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Quasi-affine systems and the Calderón condition

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Abstract. The notion of a quasi-affine system, originally introduced by Ron and Shen [20] for integer expansive dilations, is extended to the class of rational expansive dilations. It is shown that an affine system is a tight frame if and only if its quasi-affine counterpart is also a tight frame. As a consequence it is shown that for a large class of dilations (including all one dimensional real dilations) an orthonormal affine system is complete if and only if the Calderón condition holds.

1. Introduction

The main goal of this work is to show the following fact that was conjectured by G. Weiss in 1999.

Theorem 1.1. *Suppose $a \in \mathbb{R}$, $|a| > 1$, $\psi \in L^2(\mathbb{R}^n)$. If the affine system $\{\psi_{j,k}(x) = |a|^{j/2}\psi(a^j x - k) : j, k \in \mathbb{Z}\}$ is orthonormal in $L^2(\mathbb{R})$, then it is complete if and only if*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(a^j\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}. \tag{1.1}$$

Moreover, Theorem 1.1 still holds if (1.1) is replaced by a weaker condition

$$\int_{\mathbb{R}} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi = 2\ln|a|, \tag{1.2}$$

which is an immediate consequence of (1.1). The formulae (1.1) and (1.2) are often referred to as a Calderón condition, see [7, 8], [15, §7.1], [18, §3.12].

There is also a natural higher dimensional version of Theorem 1.1, where the dilation factor a is replaced by an expansive dilation matrix A. Theorem 1.1 was initially shown in the case of expansive dilations preserving lattice $\mathbb{Z}^n$ by the author [4]; shortly after that an alternative proof of this fact was also given by Z. Rzeszotnik [21]. However, it was not clear whether any of these two approaches extend to dilation matrices that do not necessarily preserve the lattice $\mathbb{Z}^n$, or some other (full rank) lattice Γ. Nevertheless, we are going to show that the methods developed

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in [4] and based on quasi-affine systems can be extended to a much larger class of expansive dilations including, e.g., all one dimensional real dilations. In order to achieve this we must extend the concept of a quasi-affine system to a class of rational expansive dilations, which was originally introduced only for integer expansive dilations by Ron and Shen [20].

Recently, a third approach showing Theorem 1.1 has been devised. R. Laugesen [16], independently from the author, has also shown Theorem 1.1, i.e., Weiss' conjecture in one dimensional case, and later in multidimensional case [17]. In contrast to the approach presented in this paper and based on shift invariant spaces and quasi-affine systems, Laugesen's methods involve almost periodic functions and semi-continuous wavelet systems; these are affine systems in which discrete translations are substituted by continuous ones. Laugesen's methods are quite versatile and can be applied not only to all expansive dilations but also to certain non-expansive dilations, i.e., a dilation A amplifying for the wavelet ψ.

This paper is organized as follows. In Section 2 we recall some basic results about shift-invariant systems. We also introduce an oversampling procedure which is subsequently used to define quasi-affine systems for rational dilations. In Section 3 we show the equivalence of affine tight frames and quasi-affine tight frames. Finally, in Section 4 we use these results to show our main result.

We start by establishing some necessary terminology. The translation by $y \in \mathbb{R}^n$ is $T_y f(x) = f(x-y)$; the dilation by $n \times n$ non-singular matrix B is $D_B f(x) = \sqrt{|\det B|} f(Bx)$.

Definition 1.2. Suppose $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ and A is an $n \times n$ expansive matrix, i.e., all eigenvalues λ of A satisfy $|\lambda| > 1$. The *affine system* $X(\Psi)$ associated with the dilation A is defined as

$$X(\Psi) = \{\psi_{j,k} : j \in \mathbb{Z},\ k \in \mathbb{Z}^n,\ \psi \in \Psi\}.$$

Here for $\psi \in L^2(\mathbb{R}^n)$ we set $\psi_{j,k}(x) = |\det A|^{j/2} \psi(A^j x - k)$ for $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$.

Definition 1.3. Suppose B is $n \times n$ non-singular matrix. We say that a measurable subset E of $\mathbb{R}^n$ is *B-multiplicatively invariant* if $B(E) = E$ modulo sets of measure zero. Given such E we introduce the closed subspace $\check{L}^2(E) \subset L^2(\mathbb{R}^n)$ by

$$\check{L}^2(E) = \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} = \{\xi : \hat{f}(\xi) \neq 0\} \subset E\}.$$

We say that $\Psi = \{\psi^1, \dots, \psi^L\} \subset \check{L}^2(E)$ is a *multiwavelet* for $\check{L}^2(E)$ associated with A, where E is A^*-multiplicatively invariant, if $X(\Psi)$ is an orthonormal basis of $\check{L}^2(E)$.

We will use the following theorem characterizing affine systems $X(\Psi)$ being tight frames, which is a generalization of a well-known characterization theorem for dyadic dilations [15, §7] and integer dilations [2, 9]. In the case of arbitrary (non-integer) expansive dilations and $E = \mathbb{R}^n$ Theorem 1.4 has been shown in one dimension in [11] and in higher dimensions in [10]. The general case of Theorem 1.4 follows verbatim by an easy adaptation of the argument given in [10].

Theorem 1.4. *Suppose that $\Psi = \{\psi^1, \dots, \psi^L\} \subset \check{L}^2(E)$. Then $X(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$ if and only if*

$$\sum_{\psi \in \Psi} \sum_{\substack{(j,m) \in \mathbb{Z} \times \mathbb{Z}^n, \\ \alpha = (A^*)^{-j} m}} \hat{\psi}((A^*)^j \xi) \overline{\hat{\psi}((A^*)^j (\xi + \alpha))} = \delta_{\alpha,0} \mathbf{1}_E(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{1.3}$$

and for all $\alpha \in \Lambda$. *Here* Λ *denotes the set of all* A^**-adic vectors, i.e.,*

$$\Lambda = \{\alpha \in \mathbb{R}^n : \alpha = (A^*)^{-j} m \quad \text{for some } (j,m) \in \mathbb{Z} \times \mathbb{Z}^n\}. \tag{1.4}$$

2. SI systems and oversampling

In this section we recall some basic results about shift invariant (SI) systems and then we describe a simple procedure of oversampling a SI system that is not necessarily SI with respect to the standard lattice $\mathbb{Z}^n$.

DEFINITION 2.1. Suppose that Γ is a (full rank) lattice, i.e, $\Gamma = P\mathbb{Z}^n$, where P is an $n \times n$ non-singular matrix. We say that a closed subspace $W \subset L^2(\mathbb{R}^n)$ is *shift invariant* (SI) with respect to the lattice Γ, if $f \in W$ implies $T_\gamma f \in W$ for all $\gamma \in \Gamma$. Given a (countable) family $\Phi \subset L^2(\mathbb{R}^n)$ and the lattice Γ we define the SI system $E^\Gamma(\Phi)$ and SI space $S^\Gamma(\Phi)$ by

$$E^\Gamma(\Phi) = \{T_\gamma \varphi : \varphi \in \Phi,\ \gamma \in \Gamma\}, \qquad S^\Gamma(\Phi) = \overline{\operatorname{span}} E^\Gamma(\Phi). \tag{2.1}$$

When $\Gamma = \mathbb{Z}^n$ we often drop the superscript Γ, and we simply say that W is SI.

