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BOUNDEDNESS OF OPERATORS ON HARDY SPACES VIA ATOMIC DECOMPOSITIONS

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ABSTRACT. An example of a linear functional defined on a dense subspace of the Hardy space $H^1(\mathbb{R}^n)$ is constructed. It is shown that despite the fact that this functional is uniformly bounded on all atoms, it does not extend to a bounded functional on the whole H^1. Therefore, this shows that in general it is not enough to verify that an operator or a functional is bounded on atoms to conclude that it extends boundedly to the whole space. The construction is based on the fact due to Y. Meyer which states that quasi-norms corresponding to finite and infinite atomic decompositions in H^p, $0 < p \leq 1$, are not equivalent.

1. INTRODUCTION

The intended purpose of this work is not only of research, but also of pedagogical nature, since it is based on an already published, but quite possibly not well-known, example of Y. Meyer.

In this note we give a rather surprising example of a linear functional defined on a dense subspace of H^1, which maps all atoms into bounded scalars, but yet it cannot be extended to a bounded functional on the whole space H^1. As a consequence of this example, it follows that in general it does not suffice to check that an operator from a Hardy space H^p, $0 < p \leq 1$, into some other quasi-Banach space X maps atoms into bounded elements of X to verify that this operator extends to a bounded operator on H^p. An untrained reader might inadvertently draw such a conclusion by reading literature on atomic decompositions of Hardy spaces. Here we list a few references, which could potentially lead someone into this not fully justified belief [3, Proof of Lemma II.2], [5, Corollary 6.3], [8, Lemma 5.1], [10, Chapter 6.7.c], [14, Chapter 6.3], [16, Chapter III.3.3], [17, Chapter 1], and [18, Proposition 6.13].

Despite this, it is important to emphasize that to verify boundedness for many important classes of operators defined on H^p spaces, it is indeed sufficient to check that atoms are mapped into bounded elements of X. Probably the best known example of a class with this property are Calderón-Zygmund operators. The complete proof of this fact (based on atomic decomposition of H^p spaces) can be found, for

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example, in [9, Chapter III.7] or [10, Theorem 6.7.1] for convolution-type Calderón-Zygmund operators and in [1, Chapter 1.9] or [15, Lemma 1 in Chapter 7.3] for non-convolution operators.

A rudimentary set of facts about real-variable theory of Hardy spaces can be found in [4, 7, 8, 9, 10, 16]. Here, we limit ourselves to the basic definition of real-variable H^p spaces due to Fefferman and Stein [7].

Definition 1. We say that a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the Hardy space $H^p(\mathbb{R}^n)$, $0 < p < \infty$, if its radial maximal function $M^0_\varphi f$ (or equivalently non-tangential maximal function $M_\varphi f$) is in L^p. Here, φ is any test function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ with $\int \varphi \neq 0$, and

$$M^0_\varphi f(x) = \sup_{t>0} |f * \varphi_t(x)|, \tag{1}$$

$$M_\varphi f(x) = \sup_{t>0} \sup_{|y-x|<t} |f * \varphi_t(y)|, \tag{2}$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$.

A fundamental result of Fefferman and Stein asserts that this definition does not depend on the choice of $\varphi \in \mathcal{S}$ (as long as $\int \varphi \neq 0$) and $H^p(\mathbb{R}^n)$ with the quasi-norm $||f||_{H^p} = ||M^0_\varphi f||_{L^p}$ (or $||f||_{H^p} = ||M_\varphi f||_{L^p}$) is a quasi-Banach space. Moreover, $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $p > 1$. For proofs of these facts we refer to [16, Chapter III.1].

2. Example of Meyer

In this section we present an example of an atom in H^p whose norm is not achieved by its finite atomic decomposition. The first example of this kind for H^1 was exhibited by Y. Meyer [13]; see also [9, Section III.8.3]. Here, we merely adapt this example to a more general $H^p(\mathbb{R}^n)$, $0 < p \leq 1$, case.

