The Wavelet Dimension Function for Real Dilations and Dilations Admitting non-MSF Wavelets

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Abstract. The wavelet dimension function for arbitrary real dilations is defined and used to address several questions involving the existence of MRA wavelets and well-localized wavelets for irrational dilations. The theory of quasi-affine frames for rational dilations and the existence of non-MSF wavelets for certain irrational dilations play an important role in this development. Expansive dilations admitting non-MSF wavelets are characterized, and an example of a wavelet with respect to a non-expansive matrix is given.

§1. Introduction

The wavelet dimension function, which is sometimes referred to as the multiplicity function, is an important subject in the theory of wavelets that has been extensively studied by a number of authors, e.g., [2,4,8,20]. However, up to the present time, the wavelet dimension function has been considered only for integer expansive dilations. In this work we plan to initiate the study of the wavelet dimension function for non-integer dilations by investigating the situation in one dimension for all real dilation factors.

This subject appears to be quite intricate with many unexpected twists that combine seemingly unrelated matters including, among other things, the development of the notion of quasi-affine systems for rational dilations and the characterization of dilations admitting non-MSF wavelets. One of the biggest surprises is the fact that the usual formula for

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the wavelet dimension function is often valid also for real dilation factors, but not always. In fact, it is valid precisely in two distinct situations: when a dilation factor is rational or a particular wavelet is MSF. Otherwise, the core space of a GMRA generated by a wavelet is not shift invariant, and the usual wavelet dimension function may not be even integer valued.

The consequences of the existence of the wavelet dimension function for real dilation factors are far reaching, as was demonstrated by Auscher [2] for expansive integer dilations. In particular, it is used in this paper to answer Daubechies' question [13] by showing the non-existence of well-localized wavelets for irrational dilations by different methods than the original argument used by the first author in [5]. Another consequence is the non-existence of MRA wavelets (not necessarily well-localized) for irrational dilations, which was expected much earlier, see [1], but lacked a proof.

We close our considerations by giving a complete characterization of expansive dilations that admit non-MSF wavelets. This turns out to be intimately connected to the topics mentioned above for several reasons. First, this characterization gives a converse to the theorem by Chui and Shi [6], which says that for dilations a such that $a^j \notin \mathbf{Q}$ for all integers $j \geq 1$, the only wavelets that exist are MSF wavelets. Secondly, it shows that the results obtained in this paper and concerning the wavelet dimension function are optimal, e.g., it provides examples of wavelets for which the usual formula for the wavelet dimension function is not valid. Thirdly, it shows that an approach used in this paper and in [5] is necessary for the complete solution to Daubechies' question, as well as providing limits on what types of negative results can be proved in higher dimensions. The idea of the proof of this characterization is to use the techniques of Meyer as explained in [15] and the procedure for constructing interpolation families of wavelet sets [14,21] to construct the Fourier transform of a wavelet. The key element that makes it possible is the recent characterization of wavelets via their Fourier transforms [9,10].

The paper is organized as follows. In Section 2 we introduce quasi-affine systems for rational dilations and we show the equivalence of affine and quasi-affine tight frames. In the next section we build the wavelet dimension function for arbitrary real dilation factors and we derive its basic properties. In Section 4 we apply the wavelet dimension function to show several results about well-localized wavelets. In Section 5 we give a characterization of all expansive dilations that admit non-MSF wavelets and we use these results to show the sharpness of the results obtained in Section 3. Finally, we give some explicit examples of wavelets illustrating the ideas underlying this paper.

We begin by recalling several definitions and theorems that will be useful in the sequel.

Definition 1.1. *Suppose $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R})$ and $a \in \mathbb{R}$, $|a| > 1$. The* affine system *$X(\Psi)$ associated with the dilation a is defined as*

$$X(\Psi) = \{\psi^l_{j,k} : j, k \in \mathbb{Z},\ l = 1, \dots, L\}.$$

Here for $\psi \in L^2(\mathbb{R})$ we set $\psi_{j,k}(x) = |a|^{j/2}\psi(a^j x - k)$ for $j, k \in \mathbb{Z}$.

Definition 1.2. *Suppose $a \in \mathbb{R}$ and $|a| > 1$. We say that a measurable subset E of $\mathbb{R}$ is a-*multiplicatively invariant *if $aE = E$ modulo sets of measure zero. Given such E we introduce the closed subspace $\check{L}^2(E) \subset L^2(\mathbb{R})$ by*

$$\check{L}^2(E) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} = \{\xi : \hat{f}(\xi) \neq 0\} \subset E\}.$$

We say that $\Psi = \{\psi^1, \dots, \psi^L\} \subset \check{L}^2(E)$ is a multiwavelet for $\check{L}^2(E)$ associated with a, *or shortly a-*multiwavelet for $\check{L}^2(E)$, *if $X(\Psi)$ is an orthonormal basis of $\check{L}^2(E)$.*

The following theorem shown by Chui and Shi when $E = \mathbb{R}$ characterizes affine systems $X(\Psi)$ that are tight frames for arbitrary real dilation factors, see [10]. The general case of Theorem 1.3 is an easy generalization of the arguments given in [10].

Theorem 1.3. *Suppose that $\Psi = \{\psi^1, \dots, \psi^L\} \subset \check{L}^2(E)$. Then $X(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$ if and only if*

$$\sum_{l=1}^{L} \sum_{\substack{(j,m) \in \mathbb{Z} \times \mathbb{Z}, \\ \alpha = a^{-j} m}} \hat{\psi}^l(a^j \xi)\overline{\hat{\psi}^l(a^j(\xi + \alpha))} = \delta_{\alpha,0}\mathbf{1}_E(\xi), \qquad \text{for a.e. } \xi \in \mathbb{R}, \tag{1.1}$$

and for all α belonging to the set of all a-adic numbers, i.e.,

$$\{\alpha \in \mathbb{R} : \alpha = a^{-j} m \quad \text{for some } (j, m) \in \mathbb{Z} \times \mathbb{Z}\}.$$

The above result can be also generalized to higher dimensions, see [9]. Indeed, let A be an expansive matrix, i.e., A is $n \times n$ matrix such that all eigenvalues of A have modulus bigger than one. An A-wavelet is a function $\psi \in L^2(\mathbb{R}^n)$ such that $\{|\det A|^{j/2}\psi(A^j x - k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.