Given a SI system $E^\Gamma(\Phi)$ we are often interested in determining whether $E^\Gamma(\Phi)$ forms a frame for $S^\Gamma(\Phi)$. One possible way of achieving this is to consider a dual Gramian $\tilde{G}(\xi)$ of $E^\Gamma(\Phi)$ introduced by Ron and Shen [19]. For simplicity we restrict our attention to the case of $\Gamma = \mathbb{Z}^n$, since this will be sufficient for our work.

Suppose that X is some subset of a Hilbert space $\mathcal{H}$. Define an operator $F : \mathcal{H} \to l^2(X)$ by $F(h) = (\langle h, x\rangle)_{x \in X}$. We say that X is a *Bessel family* if F is bounded. In addition, if F is bounded from below then we say that X is a *frame.* The *frame operator* of $X \subset \mathcal{H}$ is $G : \mathcal{H} \to \mathcal{H}$ defined as $G = F^*F$. The lower and upper frame bounds of a frame $X \subset \mathcal{H}$ are defined as $||G^{-1}||^{-1/2}$ and $||G||^{1/2}$, respectively. If the lower and upper bounds of a frame $X \subset \mathcal{H}$ are equal then we say that X is a tight frame.

DEFINITION 2.2. Suppose $\Phi \subset L^2(\mathbb{R}^n)$ is a countable set such that

$$\sum_{\varphi \in \Phi} |\hat{\varphi}(\xi)|^2 < \infty \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{2.2}$$

The *dual Gramian* of $E(\Phi)$ is a map $\tilde{G}$ from the fundamental domain $\mathbb{T}^n = (-1/2, 1/2]^n$ into self-adjoint infinite matrices $(g_{k,l})_{k,l \in \mathbb{Z}^n}$ defined for a.e. $\xi \in \mathbb{T}^n$ by

$$\tilde{G}(\xi)_{k,l} := \sum_{\varphi \in \Phi} \hat{\varphi}(\xi + k)\overline{\hat{\varphi}(\xi + l)} \qquad \text{for } k, l \in \mathbb{Z}^n. \tag{2.3}$$

Note that if a matrix $\tilde{G}(\xi) = (\tilde{G}(\xi)_{k,l})_{k,l \in \mathbb{Z}^n}$ defines a bounded operator on $l^2(\mathbb{Z}^n)$ for some $\xi \in \mathbb{T}^n$, by $\langle \tilde{G}(\xi) e_k, e_l\rangle = \tilde{G}(\xi)_{k,l}$, where $(e_k)_{k \in \mathbb{Z}^n}$ is the standard basis of $l^2(\mathbb{Z}^n)$, then $\{(\hat{\varphi}(\xi + k))_{k \in \mathbb{Z}^n} : \varphi \in \Phi\} \subset l^2(\mathbb{Z}^n)$ is a Bessel family and $\tilde{G}(\xi)$ is the frame operator of the set $\{(\hat{\varphi}(\xi + k))_{k \in \mathbb{Z}^n} : \varphi \in \Phi\} \subset l^2(\mathbb{Z}^n)$. Obviously, the converse to this is also true. Furthermore, it follows from [19, Theorem 3.3.5] that (2.2) is a necessary (but no sufficient) condition for $E(\Phi)$ to be a Bessel family.

The following result due to Ron and Shen [19] characterizes when the system of translates of a given family of functions $E(\Phi)$ is a frame (or Bessel family if $a = 0$) in terms of the dual Gramian, see also [3, Theorem 2.5(ii)].

THEOREM 2.3. *Suppose $\Phi \subset L^2(\mathbb{R}^n)$ is countable and Φ satisfies (2.2). The system $E(\Phi)$ is a frame for a SI space $S(\Phi)$ with frame bounds $0 \le a \le b < \infty$, i.e.,*

$$a||f||^2 \le \sum_{\varphi \in \Phi} \sum_{k \in \mathbb{Z}^n} |\langle f, T_k \varphi \rangle|^2 \le b||f||^2 \qquad \text{for all } f \in S(\Phi),$$

if and only if the dual Gramian $\tilde{G}(\xi)$ satisfies

$$a||v||^2 \le \langle \tilde{G}(\xi)v, v\rangle \le b||v||^2 \qquad \text{for } v \in J(\xi), \quad \text{for a.e. } \xi \in \mathbb{T}^n, \tag{2.4}$$

where

$$J(\xi) := \overline{\operatorname{span}}\{(\hat{\varphi}(\xi + k))_{k \in \mathbb{Z}^n} : \varphi \in \Phi\}, \tag{2.5}$$

is the range function of $S(\Phi)$.

Recall that, in general, a range function is a mapping

$$J : \mathbb{T}^n \to \{\text{closed subspace of } l^2(\mathbb{Z}^n)\},$$

and there is one-to-one correspondence between SI spaces of $L^2(\mathbb{R}^n)$ and measurable range functions due to a classical result of Helson [1, 3, 14]. Recall also that whenever the dual Gramian $\tilde{G}(\xi)$ is bounded it represents the frame operator of $\{(\hat{\varphi}(\xi+k))_{k \in \mathbb{Z}^n} : \varphi \in \Phi\} \subset l^2(\mathbb{Z}^n)$, and hence $\tilde{G}(\xi)$ is always a self-adjoint operator satisfying

$$\overline{\operatorname{ran} \tilde{G}(\xi)} = J(\xi) \quad \text{and} \quad \ker \tilde{G}(\xi) = J(\xi)^{\perp} \qquad \text{for a.e. } \xi \in \mathbb{T}^n, \tag{2.6}$$

where $J(\xi)$ is given by (2.5).

In this section we are primarily interested in *rational* lattices Γ, i.e., $\Gamma = P\mathbb{Z}^n$, where P is an $n \times n$ non-singular matrix with rational entries. In this situation define $\tilde{\Gamma}$, the *integral sublattice* of Γ by $\tilde{\Gamma} = \mathbb{Z}^n \cap \Gamma$. We introduce two quotient groups:

— $\Gamma/\tilde{\Gamma}$ is called an *extension* group,

— $\mathbb{Z}^n/\tilde{\Gamma}$ is called an *oversampling* group.

Intuitively, the extension group measures how much the rational latice Γ extends beyond $\mathbb{Z}^n$, whereas the oversampling group determines how much oversampling is needed to obtain a minimal superlattice of Γ containing the standard lattice $\mathbb{Z}^n$.

