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On the existence of multiresolution analysis for framelets ⋆

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Abstract. We show that a compactly supported tight framelet comes from an MRA if the intersection of all dyadic dilations of the space of negative dilates, which is defined as the shift-invariant space generated by the negative scales of a framelet, is trivial. We also construct examples of (non-tight) framelets, which are arbitrarily close to tight frame framelets, such that the corresponding space of negative dilates is equal to the entire space $L^2(\mathbb{R})$.

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1. Introduction

For a function $\psi \in L^2(\mathbb{R})$, we define its affine system by

$$\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k) \quad j, k \in \mathbb{Z}.$$

If the system is an orthonormal basis of $L^2(\mathbb{R})$, then we call ψ a wavelet. In the more general case when the system forms a frame for $L^2(\mathbb{R})$, we call ψ a framelet, or a tight framelet if the frame is tight (with constant 1). A general procedure for constructing tight framelets was presented by Ron and Shen [RS1] and also by Weiss at al. [PSWX1]. Recently, interesting examples of compactly supported tight framelets were exhibited in [RS2], [GR], [CH], [CHS], [CHSS], [Ha] and [DHRS]. The main tool used in these constructions is the multiresolution analysis (MRA) structure. The reason for this may be very simple. A well known result of Lemarié [Le] asserts that compactly supported wavelets come from an MRA and it is a common feeling that the same should be true for compactly supported tight framelets. In the paper we shall investigate this problem.

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The key ingredient in our study is the *space of negative dilates* V of a framelet ψ defined as

$$V = \overline{\operatorname{span}}\{\psi_{j,k} : j < 0, k \in \mathbb{Z}\}.$$

In the first part of this work we concentrate on tight framelets for which the space V is "small", in the sense that the intersection of all dyadic dilations of V is trivial. In this case we give necessary and sufficient conditions for a tight framelet ψ to come from a multiresolution analysis, generalizing a well-known characterization of MRA wavelets (see [Gr] and [HW]). As a corollary, we obtain that under the condition of V being "small", all compactly supported tight framelets must come from an MRA.

The second part of the paper is devoted to showing that, somewhat counter-intuitively, the space of negative dilates can indeed be very large. More precisely, we construct a (non-tight) framelet, whose space of negative dilates is the whole space $L^2(\mathbb{R})$. This framelet can be chosen in such a way that its lower and upper frame constants are arbitrarily close to one. We shall also show that the framelet has a dual framelet.

2. Preliminaries

Despite the fact that all of our results are motivated by the classical case of dyadic dilations in $\mathbb{R}$ we will adopt a more general setting of an expansive integer-valued matrix, i.e., an $n \times n$ matrix whose eigenvalues have modulus greater than 1. That is, we shall assume that we are given an $n \times n$ expansive matrix A with integer entries, which plays the role of the usual dyadic dilation.

We recall that a sequence $\{D^j(V) : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^n)$ is called an *MRA* if

(M1) V is shift-invariant
(M2) $V \subset D(V)$
(M3) $\overline{\bigcup_{j\in\mathbb{Z}} D^j(V)} = L^2(\mathbb{R}^n)$
(M4) $\bigcap_{j\in\mathbb{Z}} D^j(V) = \{0\}$
(M5) there exist a function $\varphi \in V$ such that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis of V.

Here, the *dilation* operator D is given by $D\psi(x) = |\det A|^{1/2}\psi(Ax)$ for some $n \times n$ expansive integer-valued matrix A. If only conditions (M1)–(M4) hold, then we say that the sequence $\{D^j(V)\ j \in \mathbb{Z}\}$ is a *generalized multiresolution analysis* (GMRA).

As we can see, a GMRA is based on the *core space* V. Condition (M1) means that V is invariant under integer shifts and can be concluded from (M5). It also allows us to use the theory of shift-invariant spaces for understanding the connections between the GMRA structure and wavelets or framelets. This is a subject of an extensive study by several authors, e.g. [BMM], [BG], [BW], [KKL], and [LTW].

We say that a finite family $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ is a wavelet if its associated affine system

$$\psi_{j,k}(x) = |\det A|^{j/2}\psi(A^j x - k), \qquad j \in \mathbb{Z}, k \in \mathbb{Z}^n, \psi \in \Psi$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$. In the more general case, when the affine system is a frame or tight frame (with constant 1), we say that Ψ is a framelet or a tight framelet.

It turns out that every wavelet comes from a GMRA. Indeed, for a finite family $\Psi \subset L^2(\mathbb{R}^n)$ we define its *space of negative dilates* V by

$$V = \overline{\operatorname{span}}\{\psi_{j,k} : j < 0, k \in \mathbb{Z}^n, \psi \in \Psi\}. \tag{2.1}$$

We say that a framelet Ψ is associated with an MRA, or shortly comes from an MRA, if its space V satisfies (M1)–(M5). It is not hard to check that if Ψ is a wavelet then its space of negative dilates satisfies conditions (M1)–(M4) and, therefore, it is a core space of a GMRA. If we want to see when a GMRA gives rise to a wavelet, or when condition (M5) is satisfied, then some knowledge of shift-invariant spaces is useful.