Theorem 1.4. *Let A be an expansive matrix. A function $\psi \in L^2(\mathbb{R}^n)$ is an A-wavelet if and only if $\|\psi\| = 1$ and*

$$\sum_{\substack{(j,m) \in \mathbb{Z} \times \mathbb{Z}^n, \\ \alpha = (A^*)^{-j} m}} \hat{\psi}(A^{*j}\xi)\overline{\hat{\psi}(A^{*j}(\xi + A^{*-j} m))} = \delta_{\alpha,0}, \qquad \text{for a.e. } \xi \in \mathbb{R}^n \tag{1.2}$$

and for all α belonging to the set of A-adic vectors, i.e.,

$$\Lambda = \{\alpha \in \mathbb{R}^n : \alpha = (A^*)^{-j} m \quad \text{for some } (j, m) \in \mathbb{Z} \times \mathbb{Z}^n\}. \tag{1.3}$$

§2. Quasi-affine Systems for Rational Dilations

In this section we introduce a notion of a quasi-affine system for rational dilations. Quasi-affine systems have been introduced and studied for integer dilations by Ron and Shen [19]. Their importance stems from the fact that the frame property is preserved when moving from an affine system to its corresponding quasi-affine system, and vice versa. Therefore, even though affine systems are dilation invariant (and thus not easy to study directly), instead one can work with quasi-affine systems which are shift invariant (and thus much easier to study).

The main idea behind quasi-affine systems is to oversample negative scales of the affine system at a rate adapted to the scale in order for the resulting system to be shift invariant. Even though the orthogonality of the affine system is not preserved by the corresponding quasi-affine system, however, it turns out that the frame property is preserved.

In order to define quasi-affine systems for rational dilation factors, we must oversample not only negative scales of the affine system (again at a rate proportional to the scale), but also, the positive scales. Since the resulting system coincides with the usual affine system only at the scale zero (for rational non-integral dilations), it is less clear (than in the case of integer dilations where both systems coincide at all non-negative scales) whether the quasi-affine system will share any common properties with its affine counterpart. Nevertheless, as was the case for integer dilations, the affine and quasi-affine systems share again the frame property. This turns out to be of critical importance in this work. The major consequence of this fact is the existence of the wavelet dimension function for the class of rational factors instead of the standard class of integer factors.

We start by defining the quasi-affine system for rational factors, and by showing that the property of being a tight frame is preserved when moving between these two systems. Our proof will be based on the characterization of tight affine systems by Chui and Shi [10], see Theorem 1.3.

Definition 2.1. *Suppose* $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R})$ *and* $a = p/q \in \mathbb{Q}$, *where* $\gcd(p,q) = 1$. *The* quasi-affine system $X^q(\Psi)$ associated with the dilation a *is defined as*

$$X^q(\Psi) = \{\tilde{\psi}^l_{j,k} : j,k \in \mathbb{Z},\ l = 1, \dots, L\}.$$

Here for $\psi \in L^2(\mathbb{R})$ *and* $j,k \in \mathbb{Z}$ *we set*

$$\tilde{\psi}_{j,k}(x) = \begin{cases} \frac{p^{j/2}}{q^j}\psi(a^j x - q^{-j}k) & \text{if } j \geq 0, \\ \frac{p^j}{q^{j/2}}\psi(a^j x - p^j k) & \text{if } j < 0. \end{cases}$$

We will use the following standard notation. The translation by $k \in \mathbb{R}$ is $T_k f(x) = f(x-k)$. Given a family $\Phi \subset L^2(\mathbb{R})$ define the shift invariant (SI) system $E(\Phi)$ and SI space $S(\Phi)$ by

$$E(\Phi) = \{T_k\varphi : k \in \mathbb{Z}, \varphi \in \Phi\}, \qquad S(\Phi) = \overline{\operatorname{span}} E(\Phi). \tag{2.1}$$

Theorem 2.2. *Suppose that $a \in \mathbb{Q}$ and $\Psi = \{\psi^1, \dots, \psi^L\} \subset \check{L}^2(E)$, where E is an a-multiplicatively invariant subset of $\mathbb{R}$. Then the affine system $X(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$ if and only if its quasi-affine counterpart $X^q(\Psi)$ is a tight frame with constant 1 for $\check{L}^2(E)$.*

Proof: Theorem 1.3 gives a characterization of $X(\Psi)$ being a tight frame with constant 1 in terms of equation (1.1). In the case when a is rational, Chui and Shi [10] have shown that (1.1) can be written as

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(a^j\xi)|^2 = \mathbf{1}_E(\xi), \qquad \text{for a.e. } \xi \in \mathbb{R}, \tag{2.2}$$

$$\sum_{l=1}^{L}\sum_{j=0}^{s} \hat{\psi}^l(a^j\xi)\overline{\hat{\psi}^l(a^j(\xi+q^s t))} = 0, \qquad \text{for a.e. } \xi \in \mathbb{R} \tag{2.3}$$

for all $s = 0, 1, 2, \dots$ and all $t \in \mathbb{Z}$ not divisible by p nor q ($p, q \not| \, t$).

On the other hand, since the system $X^q(\Psi)$ is shift invariant, we can phrase a necessary and sufficient condition for $X^q(\Psi)$ to be a tight frame in terms of dual Gramians following the work of Ron and Shen [18]. Our goal is to show that this condition is equivalent to (2.2) and (2.3). Indeed, for simplicity, assume that a quasi affine system $X^q(\Psi)$ is generated by a single function $\Psi = \{\psi\}$. Let Φ be a set whose integer translates generate $X^q(\Psi)$, i.e.,

$$X^q(\Psi) = E(\Phi) := \{T_k\varphi : k \in \mathbb{Z}, \varphi \in \Phi\}.$$

Let D_j denote the set of representatives of different cosets of $\mathbb{Z}/(p^j\mathbb{Z})$ if $j \ge 0$ and of $\mathbb{Z}/(q^{-j}\mathbb{Z})$ if $j < 0$. Clearly, we can take the generating set Φ to be

$$\Phi = \{\tilde{\psi}_{j,d} : j \in \mathbb{Z}, d \in D_j\}.$$

Recall from [7, 18] that the dual Gramian of the system $E(\Phi)$ at a point ξ is defined as an infinite matrix $\tilde{G}(\xi) = (\tilde{G}(\xi)_{k,l})_{k,l\in\mathbb{Z}}$ given as

$$\tilde{G}(\xi)_{k,l} = \sum_{\varphi\in\Phi} \hat{\varphi}(\xi+k)\overline{\hat{\varphi}(\xi+l)}.$$

