\mathcal{S}_N(\mathbb{R}^n)} \inf_{x\in B_0} M_N(f)(x)\\ &\le [w(B_0)]^{-1/p}\, \|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)}\, \|f\|_{H^p_{w,N}(\mathbb{R}^n;A)},\end{aligned}$$

where $\tilde{\varphi}(x) \equiv \varphi(-x)$. This implies $f \in \mathcal{S}'(\mathbb{R}^n)$ and the inclusion is continuous, which completes the proof of Proposition 3.4. □

The proof of the following proposition is a weighted variant of Proposition 3.12 in [2].

Proposition 3.5. *Let A be an expansive dilation and $w \in \mathcal{A}_\infty$. If $p \in (0,1]$ and $N \ge \lfloor (q_w/p - 1)\ln b/\ln(\lambda_-)\rfloor + 2$, then the space $H^p_{w,N}(\mathbb{R}^n; A)$ is complete.*

Proof. For every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^n)$ such that $\sum_{j\in\mathbb{N}} f_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to the tempered distribution f, the series $\sum_{j\in\mathbb{N}} f_j * \varphi$ converges to $f * \varphi$ pointwise. Thus for any $x \in \mathbb{R}^n$, we obtain

$$[M_N(f)(x)]^p \le \Big[\sum_{j\in\mathbb{N}} M_N(f_j)(x)\Big]^p \le \sum_{j\in\mathbb{N}} [M_N(f_j)(x)]^p \tag{3.2}$$

and hence $\|f\|_{H^p_{w,N}(\mathbb{R}^n;A)} \le \sum_{j\in\mathbb{N}} \|f_j\|_{H^p_{w,N}(\mathbb{R}^n;A)}$.

To prove the completeness of $H^p_{w,N}(\mathbb{R}^n; A)$, it suffices to prove that for every sequence $\{f_j\}_{j\in\mathbb{N}}$ with $\|f_j\|_{H^p_{w,N}(\mathbb{R}^n;A)} < 2^{-j}$ for any $j \in \mathbb{N}$, the series $\sum_{j\in\mathbb{N}} f_j$ converges in $H^p_{w,N}(\mathbb{R}^n; A)$. Since $\{\sum_{i=1}^j f_i\}_{j\in\mathbb{N}}$ are Cauchy sequences in $H^p_{w,N}(\mathbb{R}^n; A)$, by Proposition 3.4 and the completeness of $\mathcal{S}'(\mathbb{R}^n)$, $\{\sum_{i=1}^j f_i\}_{j\in\mathbb{N}}$ are also Cauchy sequences in $\mathcal{S}'(\mathbb{R}^n)$ and thus converge to some $f \in \mathcal{S}'(\mathbb{R}^n)$. Therefore,

$$\Big\|f - \sum_{i=1}^j f_i\Big\|^p_{H^p_{w,N}(\mathbb{R}^n;A)} = \Big\|\sum_{i=j+1}^\infty f_i\Big\|^p_{H^p_{w,N}(\mathbb{R}^n;A)} \le \sum_{i=j+1}^\infty 2^{-ip} \to 0$$

as $j \to \infty$. This completes the proof of Proposition 3.5. □

Theorem 3.6. *Let A be an expansive dilation and $w \in \mathcal{A}_\infty$. If $(p,q,s)_w$ is an admissible triplet and $N \ge N_{p,w}$, then $H^{p,q,s}_w(\mathbb{R}^n; A) \subset H^p_{w,N_{p,w}}(\mathbb{R}^n; A) \subset H^p_{w,N}(\mathbb{R}^n; A)$, and moreover, there exists a positive constant C such that for all $f \in H^{p,q,s}_w(\mathbb{R}^n; A)$,*

$$\|f\|_{H^p_{w,N}(\mathbb{R}^n;A)} \le \|f\|_{H^p_{w,N_{p,w}}(\mathbb{R}^n;A)} \le C\|f\|_{H^{p,q,s}_w(\mathbb{R}^n;A)}.$$

Proof. By (3.1), we only need to prove $H_w^{p,q,s}(\mathbb{R}^n;A) \subset H^p_{w,N_{p,w}}(\mathbb{R}^n;A)$ and for all $f \in H_w^{p,q,s}(\mathbb{R}^n;A)$, $\|f\|_{H^p_{w,N_{p,w}}(\mathbb{R}^n;A)} \lesssim \|f\|_{H_w^{p,q,s}(\mathbb{R}^n;A)}$. To this end, it suffices to prove that

$$\|M^0_{N_{p,w}}(a)\|_{L^p_w(\mathbb{R}^n)} \lesssim 1 \quad \text{for all } (p,q,s)_w\text{-atoms } a. \tag{3.3}$$

Indeed, for any $f \in H_w^{p,q,s}(\mathbb{R}^n;A)$, there exist numbers $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and $(p,q,s)_w$-atoms $\{a_i\}_{i\in\mathbb{N}}$ such that $f = \sum_{i\in\mathbb{N}} \lambda_i a_i$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\sum_{i\in\mathbb{N}} |\lambda_i|^p \lesssim \|f\|^p_{H_w^{p,q,s}(\mathbb{R}^n;A)}$. Then by (2.12) and (3.2), we have

$$\begin{aligned}
||f||^p_{H^p_{w,N_{p,w}}(\mathbb{R}^n;A)} &= \int_{\mathbb{R}^n} \left[M_{N_{p,w}}\Big(\sum_{i\in\mathbb{N}} \lambda_i a_i\Big)(x)\right]^p w(x)\,dx \\
&\lesssim \sum_{i\in\mathbb{N}} |\lambda_i|^p \int_{\mathbb{R}^n} [M^0_{N_{p,w}}(a_i)(x)]^p w(x)\,dx \lesssim \sum_{i\in\mathbb{N}} |\lambda_i|^p,
\end{aligned}$$

which implies $f \in H^p_{w,N_{p,w}}(\mathbb{R}^n;A)$ and $\|f\|_{H^p_{w,N_{p,w}}(\mathbb{R}^n;A)} \lesssim \|f\|_{H_w^{p,q,s}(\mathbb{R}^n;A)}$.

Let now a be a $(p,q,s)_w$-atom supported in the ball $x_0 + B_j$ for some $x_0 \in \mathbb{R}^n$ and $j \in \mathbb{Z}$. Write

$$\begin{aligned}
&\int_{\mathbb{R}^n} [M^0_{N_{p,w}}(a)(x)]^p w(x)\,dx \\
&\qquad = \left[\int_{x_0+B_{j+\sigma}} + \int_{(x_0+B_{j+\sigma})^\complement}\right] [M^0_{N_{p,w}}(a)(x)]^p w(x)\,dx = \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Recall that $q \in (q_w, \infty]$. Thus $w \in \mathcal{A}_q$. Using the Hölder inequality, the $L^q_w(\mathbb{R}^n)$-boundedness of $M^0_{N_{p,w}}$ (see Proposition 2.11 (ii)) and $w \in \mathcal{A}_q$ together with Proposition 2.6 (i), we have

$$\begin{aligned}
\mathrm{I} &\le ||M^0_{N_{p,w}}(a)||^p_{L^q_w(\mathbb{R}^n)} [w(x_0 + B_{j+\sigma})]^{1-p/q} \\
&\lesssim ||a||^p_{L^q_w(\mathbb{R}^n)} [w(x_0 + B_j)]^{1-p/q} \lesssim 1.
\end{aligned}$$

To estimate II, we claim that for all $m \in \mathbb{Z}_+$ and $x \in x_0 + (B_{j+\sigma+m+1} \setminus B_{j+\sigma+m})$,

$$M^0_{N_{p,w}}(a)(x) \lesssim [w(x_0 + B_j)]^{-1/p} [b(\lambda_-)^{s_0+1}]^{-m}, \tag{3.4}$$

where $s_0 = \lfloor (q_w/p - 1)\ln b/\ln(\lambda_-)\rfloor$. If this claim is true, choosing $\eta > 0$ such that $b^{-(q_w+\eta)+p}(\lambda_-)^{(s_0+1)p} > 1$, then by $w \in \mathcal{A}_{q_w+\eta}$ and Proposition 2.6 (i), we have

$$\begin{aligned}
\mathrm{II} &\lesssim \sum_{m=0}^{\infty} w(x_0 + B_{j+\sigma+m+1}) \sup_{x\in x_0+(B_{j+\sigma+m+1}\setminus B_{j+\sigma+m})} [M^0_{N_{p,w}}(a)(x)]^p \\
&\lesssim \sum_{m=0}^{\infty} [b^{-(q_w+\eta)+p}(\lambda_-)^{(s_0+1)p}]^{-m} \lesssim 1.
\end{aligned}$$

Combining the estimates for I and II yields (3.3).

To prove the estimate (3.4), we follow the techniques from the proof of Theorem 4.2 in [2]. By the Hölder inequality, Definition 3.2 (ii) and $w \in \mathcal{A}_q$, we have

$$\begin{aligned}
\int_{x_0+B_j} |a(y)|\,\mathrm{d}y &\le \|a\|_{L^q_w(\mathbb{R}^n)} \left(\int_{x_0+B_j} [w(y)]^{-q'/q}\,\mathrm{d}x \right)^{1/q'} \\
&\lesssim b^j [w(x_0+B_j)]^{-1/p}.
\end{aligned} \tag{3.5}$$

Let $x \in x_0 + (B_{j+m+\sigma+1} \setminus B_{j+m+\sigma})$, $k \in \mathbb{Z}$, and $\varphi \in S_N(\mathbb{R}^n)$. For $j > k$ and $y \in x_0 + B_j$, we have $\rho(A^{-k}(x-y)) \gtrsim b^{j-k+m}$. Observe that $N_{p,w} \ge s_0 + 2$ implies that $b(\lambda_-)^{s_0+1} \le b^{N_{p,w}}$. By this, (3.5), $\varphi \in S_{N_{p,w}}(\mathbb{R}^n)$, and $j > k$, we have

$$\begin{aligned}
|a * \varphi_k(x)| &\le b^{-k} \int_{x_0+B_j} |a(y)|\,|\varphi(A^{-k}(x-y))|\,\mathrm{d}y \\
&\lesssim b^{-N_{p,w}(j-k+m)} b^{j-k} [w(x_0+B_j)]^{-1/p} \\
&\lesssim [b(\lambda_-)^{s_0+1}]^{-m} [w(x_0+B_j)]^{-1/p}.
\end{aligned} \tag{3.6}$$