Every shift-invariant space $V \subset L^2(\mathbb{R}^n)$ has a *set of generators* Φ, that is, a countable family of functions whose integer shifts form a tight frame (with constant 1) for V. Although this family is not unique, the function

$$\sigma_V(\xi) = \sum_{\varphi \in \Phi} |\hat{\varphi}(\xi)|^2$$

does not depend (except on a set of null measure) on the choice of the family of generators. Here, the Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi\rangle}\, dx.$$

We call σ_V the *spectral function* of V. This notion was introduced by the authors in [BR]. The basic property of σ is that it is additive on countable orthogonal sums and that $\sigma_{L^2(\mathbb{R}^n)} = 1$. The spectral function also behaves nicely under dilations since $\sigma_{D(V)}(\xi) = \sigma_V((A^*)^{-1}\xi)$. Moreover, if V is generated by a single function φ then $\sigma_V(\xi) = |\hat{\varphi}(\xi)|^2(\sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(\xi + k)|^2)^{-1}$ for $\xi \in \operatorname{supp}\hat{\varphi}$ and 0 otherwise.

We also mention that there are several other equivalent ways of defining the spectral function among which we note the following formula

$$\sigma_V(\xi) = \lim_{\varepsilon\to 0} ||P_{\hat{V}}(\mathbf{1}_{(\xi-\varepsilon/2,\xi+\varepsilon/2)^n})||^2/\varepsilon^n \qquad \text{for a.e. } \xi \in \mathbb{R}^n,$$

where $P_{\hat{V}}$ denotes the orthogonal projection of the Fourier transform of V onto $L^2(\mathbb{R}^n)$.

The spectral function also allows us to define the *dimension function* of V

$$\dim_V(\xi) = \sum_{k\in\mathbb{Z}^n} \sigma_V(\xi + k).$$

The dimension function (also called the multiplicity function) is integer–valued and additive on countable orthogonal sums as well. Moreover, the minimal number of functions needed to generate V is equal to the L^∞ norm of $\dim_V$. In particular, V can be generated by a single function if and only if $\dim_V \le 1$. Moreover, condition (M5) is equivalent to the equation $\dim_V = 1$. We refer the reader to [BR] for the proofs of all these facts.

We can already see how the above information can be applied to connect GMRAs to framelets. If V is a core space of a GMRA, then the space $W = D(V) \ominus V$ is shift-invariant and has a (possibly infinite) set of generators Ψ . From (M2), (M3) and (M4) it follows that

$$L^2(\mathbb{R}^n) = \bigoplus_{j\in\mathbb{Z}} D^j(W), \tag{2.2}$$

so we conclude that Ψ is a tight framelet possibly of infinite order. That is, Ψ may have infinite number of generators and the affine system generated by the elements of Ψ forms a tight frame for $L^2(\mathbb{R}^n)$. Moreover, the framelet Ψ is *semi–orthogonal*. Semi–orthogonality means that the shift-invariant space

$$W = \overline{\text{span}}\{\psi(\cdot - k) : k \in \mathbb{Z}^n, \psi \in \Psi\}$$

generated by Ψ satisfies condition (2.2). On the other hand, it is also clear that if Ψ is a semi–orthogonal tight framelet (possibly of infinite order), then the space V of its negative dilates (given by (2.1)) satisfies conditions (M1)–(M4). Therefore, we can see that there is a perfect duality between GMRA structures and semi–orthogonal tight framelets (with possibly infinite number of generators).

The problem of characterizing GMRAs yielding wavelets can be solved in terms of the dimension function of the core space V. Clearly, a GMRA gives rise to a wavelet (with L generators) if and only if $\dim_W = L$, where $W = D(V) \ominus V$. Since $W \oplus V = D(V)$ we get

$$\sigma_W(\xi) + \sigma_V(\xi) = \sigma_{D(V)}(\xi) = \sigma_V((A^*)^{-1}\xi)$$

which implies

$$\dim_W(\xi) + \dim_V(\xi) = \sum_{d\in\mathcal{D}} \dim_V((A^*)^{-1}(\xi + d)),$$

where $\mathcal{D}$ consists of representatives of distinct cosets of $\mathbb{Z}^n/(A^*\mathbb{Z}^n)$. Therefore, the equation $\dim_W = L$ is equivalent to the *consistency equation* of Baggett

$$\sum_{d\in\mathcal{D}} \dim_V((A^*)^{-1}(\xi + d)) - \dim_V(\xi) = L, \tag{2.3}$$

provided that $\dim_V$ is finite a.e. Fortunately, if Ψ is a wavelet with L generators then $\dim_V$ is even integrable over [0,1] and

$$\int_{[0,1]^n} \dim_V(\xi)\, d\xi = L/(|\det A| - 1). \tag{2.4}$$

In this way we recover the main result of [BMM].

Theorem (2.5). *A GMRA gives rise to a wavelet if and only if the dimension function of its core space V satisfies (2.3) and (2.4).*

To see why (2.4) must be satisfied we recall the following basic fact (see [BR]) that will be also important in the next section.

Lemma (2.6). *If Ψ is a semi–orthogonal tight framelet and V is the space of negative dilates of Ψ then*

$$\sigma_V(\xi) = \sum_{\psi\in\Psi} \sum_{j=1}^{\infty} |\hat{\psi}((A^*)^j \xi)|^2 \tag{2.7}$$

and

$$dim_V(\xi) = \sum_{\psi\in\Psi} \sum_{k\in\mathbb{Z}^n} \sum_{j=1}^{\infty} |\hat{\psi}((A^*)^j (\xi + k))|^2. \tag{2.8}$$

To show (2.4) we observe that if Ψ is a wavelet then from (2.7) it follows that

$$\begin{aligned}\int_{[0,1]^n} \dim_V(\xi)\, d\xi &= \int_{\mathbb{R}^n} \sigma_V(\xi)\, d\xi \\ &= \sum_{\psi\in\Psi} \sum_{j=1}^{\infty} \int_{\mathbb{R}} |\hat{\psi}((A^*)^j \xi)|^2 \\ &= L/(|\det A| - 1),\end{aligned}$$

so (2.4) is proved.