Therefore,

$$
\begin{aligned}
\tilde{G}(\xi)_{k,l} &= \sum_{j\in\mathbb{Z}} \sum_{d\in D_j} \widehat{\tilde{\psi}_{j,d}}(\xi+k)\overline{\widehat{\tilde{\psi}_{j,d}}(\xi+l)} \\
&= \sum_{j\geq 0} \sum_{d\in D_j} p^{-j}\hat{\psi}(a^{-j}(\xi+k))\overline{\hat{\psi}(a^{-j}(\xi+l))}e^{-2\pi i q^{-j} d a^{-j}(k-l)} \\
&\quad + \sum_{j<0} \sum_{d\in D_j} q^{j}\hat{\psi}(a^{-j}(\xi+k))\overline{\hat{\psi}(a^{-j}(\xi+l))}e^{-2\pi i p^{j} d a^{-j}(k-l)} \\
&= \sum_{j\in\mathbb{Z}} \hat{\psi}(a^{-j}(\xi+k))\overline{\hat{\psi}(a^{-j}(\xi+l))} \times \begin{cases} \sum_{d\in D_j} p^{-j} e^{-2\pi i p^{-j} d(k-l)}, & j\geq 0 \\ \sum_{d\in D_j} q^{j} e^{-2\pi i q^{j} d(k-l)}, & j<0 \end{cases} \\
&= \sum_{j=m_-}^{m_+} \hat{\psi}(a^{-j}(\xi+k))\overline{\hat{\psi}(a^{-j}(\xi+l))},
\end{aligned}
$$

where

$$m_+ = \max\{j\in\mathbb{Z} : k-l\in p^j\mathbb{Z}\}, \quad m_- = \min\{j\in\mathbb{Z} : k-l\in q^{-j}\mathbb{Z}\}.$$

By a result of [18], see also [7, Theorem 2.5], $X^q(\Psi) = E(\Phi)$ is a tight frame with constant 1 for $\check{L}^2(E)$ if and only if the dual Gramian $\tilde{G}(\xi)$ is an orthogonal projection onto the range function $J(\xi)$ corresponding to $\check{L}^2(E)$ for a.e. ξ. Recall that $J(\xi)$ is given by

$$J(\xi) = \{v = (v(k))_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}) : v(k)\neq 0 \Rightarrow \xi+k\in E\}. \tag{2.4}$$

This in turn is equivalent to the fact that

$$\tilde{G}(\xi)_{k,l} = \begin{cases} 1, & \text{if } k=l \text{ and } \xi+k\in E \\ 0, & \text{otherwise} \end{cases} \qquad \text{for a.e. } \xi. \tag{2.5}$$

Now if $k = l \in \mathbb{Z}$, then $m_+ = \infty$, $m_- = -\infty$, and by the calculation above and (2.5)

$$\tilde{G}(\xi)_{k,k} = \sum_{j=-\infty}^{\infty} \hat{\psi}(a^{-j}(\xi+k))\overline{\hat{\psi}(a^{-j}(\xi+k))} = \mathbf{1}_E(\xi+k), \qquad \text{for a.e. } \xi,$$

which is just (2.2). If $k\neq l\in\mathbb{Z}$, then $l-k = p^{m_+}q^{-m_-}t$, where $t\in\mathbb{Z}$ is not divisible by p nor q. Hence by the calculation above and (2.5),

$$\begin{aligned}\tilde{G}(\xi)_{k,l} &= \sum_{j=m_-}^{m_+} \hat{\psi}(a^{-j}(\xi+k))\overline{\hat{\psi}(a^{-j}(\xi+l))} \\ &= \sum_{j=0}^{m_+-m_-} \hat{\psi}(a^{j-m_+}(\xi+k))\overline{\hat{\psi}(a^{j-m_+}(\xi+l))} \\ &= \sum_{j=0}^{m_+-m_-} \hat{\psi}(a^j a^{-m_+}(\xi+k))\overline{\hat{\psi}(a^j(a^{-m_+}(\xi+k)+a^{-m_+}(l-k)))} \\ &= \sum_{j=0}^{s} \hat{\psi}(a^j\tilde{\xi})\overline{\hat{\psi}(a^j(\tilde{\xi}+q^s t))} = 0,\end{aligned}$$

where $s = m_+ - m_-$, $\tilde{\xi} = a^{-m_+}(\xi + k)$. Hence given $s = 0, 1, \ldots$, and t not divisible by p or q, we can let $k = 0$ and $l = q^s t$ to obtain (2.3).

This shows that, if $X^q(\Psi)$ is a tight frame (with constant 1 for $\check{L}^2(E)$), then (2.2) and (2.3) hold, and hence $X(\Psi)$ is also a tight frame. Vice versa, if $X(\Psi)$ is a tight frame, then by (2.2), (2.3) and by the above calculation, the dual Gramian $\tilde{G}(\xi)$ of $X^q(\Psi)$ satisfies (2.5), i.e., $X^q(\Psi)$ is a tight frame. This completes the proof of Theorem 2.2. □

Theorem 2.2 can be generalized to the case of general (dual) frames, as is the case for integer dilations [11,19]. However, these results will not be needed here.

§3. The Wavelet Dimension Function for Real Dilation Factors

The goal of this section is to show the existence of the dimension function associated with arbitrary real dilation factors. Originally, the dimension function of wavelets, sometimes referred to as a multiplicity function, was studied only for integer dilation factors [2,4,8,20].

Suppose $W \subset L^2(\mathbb{R})$ is a shift invariant (SI) subspace of $L^2(\mathbb{R})$, i.e., W is a closed subspace of $L^2(\mathbb{R})$ such that $f \in W$ implies that $T_k f \in W$ for $k \in \mathbb{Z}$. The dimension function of W is a 1-periodic function $\dim_W : \mathbb{R} \to \mathbb{N} \cup \{0, \infty\}$ which measures the size of W over the fibers of $\mathbb{R}/\mathbb{Z}$. The precise definition in terms of a range function can be found in [3,7]. However, the reader who is not familiar with the dimension function of a general SI space can take Proposition 3.1 as a definition for the purposes of this work.