For $j \le k$, let P be the Taylor expansion of φ at the point $A^{-k}(x-x_0)$ of order s_0. Thus, by the Taylor remainder theorem, (2.6) and (2.7), we have

$$\begin{aligned}
&\sup_{y\in x_0+B_j} |\varphi(A^{-k}(x-y)) - P(A^{-k}(x-y))| \\
&\quad\lesssim \sup_{z\in B_{j-k}} \sup_{|\alpha|=s_0+1} |\partial^\alpha \varphi(A^{-k}(x-x_0)+z)|\,|z|^{s_0+1} \\
&\quad\lesssim (\lambda_-)^{(s_0+1)(j-k)} \sup_{z\in B_{j-k}} [1+\rho(A^{-k}(x-x_0)+z)]^{-N_{p,w}} \\
&\quad\lesssim (\lambda_-)^{(s_0+1)(j-k)} \min(1, b^{-N_{p,w}(j-k+m)}).
\end{aligned}$$

In the last step, we used (2.2) and the fact that

$$A^{-k}(x-x_0) + B_{j-k} \subset (B_{j-k+m+\sigma})^{\complement} + B_{j-k} \subset (B_{j-k+m})^{\complement},$$

since $m \geq 0$. By this, (3.5), $j \leq k$, and the fact that a has vanishing moments up to order s_0, we have

$$
\begin{aligned}
(3.7) \quad & |a * \varphi_k(x)| \\
& \leq b^{-k} \int_{x_0+B_j} |a(y)|\,|\varphi(A^{-k}(x-y)) - P(A^{-k}(x-y))|\,dy \\
& \lesssim [w(x_0+B_j)]^{-1/p} (\lambda_-)^{(s_0+1)(j-k)} b^{j-k} \min(1, b^{-N_{p,w}(j-k+m)}).
\end{aligned}
$$

Observe that when $j - k + m > 0$, by $b(\lambda_-)^{s_0+1} \leq b^{N_{p,w}}$ again, we have

$$
(3.8) \qquad |a * \varphi_k(x)| \lesssim [(\lambda_-)^{(s_0+1)} b]^{-m} [w(x_0+B_j)]^{-1/p}.
$$

Finally, when $j-k+m \leq 0$, (3.7) trivially yields (3.8). This shows that (3.8) holds for all $j \leq k$. Combining this together with (3.6) and taking the supremum over $k \in \mathbb{Z}$ verify the claim (3.4) and thus complete the proof of Theorem 3.6. ◻

4. Calderón-Zygmund Decompositions

In this section, we generalize the Calderón-Zygmund decomposition associated with grand maximal functions on anisotropic $\mathbb{R}^n$ in [2] to the weighted anisotropic $\mathbb{R}^n$. We follow the constructions in [17] and [2].

Throughout this section, we consider a tempered distribution f so that for all $\lambda > 0$,

$$
w(\{x \in \mathbb{R}^n : M_N(f)(x) > \lambda\}) < \infty,
$$

where $N \geq 2$ is some fixed integer. Later with regard to the weighted anisotropic Hardy space $H^p_{w,N}(\mathbb{R}^n; A)$ with $p \in (0,1]$, we restrict to

$$
N > \lfloor q_w \ln b / [p \ln(\lambda_-)] \rfloor.
$$

For a given $\lambda > 0$, we set

$$
\Omega \equiv \{x \in \mathbb{R}^n : \ M_N(f)(x) > \lambda\}.
$$

Since by Proposition 2.6 (i), $w(\mathbb{R}^n) = \infty$, which together with $w(\Omega) < \infty$ implies that Ω is a proper subset of $\mathbb{R}^n$. Observe also that Ω is open. Applying Lemma 2.3 to Ω with $d = 4\sigma$, we obtain a positive constant L independent of Ω and f, a sequence $\{x_j\}_j \subset \Omega$ and a sequence of integers $\{\ell_j\}_j$ such that

$$
\begin{aligned}
&(4.1) \quad \Omega = \textstyle\bigcup_j (x_j + B_{\ell_j}),\\
&(4.2) \quad (x_i + B_{\ell_i - 2\sigma}) \cap (x_j + B_{\ell_j - 2\sigma}) = \emptyset \quad \text{for all } i, j \text{ with } i \neq j,\\
&(4.3) \quad (x_j + B_{\ell_j + 4\sigma}) \cap \Omega^{\complement} = \emptyset \text{ and } (x_j + B_{\ell_j + 4\sigma + 1}) \cap \Omega^{\complement} \neq \emptyset \quad \text{for all } j,\\
&(4.4) \quad (x_i + B_{\ell_i + 2\sigma}) \cap (x_j + B_{\ell_j + 2\sigma}) \neq \emptyset \quad \text{implies that } |\ell_i - \ell_j| \leq \sigma,\\
&(4.5) \quad \sharp\{j : (x_i + B_{\ell_i + 2\sigma}) \cap (x_j + B_{\ell_j + 2\sigma}) \neq \emptyset\} \leq L \quad \text{for all } i.
\end{aligned}
$$

Remark 4.1. Notice that $w(\Omega) < \infty$ does not generally imply that $|\Omega| < \infty$. For example, if $n = 1$, $A = 2$, $w(x) = |x|^\alpha$ with $\alpha \in (-1, -\frac{1}{2})$, and $\Omega = \bigcup_{i\in\mathbb{N}}(i^2, i^2 + 1)$, then $w \in \mathcal{A}_1$ and $w(\Omega) < \infty$, but $|\Omega| = \infty$. Hence, Lemma 2.7 of [2] might not be applicable and the use of Lemma 2.3 is necessary.

Fix $\theta \in S(\mathbb{R}^n)$ such that $\operatorname{supp}\theta \subset B_\sigma$, $0 \le \theta \le 1$ and $\theta \equiv 1$ on B_0. For each j and all $x \in \mathbb{R}^n$, define $\theta_j(x) \equiv \theta(A^{-\ell_j}(x - x_j))$. Clearly, $\operatorname{supp}\theta_j \subset x_j + B_{\ell_j+\sigma}$ and $\theta_j \equiv 1$ on $x_j + B_{\ell_j}$. By (4.1) and (4.5), for any $x \in \Omega$, we have $1 \le \sum_j \theta_j(x) \le L$. For every i, define

$$\zeta_i(x) \equiv \frac{\theta_i(x)}{\sum_j \theta_j(x)}. \tag{4.6}$$

Then $\zeta_i \in S(\mathbb{R}^n)$, $\operatorname{supp}\zeta_i \subset x_i + B_{\ell_i+\sigma}$, $0 \le \zeta_i \le 1$, $\zeta_i \equiv 1$ on $x_i + B_{\ell_i-2\sigma}$ by (4.2), and $\sum_i \zeta_i = \chi_\Omega$. The family $\{\zeta_i\}_i$ forms a smooth partition of unity on Ω.

Let $s \in \mathbb{Z}_+$ be some fixed integer and $\mathcal{P}_s(\mathbb{R}^n)$ denote the linear space of polynomials in n variables of degrees no more than s. For each i and $P \in \mathcal{P}_s(\mathbb{R}^n)$, set

$$\|P\|_i \equiv \left[\frac{1}{\int_{\mathbb{R}^n}\zeta_i(x)\,dx}\int_{\mathbb{R}^n}|P(x)|^2\zeta_i(x)\,dx\right]^{1/2}. \tag{4.7}$$

Then $(\mathcal{P}_s(\mathbb{R}^n), \|\cdot\|_i)$ is a finite dimensional Hilbert space. Let $f \in S'(\mathbb{R}^n)$. Since f induces a linear functional on $\mathcal{P}_s(\mathbb{R}^n)$ via $Q \longmapsto 1/\int_{\mathbb{R}^n}\zeta_i(x)\,dx\langle f, Q\zeta_i\rangle$, by the Riesz lemma, there exists a unique polynomial $P_i \in \mathcal{P}_s(\mathbb{R}^n)$ for each i such that for all $Q \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\begin{aligned}\frac{1}{\int_{\mathbb{R}^n}\zeta_i(x)\,dx}\langle f, Q\zeta_i\rangle &= \frac{1}{\int_{\mathbb{R}^n}\zeta_i(x)\,dx}\langle P_i, Q\zeta_i\rangle \\ &= \frac{1}{\int_{\mathbb{R}^n}\zeta_i(x)\,dx}\int_{\mathbb{R}^n}P_i(x)Q(x)\zeta_i(x)\,dx.\end{aligned}$$

For every i, define distribution $b_i \equiv (f - P_i)\zeta_i$.

We will show that for suitable choices of s and N, the series $\sum_i b_i$ converges in $S'(\mathbb{R}^n)$, and in this case, we define $g \equiv f - \sum_i b_i$ in $S'(\mathbb{R}^n)$.

Definition 4.2. The representation $f = g + \sum_i b_i$, where g and b_i are as above, is said to be a *Calderón-Zygmund decomposition* of degree s and height λ associated with $M_N(f)$.

The rest of this section consists of a series of lemmas. In Lemma 4.3 and Lemma 4.4, we give some properties of the smooth partition of unity $\{\zeta_i\}_i$. In Lemmas 4.5 through 4.8, we derive some estimates for the bad parts $\{b_i\}_i$. Lemma 4.9 and Lemma 4.10 give controls over the good part g. Finally, Corollary 4.11 shows the density of $L^q_w(\mathbb{R}^n) \cap H^p_{w,N}(\mathbb{R}^n; A)$ in $H^p_{w,N}(\mathbb{R}^n; A)$, where $q \in (q_w, \infty)$.

Lemma 4.3 through Lemma 4.6 are essentially Lemma 5.2, Lemma 5.3, Lemma 5.4, and Lemma 5.6 of [2]; respectively. Here we omit the details.