DEFINITION 2.4. Suppose $\Phi \subset L^2(\mathbb{R}^n)$ is a countable set and Γ is a rational lattice. Define $O^{\Gamma}(\Phi)$ the *oversampled system* of $E^{\Gamma}(\Phi)$ by

$$O^{\Gamma}(\Phi) = \bigcup_{d \in \mathbb{Z}^n/\tilde{\Gamma}} T_d\left(E^{\Gamma}\left(\frac{1}{|\mathbb{Z}^n/\tilde{\Gamma}|^{1/2}}\Phi\right)\right), \tag{2.7}$$

where the union runs over representatives of distinct cosets of the oversampling group $\mathbb{Z}^n/\tilde{\Gamma}$, $\tilde{\Gamma}$ is an integral sublattice of Γ, and $|\mathbb{Z}^n/\tilde{\Gamma}|$ is the order of $\mathbb{Z}^n/\tilde{\Gamma}$.

By the above definition $O^{\Gamma}(\Phi)$ is always SI with respect to $\mathbb{Z}^n$. Indeed, by (2.7)

$$\begin{aligned} O^{\Gamma}(\Phi) &= \left\{\frac{1}{|\mathbb{Z}^n/\tilde{\Gamma}|^{1/2}} T_{d+\gamma}\varphi : \varphi \in \Phi,\ d \in \mathbb{Z}^n/\tilde{\Gamma},\ \gamma \in \Gamma\right\} \\ &= E^{\Gamma+\mathbb{Z}^n}\left(\frac{1}{|\mathbb{Z}^n/\tilde{\Gamma}|^{1/2}}\Phi\right). \end{aligned} \tag{2.8}$$

Note that if $\mathbb{Z}^n \subset \Gamma$ then no oversampling occurs, and the oversampled system $O^\Gamma(\Phi) = E^\Gamma(\Phi)$. The following lemma gives an explicit formula for the dual Gramian of $O^\Gamma(\Phi)$.

LEMMA 2.5. *Suppose $\Phi \subset L^2(\mathbb{R}^n)$ satisfies (2.2), and $\Gamma = P\mathbb{Z}^n$ is a rational lattice. The dual Gramian of $O^\Gamma(\Phi)$ is given for $k, l \in \mathbb{Z}^n$ as*

$$\tilde{G}(\xi)_{k,l} = \begin{cases} \frac{1}{|\det P|} \sum_{\varphi\in\Phi} \hat{\varphi}(\xi+k)\overline{\hat{\varphi}(\xi+l)} & \text{if } k-l \in \Gamma^*, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Here, Γ^* is the *dual* lattice of Γ, i.e.,

$$\Gamma^* = \{\eta \in \mathbb{R}^n : \langle \eta, \gamma\rangle \in \mathbb{Z} \quad \text{for } \gamma \in \Gamma\}.$$

That is, if $\Gamma = P\mathbb{Z}^n$ then $\Gamma^* = (P^*)^{-1}\mathbb{Z}^n$.

PROOF. The proof is purely computational. Note that by $(\Gamma + \mathbb{Z}^n)/\mathbb{Z}^n \simeq \Gamma/\tilde{\Gamma}$, and (2.8),

$$O^\Gamma(\Phi) = \bigcup_{d\in\Gamma/\tilde{\Gamma}} T_d\left(E^{\mathbb{Z}^n}\left(\frac{1}{|\mathbb{Z}^n/\tilde{\Gamma}|^{1/2}}\Phi\right)\right) = E^{\mathbb{Z}^n}\left(\bigcup_{d\in\Gamma/\tilde{\Gamma}}\left\{\frac{1}{|\mathbb{Z}^n/\tilde{\Gamma}|^{1/2}}T_d\Phi\right\}\right).$$

Hence, by the definition of the dual Gramian

$$\begin{aligned}\tilde{G}(\xi)_{k,l} &= \frac{1}{|\mathbb{Z}^n/\tilde{\Gamma}|}\sum_{\varphi\in\Phi}\sum_{d\in\Gamma/\tilde{\Gamma}} \widehat{T_d\varphi}(\xi+k)\overline{\widehat{T_d\varphi}(\xi+l)} \\ &= \frac{1}{|\mathbb{Z}^n/\tilde{\Gamma}|}\left(\sum_{d\in\Gamma/\tilde{\Gamma}} e^{-2\pi i\langle k-l,d\rangle}\right)\sum_{\varphi\in\Phi}\hat{\varphi}(\xi+k)\overline{\hat{\varphi}(\xi+l)}.\end{aligned}$$

Using $|\Gamma/\tilde{\Gamma}|/|\mathbb{Z}^n/\tilde{\Gamma}| = 1/|\det P|$ and Lemma 2.6 this yields (2.9). □

LEMMA 2.6. *Let Γ be a rational lattice and $\tilde{\Gamma} = \mathbb{Z}^n \cap \Gamma$ its integral sublattice. Then for any $m \in \mathbb{Z}^n$,*

$$\sum_{d\in\Gamma/\tilde{\Gamma}} e^{-2\pi i\langle m,d\rangle} = \begin{cases} |\Gamma/\tilde{\Gamma}| & \text{for } m \in \Gamma^*, \\ 0 & \text{for } m \notin \Gamma^* \end{cases} \tag{2.10}$$

PROOF. Fix $m \in \mathbb{Z}^n$. Note that the map $\sigma : \Gamma/\tilde{\Gamma} \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$ given by $\sigma(d) = e^{-2\pi i\langle m,d\rangle}$ for $d \in \Gamma/\tilde{\Gamma}$ is a well-defined group homomorphism. Since the quotent group $\Gamma/\tilde{\Gamma}$ has a finite order, $\sigma(\Gamma/\tilde{\Gamma})$ is a finite subgroup of S^1. However, any finite subgroup of S^1 consists of roots of unity of a certain order N. Hence, $\sigma(\Gamma/\tilde{\Gamma}) = \{e^{-2\pi ik/N} : k = 0, \dots, N-1\}$, where $N = N(m)$ depends on the choice of $m \in \mathbb{Z}^n$. Clearly, $N(m) = 1$ if $m \in \Gamma^*$, and $N(m) > 1$ if $m \notin \Gamma^*$. Since $\sigma^{-1}(e^{-2\pi ik/N})$ has the same cardinality for each $k \in \mathbb{Z}$ and $\sum_{k=0}^{N-1} e^{-2\pi ik/N} = \delta_{1,N}$ for $N \geq 1$ we obtain (2.10). □

3. Quasi-affine systems for rational dilations

The goal of this section is to introduce and study a class of quasi-affine systems for rational expansive dilations. Originally quasi-affine systems have been introduced and investigated only for integer expansive dilation matrices by Ron and Shen [20]. Their importance stems from the fact that the frame property carries over when moving from an affine system to its corresponding quasi-affine system, and vice versa. Furthermore, quasi-affine systems are shift invariant and thus much easier to study than affine systems which are dilation invariant.