Proposition 3.1. *Suppose $\Phi \subset L^2(\mathbb{R})$. If the system $E(\Phi)$ is a tight frame with constant 1 for the space $W = S(\Phi)$, then*

$$\dim_W(\xi) := \dim \overline{\operatorname{span}}\{(\hat{\varphi}(\xi+k))_{k\in\mathbb{Z}} : \varphi \in \Phi\} = \sum_{\varphi\in\Phi}\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+k)|^2. \tag{3.1}$$

The above proposition is folklore, and its proof follows easily from [3,18], see also [7, Theorem 2.5].

Suppose next that $\Psi \subset L^2(\mathbb{R})$ is a multiwavelet associated with a real dilation factor a, $|a| > 1$. Let $(V_j)_{j\in\mathbb{Z}}$ be a generalized multiresolution analysis (GMRA), see [4], associated with the multiwavelet Ψ which is given by

$$V_j = \overline{\operatorname{span}}\{\psi^l_{i,k} : i < j, k \in \mathbb{Z},\ l = 1, \dots, L\}. \tag{3.2}$$

Clearly, $(V_j)_{j\in\mathbb{Z}}$ is a sequence of closed subspaces of $L^2(\mathbb{R})$ that satisfies all the usual properties of a GMRA, e.g.,

$$V_j \subset V_{j+1}, \quad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}), \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \tag{3.3}$$

except one. It is not evident at all that the core space V_0 must be SI.

In the case when a is an integer (or more generally A is an expansive dilation matrix with integer entries), it is well known that V_0 must be SI. In that case, the dimension function of V_0 coincides with the wavelet dimension function $D_\Psi(\xi)$ of a multiwavelet Ψ,

$$\dim_{V_0}(\xi) = D_\Psi(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R},$$

where

$$D_\Psi(\xi) := \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}^l(a^j(\xi + k))|^2. \tag{3.4}$$

A natural question is what happens for non-integer dilation factors a. Can we still expect V_0 to be SI? If yes, is the dimension function of V_0 given by $D_\Psi(\xi)$ as above? We will see that the answer to the first question is positive if either a is rational or Ψ is a combined MSF multiwavelet. In that case, the dimension function of V_0 is indeed given by (3.4). What is more interesting, these are the only cases when we can expect the space V_0 to be SI, see Theorem 3.4. Furthermore, in Section 5 we exhibit examples of wavelets for which V_0 is not SI and the wavelet dimension function $D_\Psi(\xi)$ given by (3.4) is not integer valued. Despite these setbacks, we can still define a meaningful wavelet dimension function which measures the dimensions of certain fibers of V_0 and is integer valued, see Definition 3.6.

Theorem 3.2. *Suppose $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R})$ is a multiwavelet associated with the real dilation factor a, $|a| > 1$. Then the space*

$$V = \overline{\operatorname{span}}\{\psi^l_{jj_0,k} : j < 0, k \in \mathbb{Z}, l = 1, \dots, L\} \tag{3.5}$$

is shift invariant. Furthermore, its dimension function is given by

$$\dim_V(\xi) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}^l(a^{jj_0}(\xi + k))|^2, \tag{3.6}$$

where

$$j_0 = j_0(a, \Psi) = \begin{cases} 1, & \textit{if } \Psi \textit{ is combined MSF,} \\ \inf\{j \geq 1 : a^j \in \mathbb{Q}\}, & \textit{otherwise.} \end{cases} \tag{3.7}$$

Proof: Note that $j_0 = j_0(a, \Psi)$ is well defined for any multiwavelet Ψ and any dilation factor a, $|a| > 1$. Indeed, if a is such that $a^j \notin \mathbb{Q}$ for all $j \geq 1$, then by the result of Chui and Shi [10] for $L = 1$ and [6] for general L, Ψ must be a combined MSF multiwavelet. Recall that a multiwavelet Ψ is said to be combined minimally supported frequency if $\bigcup_{l=1}^{L} \operatorname{supp} \hat{\psi}^l$ has minimal Lebesgue measure.

Suppose first that Ψ is combined MSF, i.e., the set $K = \bigcup_{l=1}^{L} \operatorname{supp} \hat{\psi}^l$ has minimal Lebesgue measure. Therefore, see [6],

$$\overline{\operatorname{span}}\{T_k \psi^l : k \in \mathbb{Z},\ l = 1, \ldots, L\} = \check{L}^2(K)$$

and

$$\sum_{l=1}^{L} |\hat{\psi}^l(\xi)|^2 = \mathbf{1}_K(\xi).$$

Since, $\{a^j K\}_{j \in \mathbb{Z}}$ is a partition of $\mathbb{R}$, we have $V = \check{L}^2(\bigcup_{j=-\infty}^{-1} a^j K)$ is shift invariant under all translations. Therefore, by Proposition 3.1,

$$\dim_V(\xi) = \sum_{k \in \mathbb{Z}} \sum_{j=-\infty}^{-1} \mathbf{1}_{a^j K}(\xi + k) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}^l(a^j(\xi + k))|^2.$$

Suppose next that Ψ is not combined MSF. Then j_0 given by (3.7) is the smallest integer $j \geq 1$ such that a^j is rational.

Consider first the case when $j_0 = 1$, i.e., $a = p/q \in \mathbb{Q}$, where $\gcd(p, q) = 1$. For simplicity assume that $\Psi = \{\psi\}$. To see that V given by (3.5) is SI it suffices to show that for any $n \in \mathbb{Z}$, $\psi_{j,k_1}(x - n) \perp \psi_{m,k_2}(x)$, where $j < 0$, $m \geq 0$, $k_1, k_2 \in \mathbb{Z}$. Indeed, pick any $w \in \mathbb{Z}$ such that $w \equiv 0 \mod q^m$ and $w \equiv n \mod p^{-j}$. Therefore, $a^j(w - n), a^m w \in \mathbb{Z}$, and by a simple change of variables $x = y + w$,

$$\begin{aligned} &\int_{\mathbb{R}} \psi_{j,k_1}(x - n)\overline{\psi_{m,k_2}(x)}dx \\ &\qquad = a^{(j+m)/2} \int_{\mathbb{R}} \psi(a^j(x - n) - k_1)\overline{\psi(a^m x - k_2)}dx \\ &\qquad = a^{(j+m)/2} \int_{\mathbb{R}} \psi(a^j y + a^j(w - n) - k_1)\overline{\psi(a^m y + a^m w - k_2)}dy \\ &\qquad = \langle \psi_{j,-a^j(w-n)+k_1}, \psi_{m,-a^m w+k_2} \rangle = 0. \end{aligned}$$