We start by recalling a definition of an atom for H^p spaces. For the sake of simplicity, we will use only L^∞ normalization for our atoms and we will limit ourselves to the classical isotropic Hardy spaces $H^p(\mathbb{R}^n)$ given by Definition 1.

Definition 2. We say that a function a is a p-atom, where $0 < p \leq 1$, if

$$\operatorname{supp} a \subset B(x_0, r) \qquad \text{for some } x_0 \in \mathbb{R}^n,\ r > 0, \tag{3}$$

$$||a||_\infty \leq |B(x_0, r)|^{-1/p}, \tag{4}$$

$$\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0 \qquad \text{for all } |\alpha| \leq \lfloor (1/p - 1)n \rfloor. \tag{5}$$

Here, $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$.

Let $\Theta^k(\mathbb{R}^n)$ be the space of all (finite) linear combinations of p-atoms, that is,

$$\Theta^k(\mathbb{R}^n) = \{f \in L^\infty(\mathbb{R}^n) : \operatorname{supp} f \text{ is bounded and } \int_{\mathbb{R}^n} f(x) x^\alpha dx = 0 \text{ for } |\alpha| \leq k\}.$$

It is well known that $\Theta^k(\mathbb{R}^n)$ is a dense subspace of $H^p(\mathbb{R}^n)$, $0 < p \leq 1$, for sufficiently large k, that is, for $k \geq \lfloor (1/p - 1)n \rfloor$. In fact, an even smaller space $\Theta^k(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ is also a dense subspace of $H^p(\mathbb{R}^n)$ for the same range of k's.

On the space $\Theta^k(\mathbb{R}^n)$ we consider two quasi-norms corresponding to finite and infinite atomic decompositions:

$$(6)\quad ||f||_{H^p,\infty} = \inf\left\{\left(\sum_{i=1}^{\infty}|\lambda_i|^p\right)^{1/p} : f=\sum_{i=1}^{\infty}\lambda_i a_i, \qquad a_i \text{ is a } p\text{-atom for } i\in\mathbb{N}\right\}.$$

$$(7)\quad ||f||_{H^p,<\infty} = \inf\left\{\left(\sum_{i=1}^{N}|\lambda_i|^p\right)^{1/p} : f=\sum_{i=1}^{N}\lambda_i a_i,\right.$$
$$\left. a_i \text{ is a } p\text{-atom for } 1\le i\le N, \text{ and } N\in\mathbb{N}\right\}.$$

It should be emphasized that the equality $f=\sum_{i=1}^{\infty}\lambda_i a_i$ in (6) is understood in the sense of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. This follows from a standard H^p theory fact stating that for any choice of coefficients $(\lambda_i)_{i=1}^{\infty}\in\ell^p(\mathbb{N})$ and p-atoms a_i's, the series $\sum_{i=1}^{\infty}\lambda_i a_i$ converges in $||\cdot||_{H^p}$ quasi-norm, and hence in $\mathcal{S}'$.

The atomic decomposition theorem of Coifman [2] for H^p spaces states that the converse is also true, i.e., every element $f\in H^p(\mathbb{R}^n)$ can be decomposed as $f=\sum_{i=1}^{\infty}\lambda_i a_i$ for some choice of λ_i's and p-atoms a_i's. Moreover,

$$||f||_{H^p}\asymp ||f||_{H^p,\infty} \qquad \text{for all } f\in H^p,$$

and hence for all $f\in\Theta^k(\mathbb{R}^n)$, where $k\ge\lfloor(1/p-1)n\rfloor$.

A less-known result due to Y. Meyer states that the above is not true when the quasi-norm $||\cdot||_{H^p,\infty}$ is replaced by $||\cdot||_{H^p,<\infty}$. Hence, the quasi-norms $||\cdot||_{H^p,\infty}$ and $||\cdot||_{H^p,<\infty}$ are **not** equivalent on $\Theta^k(\mathbb{R}^n)$.