Lemma (2.6) can be also used to recover Gripenberg's characterization of MRA wavelets, see [Gr]. We know already that condition (M5) is equivalent to the equation $\dim_V = 1$. By (2.8) this equation becomes

$$\sum_{\psi\in\Psi} \sum_{k\in\mathbb{Z}^n} \sum_{j=1}^{\infty} |\hat{\psi}((A^*)^j (\xi + k))|^2 = 1 \qquad \text{a.e.} \tag{2.9}$$

and we conclude that a wavelet Ψ comes from an MRA if and only if (2.9) is satisfied. Note that (2.9) together with (2.8) and (2.4) imposes the restriction on the number of generators L of an MRA wavelet Ψ. Thus, a necessary condition for Ψ to be a MRA wavelet is that $L = |\det A| - 1$.

In this setting, the previously mentioned one–dimensional result of Lemarié is also clear. If ψ is a compactly supported dyadic wavelet in $L^2(\mathbb{R})$ then supp $\hat{\psi} = \mathbb{R}$ a.e., so by (2.8) we have $\dim_V > 0$. Since $\dim_V$ is integer–valued, from (2.4) it follows that $\dim_V = 1$, thus condition (M5) holds.

The main goal of the paper is to extend the characterization of MRA wavelets to tight framelets. The argument leading to (2.9) is due to Auscher who proved that the function

$$D_\Psi(\xi) := \sum_{\psi\in\Psi} \sum_{k\in\mathbb{Z}^n} \sum_{j=1}^{\infty} |\hat{\psi}((A^*)^j(\xi + k))|^2 \tag{2.10}$$

is integer–valued (provided Ψ is a wavelet) without using the theory of shift–invariant spaces, see [Au]. This key step is missing in the case of tight framelets. In fact, in [PSWX2] it is shown that in one dimension for any tight framelet ψ, the function D_ψ is integer–valued if and only if the framelet is semi–orthogonal. In the next section we will see that the lack of D_ψ being integer–valued can be overcome and that the real difficulty in extending Lemarié's result to framelets lies in a completely different spot.

3. MRA for compactly supported tight framelets

All constructions of tight framelets appearing in the literature are based on the GMRA structure. The only notable exception is the paper [GHSW], where GMRA techniques are not used at all. Nevertheless, the question **"Does every tight framelet come from a GMRA?"** is still valid.

In general, when dealing with a tight framelet, we ask if the space of negative dilates V given in (2.1) is a core space of a GMRA. The starting point for answering this question is the following observation due to Baggett.

Proposition (3.1). *If Ψ is a tight framelet then its space of negative dilates V is shift-invariant.*

Proof. It is enough to prove that the orthogonal complement $V^\perp$ of V is shift-invariant. It is clear that this complement is given by

$$V^\perp = \{\, f \in L^2(\mathbb{R}) : \|f\|_2^2 = \sum_{\psi\in\Psi} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}^n} |\langle f, \psi_{j,k}\rangle|^2 \,\}$$

(this follows from the tight frame property). Thus, we can see immediately that the space $V^\perp$ is shift–invariant. □

We remark that the above result also holds if we relax the assumption of tightness by the requirement that the framelet Ψ has a canonical dual framelet with the same number of generators, or equivalently, that Ψ has period one in the terminology of Daubechies and Han [DH]. However, the above result in general is false for non-tight framelets and even for framelets which have a dual framelet. These facts were shown by the first author and Weber in [BW].

Proposition (3.1) proves condition (M1). The other two conditions, (M2) and (M3), are clearly satisfied leaving only (M4). This crucial obstacle was noted by Baggett who posed during his talk at Washington University in 1999 the following open problem.

Question (3.2). *Let Ψ be a tight framelet with the space of negative dilates V. Is it true that*

$$\bigcap_{j\in\mathbb{Z}} D^j(V) = \{0\}\,?$$

If the answer is positive then the sequence $\{D^j(V) : j \in \mathbb{Z}\}$ forms a GMRA and we could extend most of the results of the previous section to the case of tight framelets. For a long time we thought that the answer is affirmative even in the case of framelets, but recently we constructed an example of a framelet (not tight) with the space V equal to $L^2(\mathbb{R})$. This example is presented in the next section and indicates that the above problem is indeed quite serious. The main result of this section shows that there are no other obstacles in characterizing tight framelets associated with an MRA.

Theorem (3.3). *Let Ψ be a tight framelet with $L = |\det A| - 1$ generators such that its space of negative dilates V satisfies condition (M4). Then Ψ comes from an MRA if and only if*

$$D_\Psi(\xi) = \sum_{\psi\in\Psi}\sum_{k\in\mathbb{Z}}\sum_{j=1}^{\infty} |\hat{\psi}((A^*)^j(\xi + k))|^2 > 0 \quad a.e. \tag{3.4}$$

Remark. We recall that the restriction on the number of generators $L = |\det A|-1$ in Theorem (3.3) is a necessary condition for (orthogonal) wavelet Ψ to be associated with an MRA. In the case of tight framelets it is possible to have MRA constructions of tight framelets resulting with bigger number of generators, see [CH], [CHS], and [DHRS]. However, Theorem (3.3) is false without the assumption $L = |\det A| - 1$.