Our goal is to introduce the notion of a quasi-affine frame for rational expansive dilations that overlaps with the usual definition in the case of integer dilations. The main idea of Ron and Shen [20] is to oversample negative scales of the affine system at a rate adapted to the scale in order for the resulting system to be shift invariant. Even though by doing this the orthogonality of the affine system is not carried over to the corresponding quasi-affine system, however, it turns out that the frame property is preserved.

In order to define quasi-affine systems for rational expansive dilations we need to oversample both negative and positive scales of the affine system (at a rate proportional to the scale) which results in a quasi-affine system that in general coincides with the affine system only at the scale zero. Hence, it is less clear than in the case of integer expansive dilations (where both systems coincide at all non-negative scales), whether there is any relationship between affine and quasi-affine systems. Nevertheless, it turns out the frame property still carries over between affine and quasi-affine systems.

DEFINITION 3.1. Suppose $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R})$ and A is an $n \times n$ rational expansive matrix. The *quasi-affine system* $X^q(\Psi)$ associated with the dilation A is defined as

$$X^q(\Psi) = \bigcup_{j\in\mathbb{Z}} O^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi). \tag{3.1}$$

Remark that the affine system $X(\Psi)$ can be equivalently introduced as

$$X(\Psi) = \bigcup_{j\in\mathbb{Z}} E^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi). \tag{3.2}$$

EXAMPLE 3.2. Note that if A is an integer expansive matrix then Definition 3.1 overlaps with the usual definition of a quasi-affine system, i.e.,

$$X^q(\Psi) = \{\tilde{\psi}_{j,k} : j \in \mathbb{Z},\ k \in \mathbb{Z}^n,\ \psi \in \Psi\},$$

where for $\psi \in L^2(\mathbb{R}^n)$ and $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ we set

$$\tilde{\psi}_{j,k}(x) = \begin{cases} |\det A|^{j/2}\psi(A^j x - k) & \text{if } j \ge 0, \\ |\det A|^{j}\psi(A^j (x - k)) & \text{if } j < 0. \end{cases}$$

Indeed, if A has integer entries then by (2.8)

$$O^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi) = \begin{cases} E^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi) & \text{for } j \ge 0, \\ E^{\mathbb{Z}^n}(|\det A|^{j/2} D_{A^j}\Psi) & \text{for } j < 0. \end{cases}$$

EXAMPLE 3.3. The quasi-affine system has a relatively simple algebraic form also in one dimension. Suppose $a = p/q \in \mathbb{Q}$ is a dilation factor, where $|a| > 1$, $\gcd(p,q) = 1$, $p,q \in \mathbb{Z}$. Then we claim that the quasi-affine system $X^q(\Psi)$ associated with a is given by

$$X^q(\Psi) = \{\tilde{\psi}_{j,k} : j,k \in \mathbb{Z},\ \psi \in \Psi\}.$$

Here for $\psi \in L^2(\mathbb{R})$ and $j,k \in \mathbb{Z}$ we set

$$\tilde{\psi}_{j,k}(x) = \begin{cases} \frac{p^{j/2}}{q^j}\psi(a^j x - q^{-j}k) & \text{if } j \geq 0, \\ \frac{p^j}{q^{j/2}}\psi(a^j x - p^j k) & \text{if } j < 0. \end{cases}$$

Note the above convention for $\tilde{\psi}_{j,k}$ in the case when a is an integer is consistent with Example 3.2. To show the claim not that by (2.8),

$$O^{a^{-j}\mathbb{Z}}(D_{a^j}\Psi) = \begin{cases} E^{p^{-j}\mathbb{Z}}(q^{-j/2}D_{a^j}\Psi) = q^{-j/2}D_{a^j}(E^{q^{-j}\mathbb{Z}}(\Psi)) & \text{for } j \geq 0, \\ E^{q^j\mathbb{Z}}(p^{j/2}D_{a^j}\Psi) = p^{j/2}D_{a^j}(E^{p^j\mathbb{Z}}(\Psi)) & \text{for } j < 0. \end{cases}$$

THEOREM 3.4. *Suppose A is an $n \times n$ rational expansive matrix and $\Psi = \{\psi^1, \dots, \psi^L\} \subset \check{L}^2(E)$, where E is A^*-multiplicatively invariant subset of $\mathbb{R}^n$. Then the affine system $X(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$ if and only if its quasi-affine counterpart $X^q(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$.*

PROOF. Theorem 1.4 gives a characterization of $X(\Psi)$ being a tight frame with constant 1 in terms of the equation (1.3). Note that (1.3) can be reduced to the form

$$\sum_{\psi\in\Psi} \sum_{\substack{(j,m)\in\mathbb{Z}\times\mathbb{Z}^n, \\ k=(A^*)^{-j}m}} \hat{\psi}((A^*)^j\xi)\overline{\hat{\psi}((A^*)^j(\xi+k))} = \delta_{k,0}\mathbf{1}_E(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{3.3}$$

and for all $k \in \mathbb{Z}^n$. It is clear that (1.3) implies (3.3). To see the converse implication take any $\alpha = (A^*)^{-j_0}k \in \Lambda$, $j_0 \in \mathbb{Z}$, $k \in \mathbb{Z}^n$,

$$\begin{aligned} &\sum_{\psi\in\Psi} \sum_{\substack{(j,m)\in\mathbb{Z}\times\mathbb{Z}^n, \\ \alpha=(A^*)^{-j}m}} \hat{\psi}((A^*)^j\xi)\overline{\hat{\psi}((A^*)^j(\xi+\alpha))} \\ &= \sum_{\psi\in\Psi} \sum_{\substack{(j,m)\in\mathbb{Z}\times\mathbb{Z}^n, \\ k=(A^*)^{-(j-j_0)}m}} \hat{\psi}((A^*)^{j-j_0}(A^*)^{j_0}\xi)\overline{\hat{\psi}((A^*)^{j-j_0}((A^*)^{j_0}\xi+k))} \\ &= \delta_{k,0}\mathbf{1}_E((A^*)^{j_0}\xi) = \delta_{\alpha,0}\mathbf{1}_E(\xi). \end{aligned}$$