Theorem 1. *Suppose $0<p\le 1$ and $k\ge\lfloor(1/p-1)n\rfloor$. Then for arbitrarily small $\varepsilon>0$, there exists $f\in\Theta^k(\mathbb{R}^n)$ such that*

$$(8)\qquad ||f||_{H^p,\infty}<\varepsilon \qquad \textit{and} \qquad ||f||_{H^p,<\infty}=1.$$

Proof. Let a be a p-atom supported on the unit ball $B(0,1)$ with

$$(9)\qquad \int_{\mathbb{R}^n} a(x)x^\alpha dx = 0 \qquad \text{for all } |\alpha|\le k,$$

and such that

$$(10)\qquad |a(x)|\ge c|B(0,1)|^{-1/p}>0 \qquad \text{for a.e. } x\in B(0,1).$$

To show that an atom a satisfying (10) exists, let $K=\sum_{i=0}^{k}\binom{n-1+i}{i}$ be the cardinality of the collection of all multi-indices α with $|\alpha|\le k$. Then, we claim that it suffices to construct a finite partition $\{E_i\}_{i=1}^m$ of $B(0,1)$ such that the vectors

$$(11)\qquad v_i=\left(\int_{E_i}x^\alpha dx\right)_{|\alpha|\le k}\in\mathbb{R}^K, \qquad i=1,\dots,m,$$

span the whole space $\mathbb{R}^K$ even if one of them is removed, i.e.,

$$(12)\qquad \forall 1\le i_0\le m \qquad \operatorname{span}\{v_i: 1\le i\le m, i\ne i_0\}=\mathbb{R}^K.$$

Indeed, (12) implies that there exist non-zero coefficients $c_1,\dots,c_m$ such that $\sum_{i=1}^m c_i v_i=0$. Moreover, by scaling we may also assume that $\sup_{1\le i\le m}|c_i|\le |B(0,1)|^{-1/p}$. Then, one can immediately verify that

$$a(x)=\sum_{i=1}^m c_i\mathbf{1}_{E_i}(x)$$

is a required atom satisfying (10) with $c = |B(0,1)|^{1/p} \inf_{1\le i\le m} |c_i|$. Finally, we remark that a partition $\{E_i\}_{i=1}^m$ satisfying (11) and (12) can easily be found by an inductive partitioning of the ball.

Next, we choose a collection of pairwise disjoint balls $\{B_i\}_{i\in\mathbb{N}}$ such that $B_i \subset B(0,1)$ for all $i \in \mathbb{N}$,

$$U := \bigcup_{i\in\mathbb{N}} B_i \text{ is dense in } B(0,1) \qquad \text{and} \qquad |U| = \sum_{i\in\mathbb{N}} |B_i| < c\varepsilon^p.$$

For each $i \in \mathbb{N}$, let a_i be a dilated and translated copy of the atom a with support adjusted to the ball B_i. That is, if $B_i = B(x_0, r)$, then $a_i(x) = r^{-n/p} a((x-x_0)/r)$. As a consequence of (10) each a_i is an atom supported on B_i and satisfying

$$|a_i(x)| \ge c|B_i|^{-1/p} \qquad \text{for a.e. } x \in B_i. \tag{13}$$

Let

$$f(x) = c^{-1/p} \sum_{i\in\mathbb{N}} |B_i|^{1/p} a_i(x). \tag{14}$$

Then, it is obvious that $||f||^p_{H^p,\infty} \le \sum_{i\in\mathbb{N}} |B_i|/c < \varepsilon^p$. On the other hand, we claim that $||f||_{H^p,\infty}$ must remain large.

Indeed, suppose that f has a finite atomic decomposition $f = \sum_{i=1}^N \lambda_i b_i$, where each b_i is supported on a ball $\tilde{B}_i$. Let g be a majorant of f given by

$$g = \left(\sum_{i=1}^N |\lambda_i|^p |\tilde{B}_i|^{-1} \mathbf{1}_{\tilde{B}_i} \right)^{1/p}.$$