Lemma 4.3. *There exists a positive constant C_1, depending only on N, such that for all i and $\ell \le \ell_i$,*

$$\sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} |\partial^\alpha [\zeta_i(A^\ell \cdot)](x)| \le C_1.$$

Lemma 4.4. *There exists a positive constant C_2, independent of f and λ, such that for all i,*

$$\sup_{y\in\mathbb{R}^n} |P_i(y)\zeta_i(y)| \le C_2 \sup_{y\in((x_i+B_{\ell_i+4\sigma+1})\cap\Omega^\complement)} M_N(f)(y) \le C_2\lambda.$$

Lemma 4.5. *There exists a positive constant C_3, independent of f and λ, such that for all i and $x \in x_i + B_{\ell_i+2\sigma}$, $M_N(b_i)(x) \le C_3 M_N(f)(x)$.*

Lemma 4.6. *If $N > s \ge 0$, then there exists a positive constant C_4, independent of f and λ, such that for all $t \in \mathbb{Z}_+$, i and $x \in x_i + (B_{t+\ell_i+2\sigma+1} \setminus B_{t+\ell_i+2\sigma})$, $M_N(b_i)(x) \le C_4\lambda(\lambda_-)^{-t(s+1)}$.*

Lemma 4.7. *Let $w \in \mathcal{A}_\infty$ and q_w be as in (2.8). If $p \in (0,1]$, $s \ge \lfloor q_w \ln b/[p\ln(\lambda_-)]\rfloor$ and $N > s$, then there exists a positive constant C_5 such that for all $f \in H^p_{w,N}(\mathbb{R}^n; A)$, $\lambda > 0$ and i,*

$$\int_{\mathbb{R}^n} [M_N(b_i)(x)]^p w(x)\,dx \le C_5 \int_{x_i+B_{\ell_i+2\sigma}} [M_N(f)(x)]^p w(x)\,dx. \tag{4.8}$$

Moreover, the series $\sum_i b_i$ converges in $H^p_{w,N}(\mathbb{R}^n; A)$ and

$$\int_{\mathbb{R}^n} \Big[M_N\Big(\sum_i b_i\Big)(x)\Big]^p w(x)\,dx \le LC_5 \int_\Omega [M_N(f)(x)]^p w(x)\,dx, \tag{4.9}$$

where L is as in (4.5).

Proof. By Lemma 4.5, we have

$$\int_{\mathbb{R}^n} [M_N(b_i)(x)]^p w(x)\,dx \lesssim \int_{x_i+B_{\ell_i+2\sigma}} [M_N(f)(x)]^p w(x)\,dx + \int_{(x_i+B_{\ell_i+2\sigma})^\complement} [M_N(b_i)(x)]^p w(x)\,dx.$$

Notice that $s \ge \lfloor q_w \ln b/[p\ln(\lambda_-)]\rfloor$ implies $b^{-(q_w+\eta)}(\lambda_-)^{(s+1)p} > 1$ for sufficiently small $\eta > 0$. Using Proposition 2.6 (i) with $w \in \mathcal{A}_{q_w+\eta}$, Lemma 4.6 and the fact that $M_N(f)(x) > \lambda$ for all $x \in x_i + B_{\ell_i+2\sigma}$, we have

$$\begin{aligned}
&\int_{(x_i+B_{\ell_i+2\sigma})^\complement} [M_N(b_i)(x)]^p w(x)\,\mathrm{d}x \\
&\quad = \sum_{t=0}^{\infty} \int_{x_i+(B_{t+\ell_i+2\sigma+1}\setminus B_{t+\ell_i+2\sigma})} [M_N(b_i)(x)]^p w(x)\,\mathrm{d}x \\
&\quad \lesssim \lambda^p w(x_i + B_{\ell_i+2\sigma}) \sum_{t=0}^{\infty} [b^{-(q_w+\eta)}(\lambda_-)^{(s+1)p}]^{-t} \\
&\quad \lesssim \int_{x_i+B_{\ell_i+2\sigma}} [M_N(f)(x)]^p w(x)\,\mathrm{d}x,
\end{aligned}$$

which gives (4.8).

By (4.8) and (4.5), we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \Big[\sum_i M_N(b_i)(x)\Big]^p w(x)\,\mathrm{d}x &\lesssim \sum_i \int_{x_i+B_{\ell_i+2\sigma}} [M_N(f)(x)]^p w(x)\,\mathrm{d}x \\
&\lesssim \int_\Omega [M_N(f)(x)]^p w(x)\,\mathrm{d}x,
\end{aligned}$$

which together with the completeness of $H^p_{w,N}(\mathbb{R}^n;A)$ (see Proposition 3.5) implies that $\sum_i b_i$ converges in $H^p_{w,N}(\mathbb{R}^n;A)$. So by Proposition 3.4, the series $\sum_i b_i$ converges in $\mathcal{S}'(\mathbb{R}^n)$, and therefore $M_N(\sum_i b_i)(x) \le \sum_i M_N(b_i)(x)$, which gives (4.9) and thus completes the proof of Lemma 4.7. $\square$

Lemma 4.8. *Let $w \in \mathcal{A}_\infty$, q_w be as in (2.8), $s \in \mathbb{Z}_+$, and $N \ge 2$. If $q \in (q_w, \infty]$ and $f \in L^q_w(\mathbb{R}^n)$, then the series $\sum_i b_i$ converges in $L^q_w(\mathbb{R}^n)$ and there exists a positive constant C_6, independent of f and λ, such that $\|\sum_i |b_i|\,\|_{L^q_w(\mathbb{R}^n)} \le C_6\|f\|_{L^q_w(\mathbb{R}^n)}$.*

Proof. The proof for $q = \infty$ is similar to that for $q \in (q_w, \infty)$. So we only give the proof for $q \in (q_w, \infty)$. By Lemma 4.4 and Proposition 2.6 (i),

$$\begin{aligned}
&\int_{\mathbb{R}^n} |b_i(x)|^q w(x)\,\mathrm{d}x \\
&\quad \lesssim \int_{x_i+B_{\ell_i+\sigma}} |f(x)|^q w(x)\,\mathrm{d}x + \int_{x_i+B_{\ell_i+\sigma}} |P_i(x)\zeta_i(x)|^q w(x)\,\mathrm{d}x \\
&\quad \lesssim \int_{x_i+B_{\ell_i+\sigma}} |f(x)|^q w(x)\,\mathrm{d}x + \lambda^q w(x_i + B_{\ell_i-2\sigma}).
\end{aligned}$$

Therefore, by (4.2), (4.5) and Proposition 2.11 (ii), we have

$$\sum_i \int_{\mathbb{R}^n} |b_i(x)|^q w(x)\,\mathrm{d}x \lesssim \int_\Omega |f(x)|^q w(x)\,\mathrm{d}x + \lambda^q w(\Omega)$$
$$\lesssim \int_{\mathbb{R}^n} |f(x)|^q w(x)\,\mathrm{d}x,$$

which together with (4.5) again gives $\|\sum_i |b_i|\,\|_{L^q_w(\mathbb{R}^n)} \lesssim \|f\|_{L^q_w(\mathbb{R}^n)}$ and thus completes the proof of Lemma 4.8. ◻

The following conclusion is essentially Lemma 5.9 in [2]. Here we omit the details of the proof.

Lemma 4.9. *If $N > s \geq 0$ and $\sum_i b_i$ converges in $S'(\mathbb{R}^n)$, then there exists a positive constant C_7, independent of f and λ, such that for all $x \in \mathbb{R}^n$,*

$$M_N(g)(x) \leq C_7\lambda \sum_i (\lambda_-)^{-t_i(x)(s+1)} + M_N(f)(x)\chi_{\Omega^\complement}(x),$$

where

$$t_i(x) \equiv \begin{cases} \kappa_i, & \text{if } x \in x_i + (B_{\kappa_i+\ell_i+2\sigma+1} \setminus B_{\kappa_i+\ell_i+2\sigma}) \text{ for some } \kappa_i \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.10. *Let $w \in \mathcal{A}_\infty$, q_w be as in (2.8), $p \in (0,1]$, and $q \in (q_w, \infty)$.*

(i) *If $N > s \geq \lfloor q_w \ln b/[p\ln(\lambda_-)]\rfloor$ and $M_N(f) \in L^p_w(\mathbb{R}^n)$, then $M_N(g) \in L^q_w(\mathbb{R}^n)$ and there exists a positive constant C_8, independent of f and λ, such that*

$$\int_{\mathbb{R}^n} [M_N(g)(x)]^q w(x)\,\mathrm{d}x \leq C_8\lambda^{q-p} \int_{\mathbb{R}^n} [M_N(f)(x)]^p w(x)\,\mathrm{d}x.$$

(ii) *If $N \geq 2$ and $f \in L^q_w(\mathbb{R}^n)$, then $g \in L^\infty_w(\mathbb{R}^n)$ and there exists a positive constant C_9, independent of f and λ, such that $\|g\|_{L^\infty_w(\mathbb{R}^n)} \leq C_9\lambda$.*

Proof. Since $f \in H^p_{w,N}(\mathbb{R}^n; A)$, by Lemma 4.7, $\sum_i b_i$ converges in $H^p_{w,N}(\mathbb{R}^n; A)$ and therefore in $S'(\mathbb{R}^n)$ by Proposition 3.4. Then by Lemma 4.9,

$$\int_{\mathbb{R}^n} [M_N(g)(x)]^q w(x)\,\mathrm{d}x \lesssim \lambda^q \sum_i \int_{\mathbb{R}^n} (\lambda_-)^{t_i(x)(s+1)q} w(x)\,\mathrm{d}x$$
$$+ \int_{\Omega^\complement} [M_N(f)(x)]^q w(x)\,\mathrm{d}x,$$

where $t_i(x)$ is as in Lemma 4.9. Observe that $s \ge \lfloor q_w \ln b/[p\ln(\lambda_-)]\rfloor$ implies that $b^{-(q_w+\eta)}(\lambda_-)^{(s+1)q} > 1$ for sufficiently small $\eta > 0$. Then for any i, by $w \in \mathcal{A}_{q_w+\eta}$ and Proposition 2.6 (i), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}(\lambda_-)^{-t_i(x)(s+1)q}w(x)\,\mathrm{d}x\\
&\quad= \int_{x_i+B_{\ell_i+2\sigma}} w(x)\,\mathrm{d}x + \sum_{t=0}^{\infty}\int_{x_i+(B_{\ell_i+2\sigma+t+1}\setminus B_{\ell_i+2\sigma+t})}(\lambda_-)^{-t(s+1)q}w(x)\,\mathrm{d}x\\
&\quad\lesssim w(x_i+B_{\ell_i+2\sigma})\Big\{1+\sum_{t=0}^{\infty}[b^{-(q_w+\eta)}(\lambda_-)^{(s+1)q}]^{-t}\Big\} \lesssim w(x_i+B_{\ell_i-2\sigma}).
\end{aligned}$$

Taking the sum over all i, by (4.1) and (4.2), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}[M_N(g)(x)]^q w(x)\,\mathrm{d}x\\
&\qquad\lesssim \lambda^q\sum_i w(x_i+B_{\ell_i-2\sigma}) + \int_{\Omega^\complement}[M_N(f)(x)]^q w(x)\,\mathrm{d}x\\
&\qquad\lesssim \lambda^q w(\Omega) + \int_{\Omega^\complement}[M_N(f)(x)]^q w(x)\,\mathrm{d}x\\
&\qquad\lesssim \lambda^{q-p}\int_{\Omega}[M_N(f)(x)]^p w(x)\,\mathrm{d}x + \lambda^{q-p}\int_{\Omega^\complement}[M_N(f)(x)]^p w(x)\,\mathrm{d}x\\
&\qquad\lesssim \lambda^{q-p}\int_{\mathbb{R}^n}[M_N(f)(x)]^p w(x)\,\mathrm{d}x,
\end{aligned}$$

namely, (i) holds.