On the other hand, since the system $X^q(\Psi)$ is shift invariant we can characterize $X^q(\Psi)$ to be a tight frame using Theorem 2.3 and Lemma 2.5. Our goal is to show that the resulting condition is precisely (3.3). Indeed, let $\tilde{G}_j(\xi)$ denote the dual Gramian of $O^{A^{-j}\mathbb{Z}^n}(\Psi)$ for $j \in \mathbb{Z}$. By Lemma 2.5, for $j \in \mathbb{Z}$ and $k,l \in \mathbb{Z}^n$ we have

$$\begin{aligned} \tilde{G}_j(\xi)_{k,l} &= \begin{cases} |\det A|^j \sum_{\psi\in\Psi} \widehat{D_{A^j}\psi}(\xi+k)\overline{\widehat{D_{A^j}\psi}(\xi+l)} & k-l \in (A^*)^j\mathbb{Z}^n, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \sum_{\psi\in\Psi} \hat{\psi}((A^*)^{-j}(\xi+k))\overline{\hat{\psi}((A^*)^{-j}(\xi+l))} & k-l \in (A^*)^j\mathbb{Z}^n, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

since $(A^{-j}\mathbb{Z}^n)^* = (A^*)^j\mathbb{Z}^n$. Let $\tilde{G}(\xi)$ denote the dual Gramian of $X^q(\Psi)$. By (3.1) and the additivity of dual Gramians

$$\begin{aligned}\tilde{G}(\xi)_{k,l} &= \sum_{j\in\mathbb{Z}} \tilde{G}_j(\xi)_{k,l} \\ &= \sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}} \hat{\psi}((A^*)^{-j}(\xi+k))\overline{\hat{\psi}((A^*)^{-j}(\xi+l))} \times \begin{cases} 1 & k-l\in(A^*)^j\mathbb{Z}^n, \\ 0 & \text{otherwise,}\end{cases} \\ &= \sum_{\psi\in\Psi} \sum_{\substack{(j,m)\in\mathbb{Z}\times\mathbb{Z}^n,\\ k-l=(A^*)^j m}} \hat{\psi}((A^*)^{-j}(\xi+k))\overline{\hat{\psi}((A^*)^{-j}(\xi+l))}\end{aligned} \tag{3.4}$$

By Theorem 2.3 and (2.6) it follows that $X^q(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$ if and only if the dual Gramian $\tilde{G}(\xi)$ of $X^q(\Psi)$ satisfies

$$\langle \tilde{G}(\xi)v, v\rangle = ||v||^2 \qquad \text{for } v\in J(\xi), \text{ and for a.e. } \xi\in\mathbb{T}^n,$$

where $J(\xi)$ is the range function corresponding to $\check{L}^2(E)$. In other words, $X^q(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$ if and only if the dual Gramian $\tilde{G}(\xi)$ is an orthogonal projection onto $J(\xi)$ for a.e. ξ. Recall that the range function $J(\xi)$ of $\check{L}^2(E)$ is given by

$$J(\xi) = \{v = (v(k))_{k\in\mathbb{Z}^n} \in l^2(\mathbb{Z}^n) : v(k)\neq 0 \Rightarrow \xi+k\in E\}. \tag{3.5}$$

Therefore, $X^q(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$ if and only if its dual Gramian $\tilde{G}(\xi)$ satisfies

$$\tilde{G}(\xi)_{k,l} = \begin{cases} 1 & \text{if } k=l \text{ and } \xi+k\in E \\ 0 & \text{otherwise}\end{cases} \qquad \text{for a.e. } \xi\in\mathbb{T}^n. \tag{3.6}$$

By (3.4) we conclude that (3.6) is equivalent with

$$\sum_{\psi\in\Psi} \sum_{\substack{(j,m)\in\mathbb{Z}\times\mathbb{Z}^n,\\ k-l=(A^*)^{-j} m}} \hat{\psi}((A^*)^{j}(\xi+k))\overline{\hat{\psi}((A^*)^{j}(\xi+k+(l-k)))} = \delta_{k-l,0}\mathbf{1}_E(\xi+k),$$

for a.e. $\xi\in\mathbb{T}^n$, and $k,l\in\mathbb{Z}^n$. However, this in turn is (3.3), which was shown to be equivalent with $X(\Psi)$ being a tight frame with constant 1 for $\check{L}^2(E)$ by Theorem 1.4. This completes the proof of Theorem 3.4. ☐

Theorem 3.4 can be generalized to the case of general (dual) frames. For example, as it is in the case of integer dilations [12, 20], one can show that $X(\Psi)$ is a tight frame with constants a and b if and only if its quasi-affine counterpart $X^q(\Psi)$ is a frame with the same constants. However, in this work we will only need the following simple fact.

LEMMA 3.5. *Under the assumptions of Theorem 3.4, if $X(\Psi)$ is Bessel with constant* 1 *then $X^q(\Psi)$ is also Bessel with constant* 1.

PROOF. For $j\in\mathbb{Z}$ define

$$K_j(f) = \sum_{\varphi\in E^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi)} |\langle f,\varphi\rangle|^2, \qquad K_j^q(f) = \sum_{\varphi\in O^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi)} |\langle f,\varphi\rangle|^2.$$

Given any $J \geq 0$ define

$$\Gamma_J = \bigcap_{|j| \leq J} A^j \mathbb{Z}^n,$$

which is a full rank sublattice of $\mathbb{Z}^n$. We claim that for any $J \geq 0$,

$$K_j^q(f) = \frac{1}{|\mathbb{Z}^n/\Gamma_J|} \sum_{d \in \mathbb{Z}^n/\Gamma_J} K_j(T_d f) \qquad \text{for } |j| \leq J, \tag{3.7}$$

where the sum runs over all representatives of distinct cosets of $\mathbb{Z}^n/\Gamma_J$. Indeed, pick any $|j| \leq J$ and let $\tilde{\Gamma} = \mathbb{Z}^n \cap A^{-j}\mathbb{Z}^n$ be the integral sublattice of $A^{-j}\mathbb{Z}^n$. By (2.7),

$$\begin{aligned} K_j^q(f) &= \sum_{\varphi \in O^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi)} |\langle f, \varphi\rangle|^2 \\ &= \frac{1}{|\mathbb{Z}^n/\tilde{\Gamma}|} \sum_{d \in \mathbb{Z}^n/\tilde{\Gamma}} \sum_{\varphi \in E^{A^{-j}\mathbb{Z}^n}(D_{A^j}\Psi)} |\langle f, T_d\varphi\rangle|^2 = \frac{1}{|\mathbb{Z}^n/\tilde{\Gamma}|} \sum_{d \in \mathbb{Z}^n/\tilde{\Gamma}} K_j(T_d f). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{|\mathbb{Z}^n/\Gamma_J|} \sum_{d \in \mathbb{Z}^n/\Gamma_J} K_j(T_d f) &= \frac{1}{|\tilde{\Gamma}/\Gamma_J|} \sum_{\eta \in \tilde{\Gamma}/\Gamma_J} \Big(\frac{1}{|\mathbb{Z}^n/\tilde{\Gamma}|} \sum_{d \in \mathbb{Z}^n/\tilde{\Gamma}} K_j(T_d T_\eta f) \Big) \\ &= \frac{1}{|\tilde{\Gamma}/\Gamma_J|} \sum_{\eta \in \tilde{\Gamma}/\Gamma_J} K_j^q(T_\eta f) = \frac{1}{|\tilde{\Gamma}/\Gamma_J|} \sum_{\eta \in \tilde{\Gamma}/\Gamma_J} K_j^q(f) = K_j^q(f), \end{aligned}$$

which shows (3.7). By (3.7),

$$\begin{aligned} \sum_{j \in \mathbb{Z}} K_j^q(f) = \lim_{J \to \infty} \sum_{|j| \leq J} K_j^q(f) &= \lim_{J \to \infty} \frac{1}{|\mathbb{Z}^n/\Gamma_J|} \sum_{d \in \mathbb{Z}^n/\Gamma_J} K_j(T_d f) \\ &\leq \lim_{J \to \infty} \frac{1}{|\mathbb{Z}^n/\Gamma_J|} \sum_{d \in \mathbb{Z}^n/\Gamma_J} ||T_d f||^2 = ||f||^2, \end{aligned}$$

since $X(\Psi)$ is Bessel with constant 1. This completes the proof of Lemma 3.5. □