Before we give the proof of Theorem (3.3) we must emphasize that, in general, D_Ψ is not equal to $\dim_V$ for tight framelets. This is unlike the case of (orthogonal) wavelets, where it is well-known that D_Ψ equals $\dim_V$, see [RS3], [W] or Lemma (2.6). Nevertheless, it is possible to prove that for tight framelets both these functions have the same support, i.e.,

$$\operatorname{supp} D_\Psi = \operatorname{supp} \dim_V. \tag{3.5}$$

Indeed, from Proposition (3.1) it follows immediately that V is the shift-invariant space generated by the functions

$$\{D^{-j}\psi : \psi \in \Psi, j = 1, 2, \dots\}.$$

This, combined with an equivalent definition of the dimension function of shift-invariant spaces in terms of its range function, see [BDR], [Bo] and [BR] yields

$$\dim_V(\xi) = \dim \operatorname{span}\{(\hat{\psi}((A^*)^j(\xi + k))_{k\in\mathbb{Z}^n} : \psi \in \Psi, j = 1, 2, \dots\},$$

which shows (3.5).

Proof of Theorem (3.3). First, suppose that Ψ comes from an MRA, i.e., its space of negative dilates satisfies $\dim_V = 1$. By (3.5) we have that $\operatorname{supp} D_\Psi = \mathbb{R}^n$ and thus (3.4) holds.

Conversely, assume (3.4). We need to show that (M5) is satisfied, or equivalently that $\dim_V = 1$. First we conduct the standard orthogonalization procedure on the sequence $\{D^j(V) : j \in \mathbb{Z}\}$. Let $W = D(V) \ominus V$. Since the sequence $\{D^j(V) : j \in \mathbb{Z}\}$ forms a GMRA we recall that (2.2) holds. This allows us to find a semi–orthogonal tight framelet associated to the GMRA. Indeed, W is generated by L functions, namely $\psi - P_V\psi$, $\psi \in \Psi$, where P_V is the orthogonal projection onto V. Therefore, we can find at most L generators Φ in W whose shifts form a tight frame for W. From (2.2) it follows that Φ is a semi–orthogonal tight framelet and we can use Lemma (2.6) to obtain

$$\begin{aligned}\int_{[0,1]^n} \dim_V(\xi)\, d\xi &= \int_{\mathbb{R}^n} \sigma_V(\xi)\, d\xi \\ &= \sum_{\varphi\in\Phi}\sum_{j=1}^{\infty}\int_{\mathbb{R}^n} |\hat{\varphi}((A^*)^j\xi)|^2 \\ &= \sum_{\varphi\in\Phi} \|\hat{\varphi}\|^2/(|\det A| - 1) \le 1, \end{aligned} \tag{3.6}$$

were the last inequality follows from the fact that Φ generates a tight frame (with constant 1) and consequently for each $\varphi \in \Phi$ we have $||\varphi|| \le 1$.

On the other hand, (3.4) and (3.5) imply that $\dim_V(\xi) > 0$ for a.e. ξ. Since $\dim_V$ is integer–valued, from (3.6) it follows that $\dim_V = 1$, which concludes the proof of Theorem (3.3). □

Remark. It should be noticed that Theorem (3.3) holds also for framelets Ψ which have a canonical dual framelet with the same number of generators. Indeed, in this case the space of negative dilates of Ψ is shift-invariant, see [BW], and the above proof works without any changes. However, if a framelet Ψ has merely a dual framelet (not canonical) then Theorem (3.3) fails, since the space of negative dilates may not be shift-invariant, see [BW].

As a corollary of Theorem (3.3) we can extend previously mentioned Lemarié's result to tight framelets.

Corollary (3.7). *Suppose that a tight framelet Ψ satisfies the assumptions of Theorem (3.3) and at least one generator of Ψ is compactly supported. Then Ψ must necessarily come from an MRA.*

As we can see, condition (3.4) would characterize all tight framelets with $|\det A| - 1$ generators coming from an MRA if one could prove that Question (3.2) has a positive answer. It turns out that in some cases such an answer can be given.

We already know that semi–orthogonality implies condition (M4). The condition clearly holds if the generators of a tight framelet Ψ are band–limited or, more generally, if their Fourier transform is supported in a set of a finite measure.

A less trivial result was presented in [Rz].

Proposition (3.8). *Let V be a shift-invariant space. If $\sigma_V \in L^1(\mathbb{R}^n)$ then condition (M4) holds.*

If V has a finite number of generators then $\dim_V$ is bounded and therefore σ_V is integrable over $\mathbb{R}^n$. This allows us to combine Proposition (3.1), Theorem (3.3) and Proposition (3.8) into the following

Corollary (3.9). *Suppose that a tight framelet Ψ has $L = |\det A| - 1$ generators and at least one of them is compactly supported. If the shift-invariant space V of negative dilates of Ψ is finitely generated, then Ψ comes from an MRA.*

So to speak, either V has one generator or infinitely many. We suspect that in the case of compactly supported tight framelets the latter possibility can be excluded. Nevertheless, in the final section we construct an example of a framelet (neither tight nor compactly supported) whose space V is as big as possible, namely $L^2(\mathbb{R})$.

4. Framelets with dense negative dilates

In this section we show that the answer to the generalized Baggett's problem, that is, when the adjective "tight" in Question (3.2) is dropped, is negative. We will exhibit an example of a dyadic framelet $\psi \in L^2(\mathbb{R})$, such that its space of negative dilates V given in (2.1) is the largest possible, i.e., $V = L^2(\mathbb{R})$. Furthermore, we shall show that this framelet can be chosen to be arbitrary close to a tight framelet (with respect to its frame bounds) and that it has a dual framelet.