Moreover, if $f \in L^q_w(\mathbb{R}^n)$, then g and $\{b_i\}_i$ are functions, and by Lemma 4.8, $\sum_i b_i$ converges in $L^q_w(\mathbb{R}^n)$ and thus in $\mathcal{S}'(\mathbb{R}^n)$ by Lemma 2.8. Write

$$g = f - \sum_i b_i = f\Big(1-\sum_i \zeta_i\Big) + \sum_i P_i\zeta_i = f\chi_{\Omega^\complement} + \sum_i P_i\zeta_i.$$

By Lemma 4.4 and (4.5), we have $|g(x)| \lesssim \lambda$ for all $x \in \Omega$, and by Proposition 2.11 (i) and (2.12), $|g(x)| = |f(x)| \le M_N(f)(x) \le \lambda$ for almost everywhere $x \in \Omega^\complement$, which leads to that $\|g\|_{L^\infty_w(\mathbb{R}^n)} \lesssim \lambda$ and thus yields (ii). This completes the proof of Lemma 4.10. □

Corollary 4.11. *Let A be an expansive dilation, $w \in \mathcal{A}_\infty$ and q_w be as in (2.8). If $q \in (q_w,\infty)$, $N > \lfloor q_w \ln b/[p\ln(\lambda_-)]\rfloor$ and $p \in (0,1]$, then $H^p_{w,N}(\mathbb{R}^n;A) \cap L^q_w(\mathbb{R}^n)$ is dense in $H^p_{w,N}(\mathbb{R}^n;A)$.*

Proof. Let $f \in H^p_{w,N}(\mathbb{R}^n;A)$. For any $\lambda > 0$, let $f = g^\lambda + \sum_i b_i^\lambda$ be the Calderón-Zygmund decomposition of f of degree s with $\lfloor q_w \ln b/[p\ln(\lambda_-)]\rfloor \le$

$s < N$ and height λ associated to $M_N(f)$ as in Definition 4.2. Here, we rewrite g and b_i in Definition 4.2 into g^λ and b_i^λ; respectively. By (4.9) in Lemma 4.7,

$$\Big\| \sum_i b_i^\lambda \Big\|^p_{H^p_{w,N}(\mathbb{R}^n;A)} \lesssim \int_{\{x\in\mathbb{R}^n:\, M_N(f)(x)>\lambda\}} [M_N(f)(x)]^p w(x)\,\mathrm{d}x \to 0,$$

and therefore $g^\lambda \to f$ in $H^p_{w,N}(\mathbb{R}^n;A)$ as $\lambda \to \infty$. Moreover, by Lemma 4.10 (i), $M_N(g^\lambda) \in L^q_w(\mathbb{R}^n)$, so by Proposition 2.11 (ii), $g^\lambda \in L^q_w(\mathbb{R}^n)$, which completes the proof of Corollary 4.11. $\square$

5. Weighted Atomic Decompositions of $H^p_{w,N}(\mathbb{R}^n;A)$

In this section, we shall establish the equivalence between $H^p_{w,N}(\mathbb{R}^n;A)$ and $H^{p,q,s}_w(\mathbb{R}^n;A)$ by using the Calderón-Zygmund decomposition associated to grand maximal functions in Section 4.

Let $w \in \mathcal{A}_\infty$, q_w be as in (2.8), $p \in (0,1]$ and $N > s \equiv \lfloor q_w \ln b/[p\ln(\lambda_-)]\rfloor$. Let $f \in H^p_{w,N}(\mathbb{R}^n;A)$. For each $k \in \mathbb{Z}$, as in the Definition 4.2, f has a Calderón-Zygmund decomposition of degree s and height $\lambda = 2^k$ associated to $M_N(f)$, $f = g^k + \sum_i b_i^k$, where $\Omega_k \equiv \{x \in \mathbb{R}^n : M_N(f)(x) > 2^k\}$, $b_i^k \equiv (f - P_i^k)\zeta_i^k$, and $B_i^k \equiv x_i^k + B_{\ell_i^k}$. Recall that for fixed $k \in \mathbb{Z}$, $\{x_i = x_i^k\}_i$ is a sequence in Ω^k and $\{\ell_i = \ell_i^k\}_i$ is a sequence of integers such that (4.1) through (4.5) hold for $\Omega = \Omega_k$, $\{\zeta_i = \zeta_i^k\}_i$ are given by (4.6), and $\{P_i = P_i^k\}_i$ are projections of f onto $\mathcal{P}_s(\mathbb{R}^n)$ with respect to norms given by (4.7). Moreover, for each $k \in \mathbb{Z}$ and i, j, let $P_{i,j}^{k+1}$ be the orthogonal projection of $(f - P_j^{k+1})\zeta_i^k$ onto $\mathcal{P}_s(\mathbb{R}^n)$ with respect to the norm associated to ζ_j^{k+1} given by (4.7), namely, the unique element of $\mathcal{P}_s(\mathbb{R}^n)$ such that for all $Q \in \mathcal{P}_s(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} [f(x) - P_j^{k+1}(x)]\zeta_i^k(x)Q(x)\zeta_j^{k+1}(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} P_{i,j}^{k+1}(x)Q(x)\zeta_j^{k+1}(x)\,\mathrm{d}x.$$

For convenience, we set $\hat{B}_i^k \equiv x_i^k + B_{\ell_i^k+\sigma}$. Lemma 5.1 through Lemma 5.3 below are just Lemma 6.1 through Lemma 6.3 in [2].

Lemma 5.1.

(i) *If $\hat{B}_j^{k+1} \cap \hat{B}_i^k \neq \emptyset$, then $\ell_j^{k+1} \le \ell_i^k + \sigma$ and $\hat{B}_j^{k+1} \subset x_i^k + B_{\ell_i^k+4\sigma}$.*

(ii) *For any i, $\sharp\{j : \hat{B}_j^{k+1} \cap \hat{B}_i^k \neq \emptyset\} \le 2L$, where L is as in (4.5).*

Lemma 5.2. *There exists a positive constant C_{10} independent of f such that for all i, j and $k \in \mathbb{Z}$,*

$$\sup_{y\in\mathbb{R}^n} |P_{i,j}^{k+1}(y)\zeta_j^{k+1}(y)| \le C_{10} \sup_{y\in U} M_N(f)(y) \le C_{10}2^{k+1},$$

where $U \equiv (x_j^{k+1} + B_{\ell_j^{k+1}+4\sigma+1}) \cap (\Omega_{k+1})^\complement$.

Lemma 5.3. *For every $k \in \mathbb{Z}$, $\sum_i \sum_j P_{i,j}^{k+1} \zeta_j^{k+1} = 0$, where the series converges pointwise and in $\mathcal{S}'(\mathbb{R}^n)$.*

The following lemma establishes the weighted atomic decompositions for a dense subspace of $H_{w,N}^p(\mathbb{R}^n; A)$.

Lemma 5.4. *Let $w \in \mathcal{A}_\infty$ and q_w be as in (2.8). If $q \in (q_w, \infty)$, $p \in (0, 1]$, $s \ge \lfloor q_w \ln b / [p \ln(\lambda_-)] \rfloor$ and $N > s$, then for any $f \in (L_w^q(\mathbb{R}^n) \cap H_{w,N}^p(\mathbb{R}^n; A))$, there exist numbers $\{\lambda_i^k\}_{k \in \mathbb{Z}, i} \subset \mathbb{C}$ and $(p, \infty, s)_w$-atoms $\{a_i^k\}_{k \in \mathbb{Z}, i}$ such that*

$$f = \sum_{k \in \mathbb{Z}} \sum_i \lambda_i^k a_i^k,$$

where the series converges almost everywhere and in $\mathcal{S}'(\mathbb{R}^n)$,

$$\operatorname{supp} a_i^k \subset x_i^k + B_{\ell_i^k + 4\sigma} \quad \text{for all } k \in \mathbb{Z} \text{ and } i, \tag{5.1}$$

$$\Omega_k = \bigcup_i (x_i^k + B_{\ell_i^k + 4\sigma}) \quad \text{for all } k \in \mathbb{Z}, \tag{5.2}$$

$$(x_i^k + B_{\ell_i^k - 2\sigma}) \cap (x_j^k + B_{\ell_j^k - 2\sigma}) = \emptyset \quad \text{for all } k \in \mathbb{Z} \text{ and } i, j,\ i \ne j. \tag{5.3}$$

Moreover, there exists a positive constant C, independent of f, such that

$$|\lambda_i^k a_i^k| \le C 2^k \quad \text{for all } k \in \mathbb{Z} \text{ and } i, \tag{5.4}$$

$$\sum_{k \in \mathbb{Z}, i} |\lambda_i^k|^p \le C \|f\|_{H_{w,N}^p(\mathbb{R}^n; A)}^p. \tag{5.5}$$

Proof. Let $f \in (H_{w,N}^p(\mathbb{R}^n; A) \cap L_w^q(\mathbb{R}^n))$. For each $k \in \mathbb{Z}$, f has a Calderón-Zygmund decomposition of degree $s \ge \lfloor q_w \ln b / [p \ln(\lambda_-)] \rfloor$ and height 2^k associated to $M_N(f)$, $f = g^k + \sum_i b_i^k$ as above. The conclusions (5.2) and (5.3) are immediate by (4.1) through (4.3). By (4.9) in Lemma 4.7 and Proposition 3.4, $g^k \to f$ in both $H_{w,N}^p(\mathbb{R}^n; A)$ and $\mathcal{S}'(\mathbb{R}^n)$ as $k \to \infty$. By Lemma 4.10 (ii), $\|g^k\|_{L_w^\infty(\mathbb{R}^n)} \to 0$ as $k \to -\infty$, which implies that $g^k \to 0$ almost everywhere as $k \to -\infty$, and moreover, by Lemma 2.8 (ii), $g^k \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \to -\infty$. Therefore,

$$f = \sum_{k=-\infty}^{\infty} (g^{k+1} - g^k) \tag{5.6}$$

in $\mathcal{S}'(\mathbb{R}^n)$. Moreover, since $\operatorname{supp}(\sum_i b_i^k) \subset \Omega_k$ and $w(\Omega_k) \to 0$ as $k \to \infty$, then $g^k \to f$ almost everywhere as $k \to \infty$. Thus, (5.6) also holds almost everywhere.