Consider next the case when $j_0 \geq 2$. Define a a^{j_0}-multiplicatively invariant set E by

$$E = \bigcup_{l=1}^{L} \bigcup_{j \in j_0 \mathbb{Z}} a^j (\operatorname{supp} \hat{\psi}^l). \tag{3.8}$$

By [5, Theorem 4.1], the sets $E, aE, \ldots, a^{j_0-1}E$ form a partition of $\mathbb{R}$ (modulo sets of measure zero). Therefore, Ψ is a multiwavelet associated with the rational dilation a^{j_0} for $\check{L}^2(E)$. Therefore, by the argument above, V given by (3.5) is SI. Likewise, the space

$$\check{L}^2(E) \ominus V = \overline{\operatorname{span}}\{\psi^l_{jj_0,k} : j \geq 0, k \in \mathbb{Z},\ l = 1, \ldots, L\}$$

is SI. Let $X^q(\Psi)$ denote the quasi-affine system generated by Ψ and associated with the rational dilation $\tilde{a} := a^{j_0} = p/q$. We claim that

$$V = \overline{\operatorname{span}}\{\tilde{\psi}^l_{j,k} : j < 0, k \in \mathbb{Z},\ l = 1, \ldots, L\}.$$

Indeed, the inclusion "$\subset$" is trivial. To see "$\supset$", take any $j < 0, k \in \mathbb{Z}$ and notice that

$$\tilde{\psi}^l_{j,k}(x) = \frac{p^j}{q^{j/2}} \psi^l(\tilde{a}^j x - p^j k) = \frac{p^j}{q^{j/2}} \psi^l(\tilde{a}^j (x - l_1) - l_2),$$

where $l_1, l_2 \in \mathbb{Z}$ are such that $k = q^{-j} l_1 + p^{-j} l_2$. Since V is SI, this shows $\tilde{\psi}^l_{j,k} \in V$. Analogously we can show that

$$\check{L}^2(E) \ominus V = \overline{\operatorname{span}}\{\tilde{\psi}^l_{j,k} : j \geq 0, k \in \mathbb{Z},\ l = 1, \ldots, L\}.$$

By Theorem 2.2, the quasi-affine system $X^q(\Psi)$ forms a tight frame with constant 1 for $\check{L}^2(E)$. Therefore, $\{\tilde{\psi}^l_{j,k} : j < 0, k \in \mathbb{Z},\ l = 1, \ldots, L\}$ forms a tight frame with constant 1 for V. To compute the dimension function of V, notice that

$$\{\tilde{\psi}^l_{j,k} : j < 0, k \in \mathbb{Z},\ l = 1, \ldots, L\} = E(\tilde{\Phi}),$$

where $\tilde{\Phi} = \{\tilde{\psi}^l_{j,d} : j < 0, d \in D_j,\ l = 1, \ldots, L\}$ and D_j are representatives of distinct cosets of $\mathbb{Z}/(q^{-j}\mathbb{Z})$. By Proposition 3.1,

$$\begin{aligned}
\dim_V(\xi) &= \sum_{l=1}^{L} \sum_{j<0} \sum_{k \in \mathbb{Z}} \sum_{d \in D_j} |\widehat{\tilde{\psi}^l_{j,d_j}}(\xi + k)|^2 \\
&= \sum_{l=1}^{L} \sum_{j<0} \sum_{k \in \mathbb{Z}} \sum_{d \in D_j} q^j |\hat{\psi}^l(\tilde{a}^{-j}(\xi + k))|^2 = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}^l(a^{jj_0}(\xi + k))|^2.
\end{aligned}$$

This shows (3.6) and completes the proof of Theorem 3.2. $\square$

Next we will show that Theorem 3.2 is the best possible result in the sense that the shift invariance of the core space V_0 given by (3.2) necessarily implies that either the dilation factor a is rational or Ψ is combined MSF. In addition, Theorem 5.4 will show that there indeed exist wavelets for which the core space V_0 is not SI.

Theorem 3.3. *Suppose Ψ is a multiwavelet associated with an irrational dilation factor a, $|a| > 1$. If the core space V_0 given by (3.2) is SI, then necessarily V_0 is SI under all real translations. In particular, Ψ is a combined MSF multiwavelet.*

Proof: It suffices to consider only a's such that $a^j \in \mathbb{Q}$ for some $j \geq 2$. Let $j_0 = j_0(a, \Psi)$ be the same as in (3.7). Clearly,

$$V_0 = V \oplus D_a V \oplus \ldots \oplus D_{a^{j_0-1}} V, \tag{3.9}$$

where V is given by (3.5) and $D_a f(x) = \sqrt{|a|} f(ax)$. Since $D_{a^j} V \subset \check{L}^2(a^j E)$, where E is given by (3.8), and $E, aE, \ldots, a^{j_0-1}E$ are pairwise disjoint

$$S(V_0) = S(V) \oplus S(D_a V) \oplus \ldots \oplus S(D_{a^{j_0-1}} V).$$

In particular, $S(D_a V) = D_a V$, i.e., $D_a V$ is SI. Since, $D_a V$ is also SI with respect to translates by the lattice $1/a\mathbb{Z}$ and $a \notin \mathbb{Q}$, $D_a V$ is SI with respect to all real translations. Therefore, V_0 is also SI with respect to all real translations, i.e., $V_0 = \check{L}^2(S)$ for some measurable $S \subset \mathbb{R}$. Therefore,

$$\overline{\operatorname{span}}\{\tilde{\psi}^l_{0,k} : k \in \mathbb{Z},\ l = 1, \ldots, L\} = V_1 \ominus V_0 = \check{L}^2(aS \setminus S),$$

and Ψ is combined MSF. $\square$

The proof of Theorem 3.3 shows that even though the core space V_0 in general is not SI, it can be decomposed as an orthogonal sum of a certain number of SI spaces with respect to different lattices of $\mathbb{R}$. Indeed, suppose that Ψ is a multiwavelet associated with the dilation a and $j_0 = j_0(a, \Psi) > 1$, where j_0 is the same as in (3.7). By (3.9),

$$V_0 = \bigoplus_{m=0}^{j_0-1} V^{(m)}, \qquad \text{where } V^{(m)} := D_{a^m} V, \tag{3.10}$$

and V is given by (3.6). Clearly, the space $V^{(m)}$ is SI with respect to the lattice $a^{-m}\mathbb{Z}$, and hence it is meaningful to talk about its dimension function. Hence, at least formally, we can talk about the dimension function of V_0 as the sum of the dimension functions of its components $V^{(m)}$, $m = 0, \ldots, j_0 - 1$, see (3.13). We will use the following variant of Proposition 3.1 which again can serve as a definition of the dimension function of a SI space W with respect to a general lattice $b\mathbb{Z}$.