Our construction is motivated by the following model situation, where the dyadic dilation operator D on $L^2(\mathbb{R})$ is replaced by the shift operator S on $L^2(\mathbb{T})$. More precisely, S is given by $(Sf)(e^{ix}) = e^{ix} f(e^{ix})$, where $\mathbb{T} = \{e^{ix} : x \in$

$[0, 2\pi]\}$ (S can be viewed as the shift on the dual group $\mathbb{Z}$). Given $f \in L^2(\mathbb{T})$, define the corresponding cyclic space $\mathcal{M}_f$ by $\mathcal{M}_f = \overline{\operatorname{span}}\{S^j f : j \geq 0\}$. Following the terminology of Helson [He], we say that a closed subspace $\mathcal{M} \subset L^2(\mathbb{T})$ is *simply invariant* if $\mathcal{M}$ is invariant ($S\mathcal{M} \subset \mathcal{M}$), but $\mathcal{M}$ is not doubly invariant, i.e., $S\mathcal{M} \neq \mathcal{M}$.

One may ask whether there exists $f \in L^2(\mathbb{T})$ such that $\mathcal{M}_f = L^2(\mathbb{T})$? The following result, due to Helson, which is a consequence of inner-outer factorization for Hardy spaces H^p on the unit disk, helps to answer this question.

Theorem (4.1). *Suppose that $f \in L^2(\mathbb{T})$. Then the following are equivalent:*
(i) $\mathcal{M}_f$ is simply invariant,
(ii) there exists $g \in H^2$ such that $|f(e^{ix})| = |g(e^{ix})|$ for a.e. $x \in [0, 2\pi]$,
(iii) $\int_0^{2\pi} \log|f(e^{ix})|dx > -\infty$.

Hence, Wiener's characterization of doubly invariant subspaces $\mathcal{M} \subset L^2(\mathbb{T})$ as spaces of the form $\mathcal{M} = \{f \in L^2(\mathbb{T}) : \operatorname{supp} f \subset E\}$ for some measurable $E \subset \mathbb{T}$, yields the following corollary.

Corollary (4.2). *Suppose $f \in L^2(\mathbb{T})$. Then $\mathcal{M}_f = L^2(\mathbb{T})$ if and only if $f(e^{ix}) \neq 0$ for a.e. $x \in \mathbb{T}$ and $\int_0^{2\pi} \log|f(e^{ix})|dx = -\infty$.*

Since it is not hard to construct a function satisfying the assumptions of the above corollary, we conclude that there exist functions f such that $\mathcal{M}_f = L^2(\mathbb{T})$. Since the dyadic dilation operator D is unitarily equivalent to a bilateral shift with infinite multiplicity, we can suspect that there are functions $\psi \in L^2(\mathbb{R})$, whose space of negative dilates is equal to $L^2(\mathbb{R})$. More precisely, D acts like a shift operator on dyadic Littlewood-Paley blocks

$$W_j = \check{L}^2([-2^j, -2^{j-1}] \cup [2^{j-1}, 2^j]), \qquad j \in \mathbb{Z},$$

where we use the convention that for a measurable set $Z \subset \mathbb{R}$,

$$\check{L}^2(Z) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset Z\}.$$

Indeed, the main result of this section shows the following.

Theorem (4.3). *For any $\delta > 0$, there exists a framelet $\psi \in L^2(\mathbb{R})$, with frame bounds 1 and $1 + \delta$, such that the space of negative dilates of ψ is equal to $L^2(\mathbb{R})$. Moreover, ψ has a dual framelet.*

In the proof of Theorem (4.3) we will use the following two standard results.

Theorem (4.4). *Suppose that $f \in L^2(\mathbb{R})$ is such that $\hat{f} \in L^\infty(\mathbb{R})$ and*

$$\hat{f}(\xi) = O(|\xi|^\delta) \qquad \text{as } \xi \to 0,$$
$$\hat{f}(\xi) = O(|\xi|^{-1/2-\delta}) \qquad \text{as } |\xi| \to \infty,$$

for some $\delta > 0$. Then the affine system generated by f is a Bessel sequence.

Lemma (4.5). *Suppose that $\mathcal{H}$ is a Hilbert space, $\{f_j\} \subset \mathcal{H}$ is a frame with constants C_1 and C_2,*

$$C_1||f||^2 \le \sum_j |\langle f, f_j\rangle|^2 \le C_2||f||^2 \qquad \text{for all } f \in \mathcal{H},$$

and $\{g_j\} \subset \mathcal{H}$ is a Bessel sequence with constant C_0,

$$\sum_i |\langle f, g_j\rangle|^2 \le C_0||f||^2 \qquad \text{for all } f \in \mathcal{H}.$$

If $C_0 < C_1$, then $\{f_j + g_j\}$ is a frame with constants $((C_1)^{1/2} - (C_0)^{1/2})^2$ and $((C_2)^{1/2} + (C_0)^{1/2})^2$.

Theorem (4.4) gives a sufficient condition for an affine system to be a Bessel sequence. Its proof can be found in [Ho , Theorem 13.0.1]. Lemma (4.5) is a basic perturbation result for frames, which can be found in [Ch ,Corollary 2.7] or [FZ , Theorem 3.]