By Lemma 5.3 and $\sum_i \zeta_i^k b_j^{k+1} = \chi_{\Omega_k} b_j^{k+1} = b_j^{k+1}$ for all j,

$$\begin{aligned} g^{k+1} - g^k &= \Big(f - \sum_j b_j^{k+1}\Big) - \Big(f - \sum_j b_j^k\Big) \\ &= \sum_j b_j^k - \sum_j b_j^{k+1} + \sum_i \Big(\sum_j P_{i,j}^{k+1}\zeta_j^{k+1}\Big) \\ &= \sum_i \Big[b_i^k - \sum_j (\zeta_i^k b_j^{k+1} - P_{i,j}^{k+1}\zeta_j^{k+1})\Big] \equiv \sum_i h_i^k, \end{aligned}$$

where all the series converge in $\mathcal{S}'(\mathbb{R}^n)$ and almost everywhere. Furthermore,

$$h_i^k = (f - P_i^k)\zeta_i^k - \sum_j [(f - P_j^{k+1})\zeta_i^k - P_{i,j}^{k+1}]\zeta_j^{k+1}. \tag{5.7}$$

By definitions of P_i^k and $P_{i,j}^{k+1}$, for all $Q \in \mathcal{P}_s(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} h_i^k(x) Q(x)\,\mathrm{d}x = 0. \tag{5.8}$$

Moreover, since $\sum_j \zeta_j^{k+1} = \chi_{\Omega_{k+1}}$, we rewrite (5.7) into

$$h_i^k = f\chi_{(\Omega_{k+1})^\complement}\zeta_i^k - P_i^k\zeta_i^k + \sum_j P_j^{k+1}\zeta_i^k\zeta_j^{k+1} + \sum_j P_{i,j}^{k+1}\zeta_j^{k+1}.$$

By Proposition 2.11 (i) and (2.12), $|f(x)| \le M_N(f)(x) \le 2^{k+1}$ for almost everywhere $x \in (\Omega_{k+1})^\complement$, and by Lemma 4.4, Lemma 5.1 (ii) and Lemma 5.2,

$$\|h_i^k\|_{L_w^\infty(\mathbb{R}^n)} \lesssim 2^k. \tag{5.9}$$

Recall that $P_{i,j}^{k+1} \ne 0$ implies $\hat{B}_j^{k+1} \cap \hat{B}_i^k \ne \emptyset$ and hence by Lemma 5.1 (i), $\operatorname{supp} \zeta_j^{k+1} \subset \hat{B}_j^{k+1} \subset x_i^k + B_{\ell_i^k + 4\sigma}$. Therefore, by (5.7),

$$\operatorname{supp} h_i^k \subset x_i^k + B_{\ell_i^k + 4\sigma}. \tag{5.10}$$

Let $\lambda_i^k = C2^k[w(x_i^k + B_{\ell_i^k+4\sigma})]^{1/p}$ and $a_i^k = (\lambda_i^k)^{-1}h_i^k$, where C is a positive constant independent of i, k and f. Obviously, (5.9) and (5.10) imply (5.4) and (5.1), respectively. Moreover, by (5.8), (5.9), (5.10) and a suitable choice of C, we know that a_i^k is a $(p, \infty, s)_w$-atom. By $w \in \mathcal{A}_q$, Proposition 2.6 (i) and (4.2), we have

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \sum_i |\lambda_i^k|^p &\lesssim \sum_{k=-\infty}^{\infty} \sum_i 2^{kp} w(x_i^k + B_{\ell_i^k - 2\sigma}) \lesssim \sum_{k=-\infty}^{\infty} 2^{kp} w(\Omega_k) \\ &\lesssim \|M_N(f)\|_{L_w^p(\mathbb{R}^n)}^p \lesssim \|f\|_{H_{w,N}^p(\mathbb{R}^n;A)}^p, \end{aligned}$$

which gives (5.5). This completes the proof of Lemma 5.4. □

The following is one of the main results in this paper.

Theorem 5.5. *Let A be an expansive dilation, $w \in \mathcal{A}_\infty$ and q_w be as in* (2.8). *If $q \in (q_w, \infty]$, $p \in (0,1]$, $N \geq N_{p,w}$, and $s \geq \lfloor (q_w/p - 1)\ln b/\ln(\lambda_-)\rfloor$, then $H_w^{p,q,s}(\mathbb{R}^n; A) = H_{w,N}^p(\mathbb{R}^n; A) = H_{w,N_{p,w}}^p(\mathbb{R}^n; A)$ with equivalent norms.*

Proof. Observe that by (3.5), Definition 3.3 and Theorem 3.6, we have

$$\begin{aligned} H_w^{p,\infty,\tilde{s}}(\mathbb{R}^n; A) \subset H_w^{p,q,s}(\mathbb{R}^n; A) &\subset H_{w,N_{p,w}}^p(\mathbb{R}^n; A) \\ &\subset H_{w,N}^p(\mathbb{R}^n; A) \subset H_{w,\tilde{N}}^p(\mathbb{R}^n; A), \end{aligned}$$

where $\tilde{s}$ is an integer no less than s and $\tilde{N}$ is an integer larger than N, and the inclusions are continuous. Thus, to finish the proof of Theorem 5.5, it suffices to prove that for any $N > s \geq \lfloor q_w \ln b/[p\ln(\lambda_-)]\rfloor$, $H_{w,N}^p(\mathbb{R}^n; A) \subset H_w^{p,\infty,s}(\mathbb{R}^n; A)$, and for all $f \in H_{w,N}^p(\mathbb{R}^n; A)$, $\|f\|_{H_w^{p,\infty,s}(\mathbb{R}^n;A)} \lesssim \|f\|_{H_{w,N}^p(\mathbb{R}^n;A)}$.

To this end, let $f \in H_{w,N}^p(\mathbb{R}^n; A)$. By Corollary 4.11, there exists a sequence of functions, $\{f_m\}_{m\in\mathbb{N}} \subset (H_{w,N}^p(\mathbb{R}^n; A) \cap L_w^q(\mathbb{R}^n))$, such that $\|f_m\|_{H_{w,N}^p(\mathbb{R}^n;A)} \leq 2^{-m}\|f\|_{H_{w,N}^p(\mathbb{R}^n;A)}$ and $f = \sum_{m\in\mathbb{N}} f_m$ in $H_{w,N}^p(\mathbb{R}^n; A)$. By Lemma 5.4, for each $m \in \mathbb{N}$, f_m has an atomic decomposition $f_m = \sum_{i\in\mathbb{N}} \lambda_i^m a_i^m$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\sum_{i\in\mathbb{N}} |\lambda_i^m|^p \leq C\|f_m\|_{H_{w,N}^p(\mathbb{R}^n;A)}^p$ and $\{a_i^m\}_{i\in\mathbb{N}}$ are $(p,\infty,s)_w$-atoms. Since

$$\sum_{m\in\mathbb{N}}\sum_{i\in\mathbb{N}} |\lambda_i^m|^p \lesssim \sum_{m\in\mathbb{N}} \|f_m\|_{H_{w,N}^p(\mathbb{R}^n;A)}^p \lesssim \|f\|_{H_{w,N}^p(\mathbb{R}^n;A)}^p,$$

then $f = \sum_{m\in\mathbb{N}}\sum_{i\in\mathbb{N}} \lambda_i^m a_i^m \in H_w^{p,\infty,s}(\mathbb{R}^n; A)$ and $\|f\|_{H_w^{p,\infty,s}(\mathbb{R}^n;A)} \lesssim \|f\|_{H_{w,N}^p(\mathbb{R}^n;A)}$, which completes the proof of Theorem 5.5. □

For simplicity, from now on, we denote by $H_w^p(\mathbb{R}^n; A)$ the weighted Hardy space $H_{w,N}^p(\mathbb{R}^n; A)$ associated with A and w, where $N \geq N_{p,w}$. Moreover, it is easy to see that $H_w^1(\mathbb{R}^n; A) \subset L_w^1(\mathbb{R}^n)$ via weighted atomic decomposition. However, generally speaking, the elements in $H_w^p(\mathbb{R}^n; A)$ with $p \in (0,1)$ are not necessarily functions and thus $H_w^p(\mathbb{R}^n; A) \neq L_w^p(\mathbb{R}^n)$. But, for any $q \in (q_w, \infty)$, following (5.5) in Lemma 5.4 and pointwise convergence of weighted atomic decompositions, we have $(H_w^p(\mathbb{R}^n; A) \cap L_w^q(\mathbb{R}^n)) \subset L_w^p(\mathbb{R}^n)$, and for all $f \in (H_w^p(\mathbb{R}^n; A) \cap L_w^q(\mathbb{R}^n))$, $\|f\|_{L_w^p(\mathbb{R}^n)} \leq \|f\|_{H_w^p(\mathbb{R}^n;A)}$.