By (13) and (14)

$$\begin{aligned} \mathbf{1}_U(x) \le |f(x)| \le \sum_{i=1}^N |\lambda_i||b_i(x)| \le \left(\sum_{i=1}^N |\lambda_i|^p |b_i(x)|^p \right)^{1/p} \\ \le \left(\sum_{i=1}^N |\lambda_i|^p |\tilde{B}_i|^{-1} \mathbf{1}_{\tilde{B}_i}(x) \right)^{1/p} = g(x). \end{aligned} \tag{15}$$

Since g is continuous everywhere almost everywhere (possibly with the exception of the union of boundaries of a finite collection of balls $\bigcup_{i=1}^N \partial(\tilde{B}_i)$) and U is dense in $B(0,1)$, hence $g(x) \ge 1$ for a.e. $x \in B(0,1)$. Therefore,

$$|B(0,1)| \le \int_{B(0,1)} g(x)^p dx = \sum_{i=1}^N |\lambda_i|^p.$$

Consequently, $||f||_{H^p,<\infty} \ge |B(0,1)|^{1/p}$. It is also immediate from (14) that $||f||_{H^p,<\infty} \le c^{-1/p} |B(0,1)|^{1/p}$. Since $\varepsilon > 0$ was arbitrary, by a simple rescaling we find f satisfying (8), which completes the proof of Theorem 1. □

It is perhaps worthwhile to recall the original example of Meyer, which through its simplicity better illustrates the idea of the above proof; see also [9, Chapter III.8].

Example 1. For arbitrarily small $\varepsilon > 0$, we will construct a function $f \in \Theta^0(\mathbb{R})$ such that

$$||f||_{H^1,\infty} < \varepsilon \qquad \text{and} \qquad ||f||_{H^1,<\infty} = 1.$$

Let $\{B_i\}_{i\in\mathbb{N}}$ be a collection of pairwise disjoint intervals $\subset [0,1]$ such that $U := \bigcup_{i\in\mathbb{N}} B_i$ is dense in $[0,1]$ and $|U| < \varepsilon$. Let a_i be a 1-atom supported on B_i, which equals $1/|B_i|$ on the left half of B_i and $-1/|B_i|$ on the other half. Let $f(x) = \sum_{i\in\mathbb{N}} |B_i| a_i(x)$. It is clear that $||f||_{H^1,\infty} \leq \sum_{i\in\mathbb{N}} |B_i| < \varepsilon$ and $|f(x)| = 1$ for a.e. $x \in U$.

To see that $||f||_{H^1,<\infty} = 1$, consider a finite atomic decomposition $f = \sum_{i=1}^{N} \lambda_i b_i$, where each b_i is supported on the interval $\tilde{B}_i$. Then

$$\mathbf{1}_U(x) = |f(x)| \leq \sum_{i=1}^{N} |\lambda_i||a_i(x)| \leq \sum_{i=1}^{N} |\lambda_i||\tilde{B}_i|^{-1}\mathbf{1}_{\tilde{B}_i}(x) =: g(x).$$

Since g is discontinuous only on a finite number of points and $U \subset [0,1]$ is dense, hence $g(x) \geq 1$ for a.e. $x \in [0,1]$. Integrating $g(x)$ over $[0,1]$ yields $||f||_{H^1,<\infty} \geq 1$. Since f is itself a 1-atom supported on $[0,1]$, hence $||f||_{H^1,<\infty} = 1$.

3. Unbounded linear functionals on H^1

The goal of this section is to show the existence of a linear functional on a dense subspace of H^1, which does not extend to a bounded functional on the whole H^1 despite the fact that it maps all 1-atoms into scalars with universally bounded absolute values. The existence of such a functional will follow from Meyer's example and an application of the Hahn-Banach Theorem.

Theorem 2. *There exists a linear functional l on $\Theta^0(\mathbb{R}^n)$ such that*

$$|l(f)| \leq ||f||_{H^1,<\infty} \qquad \textit{for all } f \in \Theta^0(\mathbb{R}^n), \tag{16}$$

which does not extend to a bounded functional on $H^1(\mathbb{R}^n)$, i.e.,

$$\sup_{f\in\Theta^0(\mathbb{R}^n)} |l(f)|/||f||_{H^1,\infty} = \infty. \tag{17}$$

In particular, l is uniformly bounded on all atoms in $H^1(\mathbb{R}^n)$. That is, $|l(a)| \leq 1$ for every 1-atom a.