Proof of Theorem (4.3). Define the sets $Z_1, \dots, Z_4$ by

$$\begin{aligned} Z_1 &= \bigcup_{k\in\mathbb{Z}, k\ge 0} (k + (1/8, 1/4)),\\ Z_2 &= -Z_1,\\ Z_3 &= \bigcup_{k\in\mathbb{Z}} (k + (Z_1 \cup Z_2)) = \bigcup_{k\in\mathbb{Z}} (k + (1/8, 1/4) \cup (3/4, 7/8)),\\ Z_4 &= \mathbb{R} \setminus Z_3. \end{aligned}$$

Suppose $\psi^0 = \psi^1 + \psi^2$, where $\psi^1 \in \check{L}^2(Z_3)$ and $\psi^2 \in \check{L}^2(Z_4)$. As usual, define

$$W_j^l = \overline{\text{span}}\{\psi_{j,k}^l : k \in \mathbb{Z}\} \qquad \text{for } l = 0, 1, 2.$$

We claim that

$$W_j^0 = W_j^1 \oplus W_j^2 \qquad \text{for } j \in \mathbb{Z}. \tag{4.6}$$

It suffices to show (4.6) for $j = 0$. Take any $f \in W_0^1$ and $g \in W_0^2$. Since

$$W_0^l = \{f \in L^2 : \hat{f}(\xi) = m(\xi)\hat{\psi}^l(\xi), \quad m \text{ is measurable and 1-periodic}\}, \tag{4.7}$$

$\operatorname{supp} \hat{f} \subset Z_3$, $\operatorname{supp} \hat{g} \subset Z_4$ and hence $f \perp g$. Thus, $W_0^1 \perp W_0^2$. Finally, it suffices to prove $W_0^1 \oplus W_0^2 \subset W_0^0$, since the converse inclusion is trivial. Take any $f \in W_0^1 \oplus W_0^2$. By (4.7) there are 1-periodic measurable functions m_1 and m_2 such that

$$\hat{f}(\xi) = m_1(\xi)\hat{\psi}^1(\xi) + m_2(\xi)\hat{\psi}^2(\xi) = m_1(\xi)\mathbf{1}_{Z_3}(\xi)\hat{\psi}^0(\xi) + m_2(\xi)\mathbf{1}_{Z_4}(\xi)\hat{\psi}^0(\xi).$$

Since the sets Z_3 and Z_4 are invariant under integer shifts, $m = m_1\mathbf{1}_{Z_3} + m_2\mathbf{1}_{Z_4}$ is 1-periodic. Hence, by (4.7) $f \in W_0^0$, which shows $W_0^0 = W_0^1 \oplus W_0^2$.

It now remains to choose ψ^1 and ψ^2 appropriately. The idea is that negative dilates of ψ^1 will generate functions whose Fourier transform is supported near the origin, whereas the negative dilates of ψ^2 will exhaust all functions which are supported away from the origin (in the Fourier domain). Define $\psi^1 \in \check{L}^2(Z_3)$ by

$$\widehat{\psi^1} = \mathbf{1}_{(-1/4,-1/8)\cup(1/8,1/4)}.$$

Clearly, ψ^1 is a tight framelet that is a dilated version of the usual orthonormal Shannon wavelet $\check{\mathbf{1}}_{(-1,-1/2)\cup(1/2,1)}$. Moreover, by (4.7), $W_0^1 = \check{L}^2((-1/4,-1/8)\cup(1/8,1/4))$. Hence,

$$W_j^1 = \check{L}^2((-2^{j-2},-2^{j-3})\cup(2^{j-3},2^{j-2})) \qquad \text{for any } j \in \mathbb{Z},$$

and therefore, the space of negative dilates of ψ^1 is

$$V^1 = \overline{\operatorname{span}} \bigcup_{j<0} W_j^1 = \bigoplus_{j<0} W_j^1 = \check{L}^2(-1/8, 1/8). \tag{4.8}$$

The function ψ^2 should be regarded as a perturbation term of $\psi^0 = \psi^1 + \psi^2$. However, the construction of ψ^2 requires much more work.

We will need a simple probabilistic fact. For a fixed $k \in \mathbb{Z}$, define random variables $X_l = \mathbf{1}_{(k,k+1)}\mathbf{1}_{2^{-3l}Z_4}$, $l \geq 0$ on the probability space being the interval $(k, k+1)$. An easy verification shows that for any $l_1, l_2 \geq 0$ and $l_1 \neq l_2$

$$\begin{aligned}
|(k,k+1)\cap 2^{-3l_1}Z_3 \cap 2^{-3l_2}Z_3| &= (1/4)^2,\\
|(k,k+1)\cap 2^{-3l_1}Z_3 \cap 2^{-3l_2}Z_4| &= 3/16,\\
|(k,k+1)\cap 2^{-3l_1}Z_4 \cap 2^{-3l_2}Z_4| &= (3/4)^2.
\end{aligned}$$

Therefore, $\{X_l : l \geq 0\}$ is a sequence of independent identically distributed random variables with

$$\mathbf{P}(X_l = 1) = 3/4 \quad \text{and} \quad \mathbf{P}(X_l = 0) = 1/4.$$

Let $\{\varphi_m : m \in \mathbb{N}\}$ be some enumeration of the "truncated" Gabor system

$$\{\mathbf{1}_{(k,k+1)}e^{2\pi i j\xi} : j \in \mathbb{Z}, k \in \mathbb{Z}, k \neq -1, 0\}.$$

Clearly, $\{\varphi_m : m \in \mathbb{N}\}$ is an orthonormal basis of $L^2((-\infty,-1)\cup(1,\infty))$. For any $m \in \mathbb{N}$, let $k_m \in \mathbb{Z}$ denote the left endpoint of the support of φ_m, i.e., $\operatorname{supp}\varphi_m = (k_m, k_m+1)$.