6. FINITE ATOMIC DECOMPOSITIONS

In this section, we prove that for any given finite linear combination of weighted atoms when $q < \infty$ (or continuous weighted atoms when $q = \infty$), its norm in $H_w^p(\mathbb{R}^n; A)$ can be achieved via all its finite weighted atomic decompositions.

This extends Theorem 1.1 due to Meda, Sjögren, and Vallarino [24] to the setting of weighted anisotropic Hardy spaces.

Definition 6.1. Let A be an expansive dilation, $w \in \mathcal{A}_\infty$ and $(p,q,s)_w$ be an admissible triplet. Denote by $H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$ the *vector space of all finite linear combinations of* $(p,q,s)_w$*-atoms*, and the *norm* of f in $H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$ is defined by

$$\|f\|_{H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)} = \inf\Big\{\Big[\sum_{j=1}^{k}|\lambda_j|^p\Big]^{1/p} : f = \sum_{j=1}^{k}\lambda_j a_j,\ k\in\mathbb{N},\ \{a_i\}_{i=1}^k \text{ are } (p,q,s)_w\text{-atoms}\Big\}.$$

Obviously, for any admissible triplet $(p,q,s)_w$, the set $H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$ is dense in $H^{p,q,s}_w(\mathbb{R}^n;A)$ with respect to the quasi-norm $\|\cdot\|_{H^{p,q,s}_w(\mathbb{R}^n;A)}$.

Theorem 6.2. *Let A be an expansive dilation, $w \in \mathcal{A}_\infty$, q_w be as in* (2.8)*, and $(p,q,s)_w$ be an admissible triplet.*

(i) *If $q \in (q_w,\infty)$, then $\|\cdot\|_{H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)}$ and $\|\cdot\|_{H^p_w(\mathbb{R}^n;A)}$ are equivalent quasi-norms on $H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$.*

(ii) *$\|\cdot\|_{H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)}$ and $\|\cdot\|_{H^p_w(\mathbb{R}^n;A)}$ are equivalent quasi-norms on $H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)\cap C(\mathbb{R}^n)$.*

Proof. Obviously, $H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A) \subset H^p_w(\mathbb{R}^n;A)$ and for all $f \in H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$,

$$\|f\|_{H^p_w(\mathbb{R}^n;A)} \le \|f\|_{H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)}.$$

Thus we only need to prove that there exists a positive constant C such that for all $f \in H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$ when $q \in (q_w,\infty)$ and for all $f \in (H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)\cap C(\mathbb{R}^n))$ when $q=\infty$, $\|f\|_{H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)} \le C\|f\|_{H^p_w(\mathbb{R}^n;A)}$.

Step 1. Assume that $q \in (q_w,\infty]$, and by homogeneity, $f \in H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$ and

$$\|f\|_{H^p_w(\mathbb{R}^n;A)} = 1.$$

Notice that f has compact support. Suppose that $\operatorname{supp} f \subset B_{k_0}$ for some $k_0 \in \mathbb{Z}$, where B_{k_0} is as in Section 2. For each $k \in \mathbb{Z}$, set

$$\Omega_k \equiv \{x \in \mathbb{R}^n : M_N(f)(x) > 2^k\},$$

where and in what follows $N \equiv N_{p,w}$. We use the same notation as in Lemma 5.4. Since $f \in (H^p_w(\mathbb{R}^n;A)\cap L^{\tilde q}_w(\mathbb{R}^n))$, where $\tilde q = q$ if $q<\infty$ and $\tilde q = q_w+1$ if $q=\infty$, by Lemma 5.4, there exist numbers $\{\lambda^k_i\}_{k\in\mathbb{Z},i} \subset \mathbb{C}$ and $(p,\infty,s)_w$-atoms $\{a^k_i\}_{k\in\mathbb{Z},i}$ such that $f = \sum_{k\in\mathbb{Z}}\sum_i \lambda^k_i a^k_i$ holds almost everywhere and in $\mathcal{S}'(\mathbb{R}^n)$, and moreover, (5.1) through (5.5) in Lemma 5.4 hold.

Step 2. Let $m \equiv 4\sigma$. We first claim that there exists a positive constant $\tilde{C}$ such that for all $x \in (B_{m+k_0})^{\complement}$, $M_N(f)(x) \le \tilde{C}[w(B_{k_0})]^{-1/p}$. To see this, for any fixed $x \in (B_{m+k_0})^{\complement}$, by (2.12), write

$$\begin{aligned} M_N(f)(x) &\lesssim M_N^0(f)(x) \\ &\lesssim \sup_{\varphi \in S_N(\mathbb{R}^n)} \sup_{k \ge k_0} |f * \varphi_k(x)| + \sup_{\varphi \in S_N(\mathbb{R}^n)} \sup_{k < k_0} |f * \varphi_k(x)| \\ &\equiv \mathrm{I} + \mathrm{II}. \end{aligned}$$

Let $\theta \in S(\mathbb{R}^n)$ such that $\operatorname{supp} \theta \subset B_\sigma$, $0 \le \theta \le 1$ and $\theta \equiv 1$ on B_0. For $k \ge k_0$, from $\operatorname{supp} f \subset B_{k_0}$, it follows that

$$f * \varphi_k(x) = \int_{\mathbb{R}^n} \varphi_k(x - z)\theta(A^{-k_0}z)f(z)\,\mathrm{d}z \equiv f * \phi_{k_0}(0), \tag{6.1}$$

where $\phi(z) \equiv b^{k_0-k}\varphi(A^{-k}x + A^{k_0-k}z)\theta(-z)$ and ϕ_k is defined as in (2.11). Notice that for any $|\alpha| \le N$, by (2.6), (2.7), $\lambda_- > 1$, $k \ge k_0$ and $\|\varphi\|_{S_N(\mathbb{R}^n)} \le 1$, we have

$$|\partial^\alpha[\varphi(A^{k_0-k}\cdot)](z)| \lesssim (\lambda_-)^{(k_0-k)|\alpha|}\|\varphi\|_{S_N(\mathbb{R}^n)} \lesssim 1.$$

This together with the product rule and $\operatorname{supp} \theta \subset B_\sigma$ further implies that

$$\begin{aligned} &\|\phi\|_{S_N(\mathbb{R}^n)} \\ &\quad= \sup_{|\alpha| \le N} \sup_{z \in B_\sigma} \left|\partial_z^\alpha[\varphi(A^{-k}x + A^{k_0-k}z)\theta(-z)]\right| [1 + \rho(z)]^N \lesssim 1. \end{aligned} \tag{6.2}$$

Therefore, noticing that $(\|\phi\|_{S_N(\mathbb{R}^n)})^{-1}\phi \in S_N(\mathbb{R}^n)$ and for any $u \in B_{k_0}$, $0 \in u + B_{k_0}$, by the definition of M_N, we have that for any $u \in B_{k_0}$,

$$M_N(f)(u) \ge \sup_{y \in u + B_{k_0}} \left| \left(\frac{\phi}{\|\phi\|_{S_N(\mathbb{R}^n)}} \right)_{k_0} * f(y) \right| \ge \frac{1}{\|\phi\|_{S_N(\mathbb{R}^n)}} |f * \phi_{k_0}(0)|,$$

which together with (6.1) and (6.2) further implies that

$$|f * \varphi_k(x)| \le \|\phi\|_{S_N(\mathbb{R}^n)} \inf_{u \in B_{k_0}} M_N(f)(u) \lesssim \inf_{u \in B_{k_0}} M_N(f)(u),$$

and hence, $\mathrm{I} \lesssim \inf_{u \in B_{k_0}} M_N(f)(u)$. Thus, by $\|f\|_{H_w^p(\mathbb{R}^n;A)} = 1$, we further have

$$\mathrm{I} \lesssim [w(B_{k_0})]^{-1/p} \|M_N(f)\|_{L_w^p(\mathbb{R}^n)} \lesssim [w(B_{k_0})]^{-1/p}.$$

For $k < k_0$ and $u \in B_{k_0}$, since $\operatorname{supp} f \subset B_{k_0}$ and $\theta \equiv 1$ on B_0, we have

$$f * \varphi_k(x) = \int_{\mathbb{R}^n} \varphi_k(x - z)\theta(A^{-k_0}z)f(z)\,\mathrm{d}z \equiv f * \psi_k(u),$$

where $\psi(z) \equiv \varphi(A^{-k}(x-u)+z)\theta(A^{-k_0}u - A^{k-k_0}z)$ and ψ_k is defined as in (2.11). Notice that if $z \in \operatorname{supp}\psi$, then $\rho(A^{-k_0}u - A^{k-k_0}z) < b^{\sigma}$ and therefore $\rho(z) < b^{2\sigma+k_0-k}$. Thus, using (2.1) and (2.2),

$$\begin{aligned} A^{-k}(x-u)+z \in (B_{m+k_0-k})^{\complement} + B_{k_0-k} + B_{k_0-k+2\sigma} \\ \subset (B_{4\sigma+k_0-k})^{\complement} + B_{k_0-k+3\sigma} \subset (B_{3\sigma+k_0-k})^{\complement}. \end{aligned}$$

This implies that $\rho(A^{-k}(x-u)+z) > b^{k_0-k+3\sigma}$. Since $\varphi \in S_N(\mathbb{R}^n)$ and $k < k_0$, we have

$$\|\psi\|_{S_N(\mathbb{R}^n)} \lesssim \sup_{|\alpha|\le N}\ \sup_{z\in\operatorname{supp}\psi} (\lambda_-)^{(k-k_0)|\alpha|}\left[\frac{1+\rho(z)}{1+\rho(A^{-k}(x-u)+z)}\right]^N \lesssim 1.$$

Thus, by an argument similar to I, we have II $\lesssim \inf_{u\in B_0} M_N(f)(u) \lesssim [w(B_{k_0})]^{-1/p}$. Combining the estimates of I and II verifies the claim.