4. The Calderón condition and completeness of affine systems

In this section we show that the Calderón condition

$$\sum_{\psi \in \Psi} \sum_{j \in \mathbb{Z}} |\hat{\psi}((A^*)^j \xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{4.1}$$

characterizes completeness of orthonormal affine systems for a large class of expansive dilation matrices including, e.g., for all one dimensional real dilations. This fact was conjectured by G. Weiss and it was originally shown in the case of integer expansive dilations by the author [4] and Rzeszotnik [21]. Here we extend the methods developed in [4] and based on quasi-affine systems to show this conjecture for a large class of expansive dilations. A different approach based on semi-continuous wavelet systems was used by R. Laugesen [16, 17] to show this conjecture for arbitrary expansive dilations.

We start with a result extending [4, Theorem 2.5] to the case of rational expansive dilations.

THEOREM 4.1. *Suppose A is an $n \times n$ rational expansive matrix and $\Psi = \{\psi^1, \dots, \psi^L\} \subset \check{L}^2(E)$, where E is A^*-multiplicatively invariant subset of $\mathbb{R}^n$. Assume that $X(\Psi)$ is a Bessel family with constant 1. Then the following are equivalent:*

(i) $X(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$,

(ii) the discrete Calderón formula holds

$$\sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}} |\hat{\psi}((A^*)^j\xi)|^2 = \mathbf{1}_E(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{4.2}$$

(iii) for any (some) quasi-norm ρ associated with A^*,

$$\sum_{\psi\in\Psi}\int_{\mathbb{R}^n} \frac{|\hat{\psi}(\xi)|^2}{\rho(\xi)} d\xi = \kappa(\rho, E) := \int_E \frac{\mathbf{1}_D(\xi)}{\rho(\xi)} d\xi, \tag{4.3}$$

where $D \subset E$ is any measurable set such that $\{(A^*)^j D : j \in \mathbb{Z}\}$ partitions E (modulo sets of measure zero).

Recall from [6] that a *quasi-norm* associated with an expansive dilation B is a measurable mapping $\rho : \mathbb{R}^n \to [0, \infty)$ satisfying

(i) $\rho(\xi) = 0 \iff \xi = 0$,

(ii) $\rho(B\xi) = |\det B|\rho(\xi)$ for all $\xi \in \mathbb{R}^n$,

(iii) there is $c > 0$ so that $\rho(\xi + \zeta) \le c(\rho(\xi) + \rho(\zeta))$ for all $\xi, \zeta \in \mathbb{R}^n$.

In the case when $E = \mathbb{R}^n$, $\kappa(\rho, E)$ is referred to as a *characteristic number* of a quasi-norm ρ, and it can be shown that $\kappa(\rho, E)$ does not depend on the choice of D, see [4].

PROOF OF THEOREM 4.1. The proof of Theorem 4.1 follows closely [4, Theorem 2.4]. The implication (i) $\implies$ (ii) is a consequence of Theorem 1.4. The implication (ii) $\implies$ (iii) is a consequence of

$$\sum_{\psi\in\Psi}\int_{\mathbb{R}^n} |\hat{\psi}(\xi)|^2 \frac{d\xi}{\rho(\xi)} = \sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}}\int_{(A^*)^j D} |\hat{\psi}(\xi)|^2 \frac{d\xi}{\rho(\xi)} = \sum_{\psi\in\Psi}\int_D \sum_{j\in\mathbb{Z}} |\hat{\psi}((A^*)^j\xi)|^2 \frac{d\xi}{\rho(\xi)}$$
$$= \sum_{\psi\in\Psi}\int_D \frac{\mathbf{1}_E(\xi)}{\rho(\xi)} d\xi = \kappa(\rho, E).$$

Finally, we need to show the implication (iii) $\implies$ (i). Since $X(\Psi)$ is a Bessel family with constant 1, $X^q(\Psi)$ is also a Bessel family with constant 1 by Lemma 3.5. Let $\tilde{G}(\xi)$ be the dual Gramian of $X^q(\Psi)$. By Theorem 2.3, (2.4), and (2.6) we have $||\tilde{G}(\xi)|| \le 1$ for a.e. $\xi \in \mathbb{T}^n$. In particular, for any $k \in \mathbb{Z}^n$,

$$1 \ge \sum_{l\in\mathbb{Z}^n} |\tilde{G}(\xi)_{k,l}|^2 = |\tilde{G}(\xi)_{k,k}|^2 + \sum_{l\in\mathbb{Z}^n, l\neq k} |\tilde{G}(\xi)_{k,l}|^2 \qquad \text{for a.e. } \xi \in \mathbb{T}^n. \tag{4.4}$$

By (3.4) for any $k \in \mathbb{Z}^n$ and for a.e. $\xi \in \mathbb{T}^n$,

$$\tilde{G}(\xi)_{k,k} = \sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}} |\hat{\psi}((A^*)^j(\xi + k))|^2,$$

and hence by (4.4),

$$\sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}} |\hat{\psi}((A^*)^j\xi)|^2 \le 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n.$$

Since

$$\kappa(\rho, E) = \sum_{\psi\in\Psi}\int_{\mathbb{R}^n}\frac{|\hat\psi(\xi)|^2}{\rho(\xi)}d\xi = \sum_{\psi\in\Psi}\int_D\sum_{j\in\mathbb{Z}}|\hat\psi((A^*)^j\xi)|^2\frac{d\xi}{\rho(\xi)} \le \int_D\frac{d\xi}{\rho(\xi)} = \kappa(\rho, E),$$

where D is the same as in (iii), we actually have

$$\sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}}|\hat\psi((A^*)^j\xi)|^2 = 1 \qquad \text{for a.e. } \xi\in D,$$

and thus for a.e. $\xi \in E$. The above combined with (4.4) shows (3.6), which implies that $X(\Psi)$ is a tight frame with constant 1 by the proof of Theorem 3.4. □

Even though Theorem 4.1 works nominally for rational dilations, however, it enables us to show the conjecture of Weiss for a much larger class of dilations. Theorem 4.2 extends the main result of [4, Theorem 2.4].