Let $(m_p)_{p\in\mathbb{N}}$ be a sequence of natural numbers such that each natural number occurs infinitely many times. We construct by induction a sequence of functions $\{\phi_p : p \in \mathbb{N}\}$ and a sequence of natural numbers $(l_p)_{p\in\mathbb{N}} \subset 3\mathbb{N}$. The requirement that l_p's are divisible by 3 plays an essential role later in the proof when the above probabilistic observation is used.

Let $\phi_1 = D^{-3}(\varphi_{m_1})\mathbf{1}_{Z_4}$ and $l_1 = 3$. Suppose we have constructed $\phi_1, \dots, \phi_p$ and $l_1, \dots, l_p$ for some $p \in \mathbb{N}$. Define l_{p+1} to be the smallest integer divisible by 3 such that

$$\operatorname{supp}\phi_1 \cup \ldots \cup \operatorname{supp}\phi_p \subset (-2^{l_{p+1}}, 2^{l_{p+1}}), \tag{4.9}$$

and

$$\phi_{p+1} = D^{-l_{p+1}}(\varphi_{m_{p+1}})\mathbf{1}_{Z_4}. \tag{4.10}$$

It is easy to see that the sequence $(l_p)_{p\in\mathbb{N}}$ is increasing and the supports of ϕ_p's are included in pairwise disjoint open intervals. Therefore, $\langle \phi_p, \phi_{p'} \rangle = 3/4\delta_{p,p'}$ for any $p, p' \in \mathbb{N}$, since $|Z_4 \cap (k, k+1)| = 3/4$ for any $k \in \mathbb{Z}$. Finally, define $\psi^2 \in \check{L}^2(Z_4)$ by

$$\widehat{\psi^2}(\xi) = \sum_{p\in\mathbb{N}} c_p \phi_p(\xi), \tag{4.11}$$

for some sufficiently fast decaying sequence $(c_p)_{p\in\mathbb{N}}$ of positive numbers. More precisely, we choose c_p's such that $0 < c_{p+1} < c_p/(p+1)$ for all $p \in \mathbb{N}$ and $\widehat{\psi^2}(\xi) = O(|\xi|^{-1})$ as $|\xi| \to \infty$. Hence, by Theorem (4.4), the affine system generated by ψ^2 is a Bessel sequence. Our next goal is to show the following fact.

Lemma (4.12). *Suppose that ψ^2 given by (4.11) is constructed as above. Let V^2 be the space of negative dilates of ψ^2 and P be the orthogonal projection onto $\check{L}^2((-\infty, -1) \cup (1, \infty))$, i.e.,*

$$\widehat{(Pf)}(\xi) = \hat{f}(\xi)\mathbf{1}_{(-\infty,-1)\cup(1,\infty)} \qquad \text{for } f \in L^2.$$

Then $P(V^2)$ is dense in $\check{L}^2((-\infty, -1) \cup (1, \infty))$.

Proof. Since $\tilde{V}^2 := \overline{\operatorname{span}}\{\psi^2_{-l_p,0} : p \in \mathbb{N}\} \subset V^2$ it suffices to show that $P(\tilde{V}^2)$ is dense in $\check{L}^2((-\infty, -1) \cup (1, \infty))$. Hence, we need to show that each basis element φ_m, $m \in \mathbb{N}$, of $L^2((-\infty, -1) \cup (1, \infty))$ belongs to the closure of the Fourier transform of $P(\tilde{V}^2)$. Given $r \in \mathbb{N}$,

$$\widehat{\psi^2_{-l_r,0}} = D^{l_r}(\widehat{\psi^2}) = \sum_{p\in\mathbb{N}} c_p D^{l_r}(\phi_p).$$

By (4.9), $\operatorname{supp} D^{l_r}(\phi_p) \subset (-1, 1)$ for $p < r$, and we have

$$\begin{aligned}(P(\psi^2_{-l_r,0}))\hat{} &= \sum_{p\geq r} c_p D^{l_r}(\phi_p) = \sum_{p\geq r} c_p D^{l_r-l_p}(\varphi_{m_p})\mathbf{1}_{2^{-l_r}Z_4} \\ &= c_r \mathbf{1}_{2^{-l_r}Z_4}\left[\varphi_{m_r} + \sum_{p>r} \frac{c_p}{c_r} D^{l_r-l_p}(\varphi_{m_p})\right].\end{aligned}$$

Since $c_{r+1}/c_r < 1/(r+1)$,

$$\left\| \sum_{p>r} \frac{c_p}{c_r} D^{l_r - l_p}(\varphi_{m_p}) \right\| \leq \sum_{p>r} \frac{1}{(r+1)(r+2)\dots p} \|D^{l_r-l_p}(\varphi_{m_p})\| < 2/r,$$

we conclude that $\varphi_{m_r}\mathbf{1}_{2^{-l_r}Z_4} + \eta_r$ belongs to the Fourier transform of $P(\tilde{V}^2)$ for some $\eta_r \in L^2$ with $\|\eta_r\| < 2/r$.