Step 3. Denote by k' the largest integer k such that $2^k < \tilde{C}[w(B_{k_0})]^{-1/p}$, where $\tilde{C}$ is as in Step 2. Then, we have

$$\Omega_k \subset B_{m+k_0} \quad \text{for } k > k'. \tag{6.3}$$

Set $h = \sum_{k\le k'}\sum_i \lambda_i^k a_i^k$ and $\ell = \sum_{k>k'}\sum_i \lambda_i^k a_i^k$, where the series converge almost everywhere and in $S'(\mathbb{R}^n)$. Clearly $f = h+\ell$, and $\operatorname{supp}\ell \subset \bigcup_{k>k'}\Omega_k \subset B_{m+k_0}$ for all $k > k'$, which together with $\operatorname{supp} f \subset B_{m+k_0}$ further yields $\operatorname{supp} h \subset B_{m+k_0}$.

Notice that for any $q \in (q_w, \infty]$ and $q_1 \in (1, q/q_w)$, by the Hölder inequality and $w \in \mathcal{A}_{q/q_1}$, we have

$$\int_{\mathbb{R}^n} |f(x)|^{q_1}\,dx \le b^{k_0}\|f\|^{q_1}_{L^q_w(\mathbb{R}^n)}[w(B_{k_0})]^{-q_1/q} < \infty.$$

Observing that $\operatorname{supp} f \subset B_{k_0}$ and f has vanishing moments up to order s, we have that f is a multiple of a $(1, q_1, 0)$-atom and therefore $M_N(f) \in L^1(\mathbb{R}^n)$. Then by (6.3), (5.1), (5.4) in Lemma 5.4 and Lemma 5.1 (ii), for any $|\alpha| \le s$, we have

$$\int_{\mathbb{R}^n}\sum_{k>k'}\sum_i |\lambda_i^k a_i^k(x)x^{\alpha}|\,dx \lesssim \sum_{k\in\mathbb{Z}} 2^k|\Omega_k| \lesssim \|M_N(f)\|_{L^1(\mathbb{R}^n)} < \infty.$$

This together with the vanishing moments of a_i^k implies that ℓ has vanishing moments up to order s and thus so does h by $h = f - \ell$. Using Lemma 5.1 (ii), (5.4) in Lemma 5.4 and the fact $2^{k'} \le C[w(B_{m+k_0})]^{-1/p}$, we have

$$|h(x)| \lesssim \sum_{k\le k'} 2^k \lesssim [w(B_{m+k_0})]^{-1/p}.$$

Thus there exists a positive constant C_0, independent of f, such that h/C_0 is a $(p,\infty,s)_w$-atom and by Definition 3.2, it is also a $(p,q,s)_w$-atom for any admissible triplet $(p,q,s)_w$.

Step 4. To prove (i), let $q \in (q_w, \infty)$. We first verify $\sum_{k>k'}\sum_i \lambda_i^k a_i^k \in L_w^q(\mathbb{R}^n)$. For any $x \in \mathbb{R}^n$, since $\mathbb{R}^n = \bigcup_{k\in\mathbb{Z}}(\Omega_k \setminus \Omega_{k+1})$, there exists $j \in \mathbb{Z}$ such that $x \in (\Omega_j \setminus \Omega_{j+1})$. Since $\operatorname{supp} a_i^k \subset B_{\ell_i^k+\sigma} \subset \Omega_k \subset \Omega_{j+1}$ for $k > j$, then applying Lemma 5.1 (ii) and (5.4) in Lemma 5.4, we have

$$\sum_{k>k'}\sum_i |\lambda_i^k a_i^k(x)| \lesssim \sum_{k\le j} 2^k \lesssim 2^j \lesssim M_N(f)(x).$$

Since $f \in L_w^q(\mathbb{R}^n)$, we have $M_N(f) \in L_w^q(\mathbb{R}^n)$; by the Lebesgue dominated convergence theorem, we further obtain $\sum_{k>k'}\sum_i \lambda_i^k a_i^k$ converges to ℓ in $L_w^q(\mathbb{R}^n)$.

Now, for any positive integer K, set $F_K = \{(i,k) : k > k',\ |i| + |k| \le K\}$ and $\ell_K = \sum_{(i,k)\in F_K} \lambda_i^k a_i^k$. Observing that for any $\varepsilon \in (0,1)$, if K is large enough, by $\ell \in L_w^q(\mathbb{R}^n)$, we have $(\ell - \ell_K)/\varepsilon$ is a $(p,q,s)_w$-atom. Thus, $f = h + \ell_K + (\ell - \ell_K)$ is a finite linear weighted atom combination of f. By (5.5) in Lemma 5.4 and Step 3, we have

$$\|f\|^p_{H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)} \le (C_0)^p + \sum_{(i,k)\in F_K} |\lambda_i^k|^p + \varepsilon^p \lesssim 1,$$

which ends the proof of (i).

Step 5. To prove (ii), assume that f is a continuous function in $H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$; then a_i^k is continuous by examining its definition (see also (5.7)). Since

$$M_N(f)(x) \le C_{n,N}\|f\|_{L_w^\infty(\mathbb{R}^n)} \quad \text{for } x \in \mathbb{R}^n,$$

where the constant $C_{n,N}$ only depends on n and N, then the level set Ω_k is empty for all k such that $2^k \ge C_{n,N}\|f\|_{L_w^\infty(\mathbb{R}^n)}$. We denote by k'' the largest integer for which the above inequality does not hold. Then the index k in the sum defining ℓ will run only over $k' < k \le k''$.

Let $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that if $\rho(x-y) < \delta$, then $|f(x) - f(y)| < \varepsilon$. Write $\ell = \ell_1^\varepsilon + \ell_2^\varepsilon$ with $\ell_1^\varepsilon \equiv \sum_{(i,k)\in F_1} \lambda_i^k a_i^k$ and $\ell_2^\varepsilon \equiv \sum_{(i,k)\in F_2} \lambda_i^k a_i^k$, where $F_1 \equiv \{(i,k) : b^{\ell_i^k+\sigma} \ge \delta,\ k' < k \le k''\}$ and $F_2 \equiv \{(i,k) : b^{\ell_i^k+\sigma} < \delta,\ k' < k \le k''\}$.

On the other hand, for any fixed integer $k \in (k', k'']$, by (5.3) in Lemma 5.4 and $\Omega_k \subset B_{m+k_0}$, we see that F_1 is a finite set, and thus ℓ_1^ε is continuous.

For any $(i,k) \in F_2$ and $x \in x_i^k + B_{\ell_i^k+\sigma}$, $|f(x) - f(x_i^k)| < \varepsilon$. Write $\tilde{f}(x) \equiv [f(x) - f(x_i^k)]\chi_{B_{\ell_i^k+\sigma}}(x)$ and $\tilde{P}_i^k(x) \equiv P_i^k(x) - f(x_i^k)$. By the definition of P_i^k in Section 5, for all $Q \in \mathcal{P}_s(\mathbb{R}^n)$, we have

$$\frac{1}{\int_{\mathbb{R}^n} \zeta_i^k(x)\,dx}\int_{\mathbb{R}^n} [\tilde{f}(x) - \tilde{P}_i^k(x)]Q(x)\zeta_i^k(x)\,dx = 0.$$

Since $|\tilde{f}(x)| < \varepsilon$ for all $x \in \mathbb{R}^n$ implies $M_N(\tilde{f})(x) \lesssim \varepsilon$ for all $x \in \mathbb{R}^n$, then by Lemma 4.4, we have

$$\sup_{y\in\mathbb{R}^n} |\tilde{P}_i^k(y)\zeta_i^k(y)| \lesssim \sup_{y\in\mathbb{R}^n} M_N(\tilde{f})(y) \lesssim \varepsilon. \tag{6.4}$$

Let $\tilde{P}_{i,j}^{k+1} \in \mathcal{P}_s(\mathbb{R}^n)$ be such that

$$\int_{\mathbb{R}^n} [\tilde{f}(x) - \tilde{P}_j^{k+1}(x)]\zeta_i^k(x)Q(x)\zeta_j^{k+1}(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \tilde{P}_{i,j}^{k+1}(x)Q(x)\zeta_j^{k+1}(x)\,\mathrm{d}x.$$

Since $[\tilde{f} - \tilde{P}_j^{k+1}]\zeta_i^k = [f - P_j^{k+1}]\zeta_i^k$ by $\operatorname{supp}\zeta_i^k \subset B_{\ell_i^k+\sigma}$, we have $\tilde{P}_{i,j}^{k+1} = P_{i,j}^{k+1}$. Then by Lemma 5.2, we obtain

$$\sup_{y\in\mathbb{R}^n} |\tilde{P}_{i,j}^{k+1}(y)\zeta_j^{k+1}(y)| \lesssim \sup_{y\in\mathbb{R}^n} M_N(\tilde{f})(y) \lesssim \varepsilon. \tag{6.5}$$

Thus by the definition of $\lambda_i^k a_i^k$, $\sum_j \zeta_j^{k+1} = \chi_{\Omega_{k+1}}$ and (5.7), we have

$$\begin{aligned}\lambda_i^k a_i^k &= (f - P_i^k)\zeta_i^k - \sum_j [(f - P_j^{k+1})\zeta_i^k - P_{i,j}^{k+1}]\zeta_j^{k+1}\\ &= \tilde{f}\chi_{(\Omega_{k+1})^\complement}\zeta_i^k - \tilde{P}_i^k\zeta_i^k + \sum_j \tilde{P}_j^{k+1}\zeta_j^k\zeta_j^{k+1} + \sum_j \tilde{P}_{i,j}^{k+1}\zeta_j^{k+1}.\end{aligned}$$

From this together with (6.4), (6.5) and Lemma 5.1 (ii), it follows that $|\lambda_i^k a_i^k(x)| \lesssim \varepsilon$ for all $x \in x_i^k + B_{\ell_i^k+\sigma}$ and $(i,k) \in F_2$.

Moreover, using Lemma 5.1 (ii) again, we have

$$|\ell_2^\xi| \le C \sum_{k'<k\le k''} \varepsilon \lesssim (k'' - k')\varepsilon.$$

Since ε is arbitrary, we can thus split ℓ into a continuous part and a part that is uniformly arbitrarily small. It follows that ℓ is continuous. Then, $h = f - \ell$ is a C_0 multiple of a continuous $(p,\infty,s)_w$-atom by Step 3.