Proof. Suppose $\{x_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^n$ is any sequence such that $B(x_i,1) \cap B(x_j,1) = \emptyset$ for every $i \neq j$. For each $i \in \mathbb{N}$, let $a_i(x)$ be a function in $\Theta^0(\mathbb{R}^n)$ supported on the ball $B(x_i,1)$ and satisfying

$$||a_i||_{H^1,\infty} < 1/i \qquad \text{and} \qquad ||a_i||_{H^1,<\infty} = 1. \tag{18}$$

In addition, from the proof of Theorem 1 we can also assume that

$$|a_i(x)| \geq c/|B(0,1)| > 0 \qquad \text{for } x \in U_i, \text{ where } U_i \subset B(x_i,1) \text{ is dense.} \tag{19}$$

Here, c is a constant independent of $i \in \mathbb{N}$. In fact, we can choose a_i's such that $c = 1$ is the largest possible by taking atoms taking only two non-zero and opposite values as in Example 1.

Let $V = \operatorname{span}\{a_i(x) : i \in \mathbb{N}\} \subset \Theta^0(\mathbb{R}^n)$ be the space of all finite linear combinations of the above functions. We claim that

$$c\sum_{i\in\mathbb{N}} |c_i| \leq ||f||_{H^1,<\infty} \leq \sum_{i\in\mathbb{N}} |c_i| \qquad \text{for all } f(x) = \sum_{i\in\mathbb{N}} c_i a_i(x) \in V. \tag{20}$$

Therefore, V is isomorphic to the subspace of $\ell^1(\mathbb{N})$ consisting of sequences with finite support.

To show (20), we proceed as in the proof of Theorem 1. Suppose that $f(x) = \sum_{i\in\mathbb{N}} c_i a_i(x) \in V$ has a finite atomic decomposition $f(x) = \sum_{j=1}^N \lambda_j b_j(x)$, where each b_j is supported on a ball B_j. By (19)

$$\frac{c}{|B(0,1)|}\sum_{i\in\mathbb{N}} |c_i|\mathbf{1}_{U_i}(x) \le |f(x)| \le \sum_{j=1}^N |\lambda_j||b_j(x)| \le \sum_{j=1}^N |\lambda_j||B_j|^{-1}\mathbf{1}_{B_j}(x) =: g(x).$$

Since g is continuous everywhere almost everywhere (possibly with the exception of the union of boundaries of a finite collection of balls $\bigcup_{j=1}^N \partial(B_j)$) and each U_i is dense in $B(x_i,1)$, hence

$$g(x) \ge \frac{c}{|B(0,1)|}\sum_{i\in\mathbb{N}} |c_i|\mathbf{1}_{B(x_i,1)}(x) \qquad \text{for a.e. } x \in \mathbb{R}^n.$$

Therefore,

$$c\sum_{i\in\mathbb{N}} |c_i| \le \int_{\mathbb{R}^n} g(x)dx = \sum_{j=1}^N |\lambda_j|.$$

This shows the lower bound in (20). The upper bound in (20) is trivial by the triangle inequality and (18).

Define a linear functional l initially on V by

$$l(f) = \sum_{i\in\mathbb{N}} c_i \qquad \text{for } f(x) = \sum_{i\in\mathbb{N}} c_i a_i(x) \in V.$$

By (20), l is a bounded functional on a subspace of V of a normed space $\Theta^0(\mathbb{R}^n)$ equipped with the norm $||\cdot||_{H^1,<\infty}$. Moreover, the norm of l is at most 1. Therefore, by the Hahn-Banach Theorem, l extends to a bounded functional on the whole space $\Theta^0(\mathbb{R}^n)$ such that (16) holds. Since, $l(a_i)/||a_i||_{H^1,\infty} \ge i$ and $i \in \mathbb{N}$ is arbitrary, we also have (17), which completes the proof of Theorem 2. $\square$