For a fixed $m \in \mathbb{N}$, let $R = \{r \in \mathbb{N} : m_r = m\}$. By our construction $R = \{r_1, r_2, \dots\}$ is infinite. Define random variables $X_l = \mathbf{1}_{(k_m, k_m+1)}\mathbf{1}_{2^{-l}Z_4}$, $l \geq 0$ on the probability space being the interval (k_m, k_m+1). Earlier we have observed that $\{X_l : l \in 3\mathbb{Z},\ l \geq 0\}$ is a sequence of independent identically distributed random variables with

$$\mathbf{P}(X_l = 1) = 3/4 \quad \text{and} \quad \mathbf{P}(X_l = 0) = 1/4.$$

Therefore, by the Strong Law of Large Numbers

$$(X_{l_{r_1}}(\xi) + \dots + X_{l_{r_n}}(\xi))/n \to \mathbf{E}(X_l) = 3/4 \quad \text{as } n \to \infty$$
$$\text{for a.e. } \xi \in (k_m, k_m+1).$$

Hence, by the Lebesgue Dominated Convergence Theorem

$$\frac{(\varphi_{m_{r_1}}\mathbf{1}_{2^{-l_{r_1}}Z_4} + \eta_{r_1}) + \dots + (\varphi_{m_{r_n}}\mathbf{1}_{2^{-l_{r_n}}Z_4} + \eta_{r_n})}{n}$$
$$= \frac{(\varphi_m X_{l_{r_1}} + \eta_{r_1}) + \dots + (\varphi_m X_{l_{r_n}} + \eta_{r_n})}{n} \to 3/4\varphi_m \qquad \text{in } L^2(\mathbb{R}) \text{ as } n \to \infty,$$

since $(\eta_{r_1} + \dots + \eta_{r_n})/n \to 0$ in $L^2(\mathbb{R})$ as $n \to \infty$. Therefore, φ_m belongs to the closure of the Fourier transform of $P(\tilde{V}^2)$. Since $m \in \mathbb{N}$ is arbitrary and $\{\varphi_m : m \in \mathbb{N}\}$ is an orthonormal basis of $L^2((-\infty, -1) \cup (1, \infty))$, this completes the proof of the lemma. □

Corollary (4.13). *Suppose that V^2 is the same as in Lemma (4.12). Let P_j be the orthogonal projection onto $\check{L}^2((-\infty, -2^j) \cup (2^j, \infty))$, i.e.,*

$$\widehat{(P_j f)}(\xi) = \hat{f}(\xi)\mathbf{1}_{(-\infty,-2^j)\cup(2^j,\infty)} \qquad \textit{for } f \in L^2.$$

Then $P_j(V^2)$ is dense in $\check{L}^2((-\infty, -2^j) \cup (2^j, \infty))$ for any $j \in \mathbb{Z}$.

Proof. Since the case $j \geq 0$ follows immediately from Lemma (4.12), we may assume that $j < 0$. A straightforward calculation shows that $P_j = D^j P D^{-j}$. Take any $f \in \check{L}^2((-\infty, -2^j) \cup (2^j, \infty))$. Since $D^{-j}f \in \check{L}^2((-\infty, -1) \cup (1, \infty))$, by Lemma (4.12) there exists a sequence $\{f_k : k \in \mathbb{N}\} \subset V^2$ such that $P_0 f_k \to D^{-j}f$ as $k \to \infty$. Hence, $P_j D^j f_k \to f$ as $k \to \infty$. Since $D^j f_k \in V^2$ for $j \leq 0$, this shows the corollary. □

We are now ready to conclude the proof of Theorem (4.3). Let V^0 be the space of negative dilates of ψ^0. By (4.6),

$$V^0 = \overline{\operatorname{span}}\left(\bigcup_{j<0} W_j^0\right) = \overline{\operatorname{span}}\left(\bigcup_{j<0}(W_j^1 \cup W_j^2)\right) = \overline{\operatorname{span}}(V^1 \cup V^2).$$

Therefore, by (4.8) and $\overline{P_{-3}(V^2)} = \check{L}^2((-\infty, -1/8) \cup (1/8, \infty))$, we have that V^0 is dense in $L^2(\mathbb{R})$. Since V^0 is closed it must be equal to $L^2(\mathbb{R})$. It remains to show that one can also find a framelet with this property.

Recall that $\psi^0 = \psi^1 + \psi^2$, where ψ^1 is a tight framelet and ψ^2 is a Bessel sequence. Therefore, by Lemma (4.5), there exists $\varepsilon > 0$ such that $\psi' = \psi^1 + \varepsilon\psi^2$ is a framelet with frame constants $1 - \delta/3$ and $1 + \delta/3$. Moreover, since $\varepsilon\psi^2$ is of the form (4.11), the space of negative dilates of ψ' is also $L^2(\mathbb{R})$. Therefore, $\psi = (1 - \delta/3)^{-1/2}\psi'$ is a framelet with constants 1 and $1 + \delta$ whose space of negative dilates is $L^2(\mathbb{R})$. Finally, we employ a set of equations which characterizes dual framelets. We recall that functions $\phi, \psi \in L^2(\mathbb{R})$ whose respective affine systems are Bessel sequences form a pair of dual framelets if and only if

$$\sum_{j\in\mathbb{Z}} \hat{\phi}(2^j\xi)\overline{\hat{\psi}(2^j\xi)} = 1 \qquad \text{a.e. } \xi,$$

$$\sum_{j=0}^{\infty} \hat{\phi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi + q))} = 0 \qquad \text{a.e. } \xi \text{ and for odd } q.$$

Thus, an easy verification shows that $\phi = (1 - \delta/3)^{1/2}\psi^1$ is a dual framelet to $\psi = (1 - \delta/3)^{-1/2}(\psi^1 + \varepsilon\psi^2)$. This completes the proof of Theorem (4.3). □

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