Step 6. To find a finite atomic decomposition of ℓ, we use again the splitting $\ell = \ell_1^\xi + \ell_2^\xi$. Clearly, for each ε, ℓ_1^ξ is a finite linear combination of continuous $(p,q,s)_w$-atoms, and the ℓ^p norm of the coefficients is controlled by $\|f\|_{H_w^p(\mathbb{R}^n;A)}$ in view of (5.5) in Lemma 5.4. Observe that $\ell_2^\xi = \ell - \ell_1^\xi$ is continuous. Moreover, since $\operatorname{supp}\ell_2^\xi \subset B_{m+k_0}$, ℓ_2^ξ has vanishing moments up to order s and satisfies $|\ell_2^\xi| \lesssim (k'' - k')\varepsilon$. Choosing ε small enough, we can make ℓ_2^ξ into an arbitrarily small multiple of a continuous $(p,\infty,s)_w$-atom.

To sum up, $f = h + \ell_1^\xi + \ell_2^\xi$ gives the desired finite atomic decomposition of f with coefficients controlled by $\|f\|_{H_w^p(\mathbb{R}^n;A)}$. This finishes the proof of (ii) and hence, the proof of Theorem 6.2. □

7. APPLICATIONS

As an application of finite atomic decompositions, we establish boundedness in $H^p_w(\mathbb{R}^n; A)$ of quasi-Banach-valued sublinear operators.

Recall that a *quasi-Banach space* $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is nonnegative, non-degenerate (i.e., $\|f\|_{\mathcal{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, i.e., there exists a positive constant K no less than 1 such that for all $f, g \in \mathcal{B}$, $\|f + g\|_{\mathcal{B}} \le K(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}})$.

Definition 7.1. Let $\gamma \in (0, 1]$. A quasi-Banach space $\mathcal{B}_\gamma$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_\gamma}$ is said to be a *γ-quasi-Banach space* if $\|f + g\|^\gamma_{\mathcal{B}_\gamma} \le \|f\|^\gamma_{\mathcal{B}_\gamma} + \|g\|^\gamma_{\mathcal{B}_\gamma}$ for all $f, g \in \mathcal{B}_\gamma$.

Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach spaces ℓ^γ, $L^\gamma_w(\mathbb{R}^n)$ and $H^\gamma_w(\mathbb{R}^n; A)$ with $\gamma \in (0, 1)$ are typical γ-quasi-Banach spaces. Moreover, according to the Aoki-Rolewicz theorem (see [1] or [28]), any quasi-Banach space is, essentially, a γ-quasi-Banach space, where $\gamma = [\log_2(2K)]^{-1}$.

For any given γ-quasi-Banach space $\mathcal{B}_\gamma$ with $\gamma \in (0, 1]$ and a linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_\gamma$ is said to be *$\mathcal{B}_\gamma$-sublinear* if for any $f, g \in \mathcal{Y}$ and λ, $\nu \in \mathbb{C}$, we have

$$\|T(\lambda f + \nu g)\|_{\mathcal{B}_\gamma} \le \left(|\lambda|^\gamma \|T(f)\|^\gamma_{\mathcal{B}_\gamma} + |\nu|^\gamma \|T(g)\|^\gamma_{\mathcal{B}_\gamma}\right)^{1/\gamma}$$

and $\|T(f) - T(g)\|_{\mathcal{B}_\gamma} \le \|T(f - g)\|_{\mathcal{B}_\gamma}$.

We remark that if T is linear, then T is $\mathcal{B}_\gamma$-sublinear. Moreover, if $\mathcal{B}_\gamma = L^q_w(\mathbb{R}^n)$, and T is nonnegative and sublinear in the classical sense, then T is also $\mathcal{B}_\gamma$-sublinear.

Theorem 7.2. *Let A be an expansive dilation, $w \in \mathcal{A}_\infty$, $0 < p \le \gamma \le 1$, and $\mathcal{B}_\gamma$ be a γ-quasi-Banach space. Suppose one of the following holds:*

(i) *$q \in (q_w, \infty)$, and $T : H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n; A) \to \mathcal{B}_\gamma$ is a $\mathcal{B}_\gamma$-sublinear operator such that*

$$S \equiv \sup\{\|T(a)\|_{\mathcal{B}_\gamma} : a \text{ is any } (p, q, s)_w\text{-atom}\} < \infty;$$

(ii) *T is a $\mathcal{B}_\gamma$-sublinear operator defined on continuous $(p, \infty, s)_w$-atoms such that*

$$S \equiv \sup\{\|T(a)\|_{\mathcal{B}_\gamma} : a \text{ is any continuous } (p, \infty, s)_w\text{-atom}\} < \infty.$$

Then there exists a unique bounded $\mathcal{B}_\gamma$-sublinear operator $\tilde{T}$ from $H^p_w(\mathbb{R}^n; A)$ to $\mathcal{B}_\gamma$ which extends T.

Proof. Suppose that the assumption (i) holds. For any $f \in H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n; A)$, by Theorem 6.2 (i), there exist numbers $\{\lambda_j\}^\ell_{j=1} \subset \mathbb{C}$ and $(p, q, s)_w$-atoms $\{a_j\}^\ell_{j=1}$

such that $f = \sum_{j=1}^{\ell} \lambda_j a_j$ pointwise and $\sum_{j=1}^{\ell} |\lambda_j|^p \lesssim \|f\|^p_{H^p_w(\mathbb{R}^n;A)}$. Then by the assumption (i), we have

$$\|T(f)\|_{\mathcal{B}_\gamma} \lesssim \Big[\sum_{j=1}^{\ell} |\lambda_j|^p\Big]^{1/p} \lesssim \|f\|_{H^p_w(\mathbb{R}^n;A)}.$$

Since $H^{p,q,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$ is dense in $H^p_w(\mathbb{R}^n;A)$, a density argument gives the desired result.

Suppose that the assumption (ii) holds. Similarly to the proof of (i), using Theorem 6.2 (ii), we also have that for all $f \in (H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A) \cap C(\mathbb{R}^n))$, $\|T(f)\|_{\mathcal{B}_\gamma} \lesssim \|f\|_{H^p_w(\mathbb{R}^n;A)}$. To extend T to the whole $H^p_w(\mathbb{R}^n;A)$, we only need to prove that $H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A) \cap C(\mathbb{R}^n)$ is dense in $H^p_w(\mathbb{R}^n;A)$. Since $H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$ is dense in $H^p_w(\mathbb{R}^n;A)$, it suffices to prove $H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A) \cap C(\mathbb{R}^n)$ is dense in $H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$ in the quasi-norm $\|\cdot\|_{H^p_w(\mathbb{R}^n;A)}$. Actually, we will show that $H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A) \cap C^\infty(\mathbb{R}^n)$ is dense in $H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$.

To see this, let $f \in H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A)$. Since f is a finite linear combination of functions with bounded supports, there exists $\ell \in \mathbb{Z}$ such that $\operatorname{supp} f \subset B_\ell$. Take $\varphi \in S(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset B_0$ and $\int_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}x = 1$. By (2.1), it is easy to check that $\operatorname{supp}(\varphi_k * f) \subset B_{\ell+\sigma}$ for $k < \ell$, and $f * \varphi_k$ has vanishing moments up to order s, where $\varphi_k(x) = b^{-k}\varphi(A^{-k}x)$ for all $x \in \mathbb{R}^n$. Hence, $f * \varphi_k \in H^{p,\infty,s}_{w,\mathrm{fin}}(\mathbb{R}^n;A) \cap C^\infty(\mathbb{R}^n)$.

Likewise, $\operatorname{supp}(f - f * \varphi_k) \subset B_{\ell+\sigma}$ for $k < \ell$, and $f - f * \varphi_k$ has vanishing moments up to order s. Take any $q \in (q_w, \infty)$. By Proposition 2.9 (ii),

$$\|f - f * \varphi_k\|_{L^q_w(\mathbb{R}^n)} \to 0 \quad \text{as } k \to -\infty.$$

Hence, $f - f * \varphi_k = c_k a_k$ for some $(p,q,s)_w$-atom a_k, and the constants $c_k \to 0$ as $k \to -\infty$. Thus, $\|f - f * \varphi_k\|_{H^p_w(\mathbb{R}^n;A)} \to 0$ as $k \to -\infty$. This completes the proof of Theorem 7.2. □

Remark 7.3. It is obvious that if T is a bounded $\mathcal{B}_\gamma$-sublinear operator from $H^p_w(\mathbb{R}^n;A)$ to $\mathcal{B}_\gamma$, then for any admissible triplet $(p,q,s)_w$, T maps all $(p,q,s)_w$-atoms into uniformly bounded elements of $\mathcal{B}_\gamma$. Theorem 7.2 shows that the converse is true when $q < \infty$. However, such converse is generally false for $q = \infty$ due to the example in [3, Theorem 2]. That is, there exists an operator T_∞ uniformly bounded on $(1,\infty,0)$-atoms, which does not have a bounded extension to $H^1(\mathbb{R}^n)$.

Despite this, Theorem 7.2 (ii) shows that the uniform boundedness of T on a smaller class of *continuous* $(p,\infty,s)_w$-atoms, implies the existence of a bounded extension on the whole space $H^p_w(\mathbb{R}^n;A)$. In particular, the restriction of the operator T_∞ to the subspace $H^{1,\infty,0}_{\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ does have a bounded extension, denoted by $\tilde{T}_\infty$, to $H^1(\mathbb{R}^n)$, whereas T_∞ itself does not have this property. To be precise, T_∞ and $\tilde{T}_\infty$ coincide on continuous $(1,\infty,0)_w$-atoms, but not on all $(1,\infty,0)_w$-atoms; see also [24]. This shows the necessity of using only continuous atoms when $q = \infty$ in Theorem 7.2 (ii). Consequently, such a bounded extension must be obtained in a rather delicate and non-trivial way using only finite decompositions into continuous atoms.

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MARCIN BOWNIK:
Department of Mathematics
University of Oregon
Eugene, OR 97403-1222, U.S.A.
E-MAIL: mbownik@uoregon.edu

BAODE LI, DACHUN YANG (corresponding author) & YUAN ZHOU:
School of Mathematical Sciences
Beijing Normal University
Laboratory of Mathematics and Complex Systems
Ministry of Education
Beijing 100875, P.R. China.
E-MAIL: baodeli@mail.bnu.edu.cn
E-MAIL: dcyang@bnu.edu.cn
E-MAIL: yuanzhou@mail.bnu.edu.cn

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