Remark 1. We remark that the proof of Theorem 2 can be easily modified to show the existence of a linear functional l defined on some subspace of $V \subset \Theta^k(\mathbb{R}^n)$, where $k \ge \lfloor(1/p-1)n\rfloor$, $0 < p \le 1$, which is bounded on V equipped with the quasi-norm $||\cdot||_{H^p,<\infty}$, but is unbounded as a functional on V with the quasi-norm $||\cdot||_{H^p,\infty}$. However, since the Hahn-Banach Theorem is not valid on general quasi-normed spaces, there is no guarantee that this functional can be boundedly extended to the whole $\Theta^k(\mathbb{R}^n)$.

In fact, Duren, Romberg, and Shields [6] characterized the duals of classical Hardy spaces on the unit complex disc $H^p(\mathbb{D})$ for $0 < p < 1$ and used it to show that the Hahn-Banach Theorem fails for these spaces. Furthermore, Kalton [11, 12] showed that a quasi-Banach space X has the Hahn-Banach Extension property (continuous linear functionals on a closed subspace extend to the whole space) if and only if it is a Banach space.

Finally, we discuss how Theorem 2 relates to the problem of showing boundedness of operators on Hardy spaces via atomic decompositions. A typical argument invoked for that purpose is as follows.

Suppose T is a linear operator defined on some dense subspace D of $H^p(\mathbb{R}^n)$, $0 < p \le 1$, into some quasi-Banach space X, with the property that $||T(a)||_X \le C < \infty$ for all p-atoms a and some universal constant C. Here, we implicitly require that $\Theta^k(\mathbb{R}^n) \subset D$, where $k = \lfloor(1/p-1)n\rfloor$ and $||\cdot||_X$ satisfies p-triangle inequality

$||f+g||_X^p \leq ||f||_X^p + ||g||_X^p$. To show that T extends to a bounded operator from H^p to X, consider an arbitrary element $f \in \Theta^k(\mathbb{R}^n)$. By the atomic decomposition theorem for H^p spaces, we can represent $f = \sum_{i\in\mathbb{N}} \lambda_i a_i$, where the a_i's are p-atoms and $\sum_{i\in\mathbb{N}} |\lambda_i|^p \leq C_0 ||f||_{H^p}$ for some universal constant C_0. Since

$$Tf = T\left(\sum_{i\in\mathbb{N}} \lambda_i a_i\right) = \sum_{i\in\mathbb{N}} \lambda_i T(a_i), \tag{21}$$

hence

$$||Tf||_X^p \leq \sum_{i\in\mathbb{N}} ||\lambda_i T(a_i)||_X^p \leq C \sum_{i\in\mathbb{N}} |\lambda_i|^p \leq CC_0 ||f||_{H^p}. \tag{22}$$

Since f was arbitrary, (22) shows that T extends to a bounded operator $T : H^p \to X$.

The main problem with this argument is that in general there is no guarantee that (21) is valid due the fact that the sum in (21) is infinite. Theorem 2 shows that this is not only a theoretical possibility, but (21) may indeed fail in certain situations (at least when $p = 1$).

The above argument also has a variant, where infinite atomic decomposition is replaced with a finite one $f = \sum_{i=1}^N \lambda_i a_i$, where the a_i's are p-atoms and $\sum_{i=1}^N |\lambda_i|^p \leq C_0 ||f||_{H^p}$ for some constant C_0. This time the problem lies with the fact that C_0 cannot be chosen universally for all $f \in \Theta^k(\mathbb{R}^n)$ as it is evidenced by Theorem 1.

Therefore, in light of Theorems 1 and 2 we must undoubtedly admit that in general it is not enough to verify that an operator or a functional is merely bounded on p-atoms to conclude that it extends boundedly to the whole space H^p, $0 < p \leq 1$. It is also necessary to verify an identity such as (21), asserting that T behaves well with respect to infinite atomic decompositions. This in turn is not always a trivial task, e.g. in the case of Calderón-Zygmund operators it requires use of certain approximation arguments. For further details, we refer to [1, 9, 15].

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