Proposition 3.4. *Suppose $b > 0$ and $W \subset L^2(\mathbb{R})$ is a SI space with respect to the lattice $b\mathbb{Z}$, i.e., $f \in W$ implies that $T_{kb}f \in W$ for any $k \in \mathbb{Z}$. Then D_bW is SI (with respect to $\mathbb{Z}$), and the dimension function of W is a $1/b$-periodic function satisfying*

$$\dim_W^{b\mathbb{Z}}(\xi) := \dim \operatorname{span}\{(\hat{\varphi}(\xi + \frac{k}{b}))_{k\in\mathbb{Z}} : \varphi \in \Phi\} = \dim_{D_bW}(b\xi). \quad (3.11)$$

Finally, we are ready to define the wavelet dimension function for arbitrary real dilation factors.

Definition 3.5. *Suppose $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R})$ is a multiwavelet associated with the real dilation factor a, $|a| > 1$. Let $j_0 = j_0(a, \Psi)$ be given by (3.7). Define the wavelet dimension function of Ψ as*

$$D_\Psi(\xi) := \sum_{l=1}^{L}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} |\hat{\psi}^l(a^j(\xi + a^{j-\lfloor j/j_0\rfloor j_0}k))|^2. \quad (3.12)$$

The following result justifies the above definition and shows the connection of D_Ψ with the core space V_0 of a multiwavelet Ψ.

Theorem 3.6. *Suppose $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R})$ is a multiwavelet associated with the real dilation factor a, $|a| > 1$. Let V_0 be the core space of Ψ given by (3.2), and suppose V is defined by (3.6) and $j_0 = j_0(a, \Psi)$ by (3.7). Then (3.10) holds and*

$$\sum_{m=0}^{j_0-1} \dim_{V^{(m)}}^{a^{-m}\mathbb{Z}}(\xi) = D_\Psi(\xi) \qquad \textit{for a.e. } \xi \in \mathbb{R}, \quad (3.13)$$

where $V^{(m)} = D_{a^m}V$. In particular, $D_\Psi(\xi)$ given by (3.12) is almost everywhere integer valued.

Proof: By Theorem 3.2 and Proposition 3.4,

$$\begin{aligned}\dim_{V^{(m)}}^{a^{-m}\mathbb{Z}}(\xi) &= \dim_{D_{a^{-m}}V^{(m)}}(a^{-m}\xi)\\ &= \dim_V(a^{-m}\xi) = \sum_{l=1}^{L}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} |\hat{\psi}^l(a^{jj_0-m}(\xi + a^m k))|^2.\end{aligned}$$

Summing the above over $m = 0, \ldots, j_0 - 1$, we immediately obtain (3.13). $\square$

§4. Applications of the Wavelet Dimension Function

The wavelet dimension function is a very powerful tool which enables us to show many properties of multiwavelets which are well-localized both in time and frequency. For the purposes of this work we say that $\psi \in L^2(\mathbb{R})$ is well-localized in time and frequency if it satisfies the condition $(\Re^0)$ below, see [16, §7.6].

Definition 4.1. *We say that a function $\psi \in L^2(\mathbb{R})$ satisfies* condition $(\Re^0)$ *if there exist $c, \delta > 0$ such that*

$$\begin{aligned} &|\hat{\psi}(\xi)| \text{ is continuous on } \mathbb{R}, \\ &|\hat{\psi}(\xi)| \leq c|\xi|^{-1/2-\delta} \qquad \text{for all } \xi \in \mathbb{R}. \end{aligned} \tag{4.1}$$

We say that a collection $\Psi = \{\psi^1, \dots, \psi^L\}$ satisfies $(\Re^0)$ if each ψ^l satisfies $(\Re^0)$.

Daubechies in her book [13] has asked whether there are any well-localized wavelets with irrational dilations. This question was answered by the first author [5] by showing that well-localized multiwavelets can only exist for rational dilation factors. We will give an alternative proof of this result using the dimension function techniques, see Theorem 4.4. On the other hand, Auscher [1] has constructed Meyer-type wavelets for any rational dilation factor. Therefore, his construction is sharp in the sense that it cannot be extended to irrational dilation factors. However, it is not clear what is the minimal size L of a well-localized multiwavelet Ψ associated with rational non-integral dilation factor a. Auscher [2] has shown that if a is an integer then L has to be a multiple of $|a| - 1$. Theorem 4.3 extends this result to rational dilation factors.

Lemma 4.2. *Suppose $\psi \in L^2(\mathbb{R})$ satisfies $(\Re^0)$ and $|a| > 1$. Then $D_\psi(\xi)$ given by*

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}(a^j(\xi + k))|^2$$

is a 1-periodic continuous function on $\mathbb{R} \setminus \mathbb{Z}$.