THEOREM 4.2. *Suppose A is an $n\times n$ expansive matrix such that for any $j\in\mathbb{Z}$, $\mathbb{Z}^n\cap(A^*)^j\mathbb{Z}^n$ is either a trivial or a full rank sublattice of $\mathbb{Z}^n$. Suppose that $\Psi = \{\psi^1,\dots,\psi^L\}\subset L^2(\mathbb{R}^n)$ is such that the affine system $X(\Psi)$ is orthonormal (but not necessarily complete). Then the following are equivalent:*
(i) $X(\Psi)$ is complete, i.e., Ψ is a wavelet,
(ii) the discrete Calderón formula holds

$$\sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}}|\hat\psi((A^*)^j\xi)|^2 = 1 \qquad \text{for a.e. } \xi\in\mathbb{R}^n, \tag{4.1}$$

(iii) for any (some) quasi-norm ρ associated with A^*,

$$\sum_{\psi\in\Psi}\int_{\mathbb{R}^n}|\hat\psi(\xi)|^2\frac{d\xi}{\rho(\xi)} = \kappa(\rho), \tag{4.5}$$

where $\kappa(\rho)$ is the characteristic number of ρ.

Before we start the proof of Theorem 4.2 we need to show Lemma 4.3.

LEMMA 4.3. *Under the assumptions of Theorem 4.2, suppose that for some $j\in\mathbb{Z}$, $\mathbb{Z}^n\cap(A^*)^j\mathbb{Z}^n = \{0\}$. Then K and $(A^*)^jK$ are disjoint (modulo null sets), where $K = \bigcup_{\psi\in\Psi}\operatorname{supp}\hat\psi$.*

PROOF. By the orthogonality of $X(\Psi)$

$$\begin{aligned}\delta_{l,l'}\delta_{j,0}\delta_{k,k'} = \langle\psi^l_{j,k},\psi^{l'}_{0,k'}\rangle &= |\det A|^{j/2}\int_{\mathbb{R}^n}\psi^l(A^jx-k)\overline{\psi^{l'}(x-k')}dx\\ &= b^{j/2}\int_{\mathbb{R}^n}\psi^l(A^jx+A^jk'-k)\overline{\psi^{l'}(x)}dx,\end{aligned}$$

for all $k,k'\in\mathbb{Z}^n$, $l,l' = 1,\dots,L$. Thus, by Plancherel's formula

$$\begin{aligned}0 &= \int_{\mathbb{R}^n}\overline{\hat\psi^{l'}(\xi)}\hat\psi^l((A^*)^{-j}\xi)e^{2\pi i\langle(A^*)^{-j}\xi, A^jk'-k\rangle}d\xi\\ &= \int_{\mathbb{R}^n}\overline{\hat\psi^{l'}(\xi)}\hat\psi^l((A^*)^{-j}\xi)e^{2\pi i\langle\xi, k'-A^{-j}k\rangle}d\xi.\end{aligned} \tag{4.6}$$

The condition $\mathbb{Z}^n\cap(A^*)^j\mathbb{Z}^n = \{0\}$ means that the rows of the matrix A^j (treated as vectors in $\mathbb{R}^n$) together with the standard basis vectors $(1,0,\dots,0)$,

..., $(0,\dots,0,1)$ are linearly independent over $\mathbb{Q}$. By Lemma 4.4 we conclude that $\mathbb{Z}^n + A^{-j}\mathbb{Z}^n$ is dense in $\mathbb{R}^n$. Therefore, by the Fourier Inversion Formula and (4.6), $\hat\psi^{l'}(\xi)\hat\psi^{l}((A^*)^{-j}\xi)=0$ for a.e. $\xi\in\mathbb{R}^n$ and $l,l'=1,\dots,L$. This completes the proof of Lemma 4.3. □

LEMMA 4.4. *Suppose B is an $n\times n$ real matrix such that the rows of B (treated as vectors in $\mathbb{R}^n$) together with the standard basis vectors $(1,0,\dots,0)$, ..., $(0,\dots,0,1)$ are linearly independent over $\mathbb{Q}$. Then the set $\mathbb{Z}^n + B\mathbb{Z}^n$ is dense in $\mathbb{R}^n$.*

PROOF. The proof of Lemma 4.4 uses an argument involving the Weyl Criterion of uniform distribution mod 1, and it can be found in [6, Chapter 2, Lemma 3.2]. □

PROPOSITION 4.5. *Suppose A is an $n\times n$ expansive dilation matrix and $\Psi=\{\psi^1,\dots,\psi^L\}\subset L^2(\mathbb{R}^n)$ is such that the affine system $X(\Psi)$ is a Bessel family with constant 1. Then*

$$\sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}}|\hat\psi((A^*)^j\xi)|^2\le 1 \qquad \text{for a.e. } \xi\in\mathbb{R}^n. \tag{4.7}$$

PROOF. The argument follows verbatim the proof of the necessity " $\Longrightarrow$ " part of Theorem 1.4, see [10]. □

PROOF OF THEOREM 4.2. As in Theorem 4.1 the implications (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) follow in the same manner. The implication (iii) $\Longrightarrow$ (ii) is a consequence of Proposition 4.5 and

$$\kappa(\rho)=\sum_{\psi\in\Psi}\int_{\mathbb{R}^n}\frac{|\hat\psi(\xi)|^2}{\rho(\xi)}d\xi=\sum_{\psi\in\Psi}\int_D\sum_{j\in\mathbb{Z}}|\hat\psi((A^*)^j\xi)|^2\frac{d\xi}{\rho(\xi)}\le\sum_{\psi\in\Psi}\int_D\frac{1}{\rho(\xi)}d\xi=\kappa(\rho),$$

where $D\subset\mathbb{R}^n$ is such that $\{(A^*)^jD : j\in\mathbb{Z}\}$ partitions $\mathbb{R}^n$. Finally we are left with (ii) $\Longrightarrow$ (i).

Assume first that

$$\mathbb{Z}^n\cap(A^*)^j\mathbb{Z}^n=\{0\} \qquad \text{for all } j\in\mathbb{Z}\setminus\{0\}. \tag{4.8}$$

Let $K=\bigcup_{\psi\in\Psi}\operatorname{supp}\hat\psi$. By Lemma 4.3 we have that the sets $\{(A^*)^jK : j\in\mathbb{Z}\}$ are pairwise disjoint (modulo null sets) and therefore by (4.1),

$$|K|=\int_K\sum_{\psi\in\Psi}\sum_{j\in\mathbb{Z}}|\hat\psi((A^*)^j\xi)|^2d\xi=\int_K\sum_{\psi\in\Psi}|\hat\psi(\xi)|^2=L, \tag{4.9}$$

For $j\in\mathbb{Z}$ let

$$W_j=\overline{\operatorname{span}}\{\psi_{j,k} : k\in\mathbb{Z}^n,\ \psi\in\psi\}.$$

Since $W_0\subset\check L^2(K)$, by the basic properties of the dimension function,

$$L=\dim_{W_0}(\xi)\le\dim_{\check L^2(K)}(\xi) \qquad \text{for a.e. } \xi\in\mathbb{T}^n,$$

since $W_0\subset\check L^2(K)$; for the definition and properties of the the dimension function $\dim_W(\xi)$ of a SI space W we refer to [1, 3]. Here we only recall that the dimension function of a SI space W is a map $\dim_W : \mathbb{T}^n \to \mathbb{N} \cup \{0, \infty\}$ that measure the size of W over the fibers $\mathbb{R}^n/\mathbb{Z}^n$. Hence by (4.9),

$$\int_{\mathbb{T}^n} \dim_{\check{L}^2(K)}(\xi) d\xi = |K| = L,$$

and we actually have $W_0 = \check{L}^2(K)$. By (4.1) we also have $\bigcup_{j\in\mathbb{Z}} (A^*)^j K = \mathbb{R}^n$ and thus $\bigoplus_{j\in\mathbb{Z}} W_j = \bigoplus_{j\in\mathbb{Z}} \check{L}^2((A^*)^j K) = L^2(\mathbb{R}^n)$, i.e., $X(\Psi)$ is complete. Moreover, this argument shows that Ψ has to be a combined MSF multiwavelet, see [5].

Assume next that for some integer $j \geq 1$, $\mathbb{Z}^n \cap (A^*)^j \mathbb{Z}^n$ is a full rank sublattice of $\mathbb{Z}^n$. Let j_0 be the smallest among such integers. Clearly, $\mathbb{Z}^n \cap (A^*)^j \mathbb{Q}^n = \{0\}$ for $0 < j < j_0$. Moreover, $(A^*)^{j_0}$ has rational entries and hence all matrices $(A^*)^j$ for $j \in j_0\mathbb{Z}$ are rational. Therefore, for any $j \notin j_0\mathbb{Z}$,

$$\mathbb{Z}^n \cap (A^*)^j \mathbb{Z}^n = \mathbb{Z}^n \cap ((A^*)^{j-\lfloor j/j_0 \rfloor j_0} (A^*)^{\lfloor j/j_0 \rfloor j_0} \mathbb{Z}^n) \subset \mathbb{Z}^n \cap (A^*)^{j-\lfloor j/j_0 \rfloor j_0} \mathbb{Q}^n = \{0\}$$

By Lemma 4.3, K and $(A^*)^j K$ are disjoint for $j \notin j_0\mathbb{Z}$, where $K = \bigcup_{\psi\in\Psi} \operatorname{supp} \hat{\psi}$. Therefore, the sets $E, A^*E, \ldots, (A^*)^{j_0-1}E$ are pairwise disjoint (modulo null sets), where

$$E = \bigcup_{j\in j_0\mathbb{Z}} (A^*)^j K.$$

Moreover, by (4.1), $E \cup A^*E \cup \ldots \cup (A^*)^{j_0-1}E = \mathbb{R}^n$, and

$$\sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}} |\hat{\psi}((A^*)^{jj_0}\xi)|^2 = \mathbf{1}_E(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}^n. \tag{4.10}$$

Therefore, to complete the proof, it suffices to show that the affine system

$$\tilde{X}(\Psi) = \{D_{A^j} T_k \psi : j \in j_0\mathbb{Z},\ k \in \mathbb{Z}^n,\ \psi \in \Psi\}$$

associated with the rational dilation A^{j_0} is complete in $\check{L}^2(E)$. Let $\tilde{X}^q(\Psi)$ be the corresponding quasi-affine system, as given by Definition 3.1. Let $\tilde{G}(\xi)$ be the dual Gramian of this system as in the proof of Theorem 3.4. By (3.4) for $k, l \in \mathbb{Z}^n$,

$$\tilde{G}(\xi)_{k,l} = \sum_{\psi\in\Psi} \sum_{\substack{(j,m)\in\mathbb{Z}\times\mathbb{Z}^n, \\ k-l=(A^*)^{-jj_0}m}} \hat{\psi}((A^*)^{jj_0}(\xi+k)) \overline{\hat{\psi}((A^*)^{jj_0}(\xi+l))}.$$

Therefore, by (4.10), for $k \in \mathbb{Z}^n$,

$$\tilde{G}(\xi)_{k,k} = \sum_{\psi\in\Psi} \sum_{j\in\mathbb{Z}} |\hat{\psi}((A^*)^{jj_0}(\xi+k))|^2 = \mathbf{1}_E(\xi+k) \qquad \text{for a.e. } \xi \in \mathbb{T}^n.$$

Moreover, for $k, l \in \mathbb{Z}^n$,

$$\tilde{G}(\xi)_{k,l} = 0 \qquad \text{for a.e. } \xi \in \mathbb{T}^n,\ \xi+k \notin E \text{ or } \xi+l \notin E.$$

By Lemma 3.5, $\tilde{X}^q(\Psi)$ is Bessel with constant 1, hence by Theorem 2.3, $||\tilde{G}(\xi)|| \leq 1$ for a.e. ξ. Therefore, for $\xi + k \in E$,

$$1 \geq \sum_{l\in\mathbb{Z}^n} |\tilde{G}(\xi)_{k,l}|^2 = 1 + \sum_{l\in\mathbb{Z}^n, l\neq k} |\tilde{G}(\xi)_{k,l}|^2,$$

hence $\tilde{G}(\xi)_{k,l} = 0$ for all $l \neq k$. This shows (3.6) and hence $\tilde{G}(\xi)$ is an orthogonal projection onto the range function $J(\xi)$ corresponding to $\check{L}^2(E)$, where $J(\xi)$ is given by (3.5). Therefore, $\tilde{X}^q(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$ and by Theorem 3.4, $\tilde{X}(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$, and thus $\tilde{X}(\Psi)$ is complete in $\check{L}^2(E)$. This completes the proof of Theorem 4.2. □

As an immediate corollary of Theorem 4.2 we have

COROLLARY 4.6. *Suppose that a is real, $|a| > 1$, and $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R})$ is such that $X(\Psi)$ is orthonormal (but not necessarily complete). Then the following are equivalent:*

(i) $X(\Psi)$ is complete, i.e., Ψ is a wavelet,

(ii) the discrete Calderón formula holds

$$\sum_{\psi \in \Psi} \sum_{j \in \mathbb{Z}} |\hat{\psi}(a^j \xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R},$$

(iii) the continuous Calderón formula holds

$$\sum_{\psi \in \Psi} \int_{\mathbb{R}} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi = 2 \ln |a|.$$

We remark that the assumption that $X(\Psi)$ is orthogonal can be relaxed by $X(\Psi)$ being Bessel with constant 1 (at least when a is rational by Theorem 4.1, or when a is positive, see [16]).

Finally, we remark that the equivalence (i) $\iff$ (iii) in Theorem 4.2 can be used to show the completeness theorem of Garrigós and Speegle [13] for a much larger class of dilations (than with integer entries) satisfying the hypothesis of Theorem 4.2. The argument follows verbatim [4, §3].

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