Proof: Notice first that $s(\xi) = \sum_{j=1}^{\infty} |\hat{\psi}(a^j \xi)|^2$ is continuous everywhere except possibly at the origin and that $s(\xi) \leq c/(|a|-1)|\xi|^{-1-2\delta}$. Therefore, $\sum_{k \in \mathbb{Z}} s(\xi + k)$ is 1-periodic and continuous on $\mathbb{R} \setminus \mathbb{Z}$. □

Theorem 4.3. *Suppose $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R})$ is a multiwavelet associated with the rational dilation factor $a = p/q$, where $\gcd(p, q) = 1$. If Ψ is well-localized in time and frequency, then L is a multiple of $|p| - |q|$.*

Proof: By Lemma 4.2, the dimension function of Ψ given by

$$D_\Psi(\xi) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} |\hat{\psi}^l(a^j(\xi+k))|^2$$

is continuous on $\mathbb{R}\setminus\mathbb{Z}$. By Theorem 3.2, $D_\Psi(\xi)$ is integer valued for a.e. ξ, hence $D_\Psi(\xi)=M$ for some non-negative integer M. Since

$$\int_0^1 D_\Psi(\xi)d\xi = L/(|a|-1) = L|q|/(|p|-|q|) = M,$$

L has to be a multiple of $|p|-|q|$, because $\gcd(|q|,|p|-|q|)=1$. $\square$

Theorem 4.3 indicates that it would be extremely difficult to have well-localized wavelet bases for irrational dilations. Indeed, as we approach an irrational dilation a by a sequence of rational dilation factors $(p_n/q_n)_{n\in\mathbb{N}}$, we see that the size of well-localized multiwavelets associated with p_n/q_n has to be a multiple of $|p_n|-|q_n|$, and therefore it must grow to infinity as $n\to\infty$. This heuristic argument is made precise in the proof of Theorem 4.4. The following proof is a variant of [5, Theorem 4.1].

Theorem 4.4. *There are no multiwavelets Ψ for irrational dilation factors which are well-localized in the sense of the condition ($\Re^0$).*

Proof: Let $|a|>1$ be irrational. It suffices to consider only the case when $j_0 = j_0(a,\Psi)$ given by (3.7) is >1. Otherwise, Ψ is combined MSF and at least one of $\hat{\psi}^l$'s is not continuous, see [6].

Assume that Ψ satisfies ($\Re^0$). Let V be the SI space given by (3.5). By Theorem 3.2 and Lemma 4.2, $\dim_V(\xi)$ is 1-periodic and continuous on $\mathbb{R}\setminus\mathbb{Z}$. Since $\dim_V(\xi)$ is integer valued, hence $\dim_V(\xi)=M$ for some $M\in\mathbb{N}$. Let

$$s(\xi) = \sum_{l=1}^{L}\sum_{j=1}^{\infty} |\hat{\psi}(a^{jj_0}\xi)|^2, \quad t(\xi) = \sum_{l=1}^{L}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}\setminus\{0\}} |\hat{\psi}(a^{jj_0}(\xi+k))|^2.$$

By Lemma 4.2, $s(\xi)$ is continuous on $\mathbb{R}\setminus\{0\}$, $t(\xi)$ is continuous on $(\mathbb{R}\setminus\mathbb{Z})\cup\{0\}$, and $\dim_V(\xi) = s(\xi)+t(\xi) = M$. Let

$$E = \bigcup_{l=1}^{L}\sum_{j\in\mathbb{Z}} a^{jj_0}(\operatorname{supp}\hat{\psi}^l).$$

By the orthogonality argument, see [5, Lemma 2.2], the sets $E, aE, \ldots, a^{j_0-1}E$ are pairwise disjoint. Moreover, by [5, Corollary 3.3] the following Calderón formula holds:

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(a^{jj_0}\xi)|^2 = \mathbf{1}_E(\xi), \qquad \text{for a.e. } \xi\in\mathbb{R}. \tag{4.2}$$

Therefore, for a.e. $\xi \in E$, we have

$$\begin{aligned} 0 + t(0) &= \lim_{j \to -\infty} (s(a^{jj_0+1}\xi) + t(a^{jj_0+1}\xi)) = M \\ &= \lim_{j \to -\infty} (s(a^{jj_0}\xi) + t(a^{jj_0}\xi)) = 1 + t(0), \end{aligned}$$

since $a\xi \notin E$ by (4.2). This is a contradiction. $\square$

As an immediate corollary of Theorems 3.2, 4.3 and 4.4, we have that well-localized multiwavelets always come from an MRA possibly of higher multiplicity.

Corollary 4.5. *Suppose* $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R})$ *is a well-localized multiwavelet associated with a* real *dilation factor* a, $|a| > 1$. *Then the dilation factor* a *is rational and* $d = L/(|a| - 1)$ *is an integer. Moreover,* Ψ *is associated with an MRA* $(V_j)_{j \in \mathbb{Z}}$ *of multiplicity* d, *i.e., there exists* $\Phi = \{\phi^1, \ldots, \phi^d\} \subset V_0$, *such that* $E(\Phi)$ *is an orthonormal basis of the core space* V_0 *given by (3.2).*

In particular, we obtain the following simple corollary of Theorems 4.3 and 4.4.

Corollary 4.6. *If a wavelet* ψ *associated with a* real *dilation factor* a *is well-localized in time and frequency, then* $a = \pm(q+1)/q$ *for some integer* $q \geq 1$.

Another consequence of the wavelet dimension function technique is the converse to Corollary 4.5 which says that only for rational dilation factors, multiwavelets (not necessarily well-localized) can be associated with an MRA (possibly of higher multiplicity). This has been already observed (without proof) by Auscher [1]. In fact, Theorem 4.7 states that among all (single) wavelets with arbitrary *real* dilation factors, only dyadic (or negative dyadic) wavelets can be associated with an MRA.

Theorem 4.7. *Suppose a multiwavelet* $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R})$ *with a* real *dilation factor* a, $|a| > 1$, *is associated with an MRA* $(V_j)_{j \in \mathbb{Z}}$ *of multiplicity* d, *i.e., there exists* $\Phi = \{\phi^1, \ldots, \phi^d\} \subset V_0$, *such that* $E(\Phi)$ *is an orthonormal basis of the core space* V_0. *Then necessarily* $|a| = 1 + L/d$.

Proof: Since V_0 is SI, by Theorem 3.3, either Ψ is combined MSF or a is rational. In either case, by Theorem 3.2,

$$d = \int_0^1 \dim_{V_0}(\xi) d\xi = \int_0^1 D_\Psi(\xi) d\xi = L/(|a| - 1),$$

since $\dim_{V_0}(\xi) = d$ for a.e. $\xi \in \mathbb{R}$. Therefore, $L = (|a| - 1)d$. $\square$

Finally, we can address the question of the existence of well-localized wavelets for the Hardy space $H^2(\mathbb{R})$ which was originally posed by Meyer [17] in the case of dyadic wavelets. Auscher [2] has given a negative answer in the case of integer dilation factors. However, the non-integer case has not been addressed until recently. Using the same methods as in [5] one can show that there are no well-localized wavelets for $H^2(\mathbb{R})$ with irrational dilation factors. However, to show the same for rational dilations, it appears that one has to use the wavelet dimension function for rational dilation factors introduced in Section 3. The proof will then follow along the same lines as Auscher's original argument [2], see also [16, §7, Theorem 6.20].

Theorem 4.8. *There are no well-localized (in the sense of the condition* $(\Re^0)$*) multiwavelets* Ψ *for the Hardy space* $H^2(\mathbb{R})$ *and any real dilation factor.*

Proof: By [2,5] it remains to consider only the case of a rational dilation factor a. By Theorem 2.3 with $E = (0, \infty)$ and an $H^2(\mathbb{R})$ variant of Theorem 3.2, the space V_0 given by (3.2) is SI and its dimension function D_Ψ is given by (3.4).

Assume that Ψ satisfies $(\Re^0)$. By Lemma 4.2, $D_\Psi(\xi)$ is 1-periodic and continuous on $\mathbb{R} \setminus \mathbb{Z}$. Since $D_\Psi(\xi)$ is integer valued for a.e. ξ, hence $D_\Psi(\xi) = M$ for some $M \in \mathbb{N}$. Let

$$s(\xi) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} |\hat{\psi}(a^j \xi)|^2, \quad t(\xi) = \sum_{l=1}^{L} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z} \setminus \{0\}} |\hat{\psi}(a^j(\xi + k))|^2.$$

By Lemma 4.2, $s(\xi)$ is continuous on $\mathbb{R} \setminus \{0\}$, $t(\xi)$ is continuous on $(\mathbb{R} \setminus \mathbb{Z}) \cup \{0\}$, and $D_\Psi(\xi) = s(\xi) + t(\xi) = M$. By Theorem 1.3,

$$\sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^l(a^j \xi)|^2 = \mathbf{1}_{(0,\infty)}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}.$$

Therefore, for a.e. $\xi > 0$,

$$\begin{aligned} 0 + t(0) &= \lim_{j \to -\infty} (s(-a^j \xi) + t(-a^j \xi)) \\ &= M = \lim_{j \to -\infty} (s(a^j \xi) + t(a^j \xi)) = 1 + t(0), \end{aligned}$$

which is a contradiction. □

§5. Dilations Admitting non-MSF Wavelets

In this section we characterize dilations that admit non-MSF wavelets. Since the arguments employed in this section work beyond the one-dimensional setting we consider higher dimensional wavelets. We prove that in the case of $n \times n$ expansive matrices A, there exists an A-wavelet which is not an MSF wavelet if and only if there exists an integer $j \neq 0$ such that $A^{*j}(\mathbb{Z}^n) \cap \mathbb{Z}^n \neq \emptyset$. We begin by recalling several definitions and theorems that will be useful in the sequel.

We say that a set E tiles $\mathbb{R}^n$ by translations if $\{E + k : k \in \mathbb{Z}^n\}$ is a partition of $\mathbb{R}^n$ and by A^*-dilations if $\{A^{*j}(E) : j \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^n$. An MSF wavelet is a wavelet ψ such that the support of $\hat{\psi}$ has a minimal Lebesgue measure; in other words, $|\hat{\psi}|$ is the indicator function of a set. The support of an MSF wavelet (in the Fourier domain) is called an A-wavelet set, and is characterized by the fact that it tiles $\mathbb{R}^n$ by translations and A^*-dilations. That is,

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} \mathbf{1}_E(A^{*j}\xi) = 1, \qquad \text{for a.e. } \xi \in \mathbb{R}^n,\\
&\sum_{k\in\mathbb{Z}^n} \mathbf{1}_E(\xi + k) = 1, \qquad \text{for a.e. } \xi \in \mathbb{R}^n,
\end{aligned} \tag{5.1}
$$

where E is the support of $\hat{\psi}$. Note that this characterization does not depend on A being expansive or integer valued.

Assume now that A is expansive. Fix $C = [-1/2, 1/2]^n$, and suppose O is a bounded set which is bounded away from the origin and tiles $\mathbb{R}^n$ by A^*-dilations. We define the translation projection τ and the dilation projection d as

$$\tau(E) = \cup_{k\in\mathbb{Z}^n}((E + k) \cap C), \qquad d(E) = \cup_{j\in\mathbb{Z}}(A^{*j}(E) \cap O).$$

The following theorem gives a sufficient condition for sets to be contained in wavelet sets, which is a higher dimensional analogue of [22, Theorem 1.1]. The idea of the proof of Theorem 5.1 is to employ the iterative procedure as in [22], which can be intuitively described as "filling up arbitrary large pieces" of C while "filling up arbitrary small pieces" of O, and vice versa.

Theorem 5.1. *Let $E \subset \mathbb{R}^n$ and A be an expansive matrix. Then E is a subset of an A-wavelet set provided that:*

(i) $E \cap (E + k) = \emptyset$ *for all* $k \in \mathbb{Z}^n$,
(ii) $A^{*j}(E) \cap E = \emptyset$ *for all* $j \in \mathbb{Z}$,
(iii) there exists an $\epsilon_0 > 0$ *such that* $B_{\epsilon_0}(0) \cap \tau(E) = \emptyset$, *and*
(iv) $O \setminus d(E)$ *has a non-empty interior.*

Proof: We will make use of the following two elementary facts which are a consequence of A^* being an expansive matrix:

$$\forall \epsilon > 0 \ \ \exists M(\epsilon) \in \mathbb{Z}, \delta(\epsilon) > 0 : \quad A^{*M(\epsilon)}(O) \subset \big(B_\epsilon(0) \setminus B_{\delta(\epsilon)}(0)\big), \quad (5.2)$$

$$\forall S \subset O \text{ with non-empty interior } \exists k(S) \in \mathbb{Z}^n : \quad d(C + k(S)) \subset S. \quad (5.3)$$

By (iv), we can write $O \setminus d(E)$ as the disjoint union of sets $\{E_i\}_{i=1}^\infty$, each of which has non-empty interior. We define by induction a decreasing sequence $(\epsilon_i)_{i=1}^\infty$ of positive numbers, a sequence $(F_i)_{i=1}^\infty$ of subsets of C, and a sequence $(G_i)_{i=1}^\infty$ of subsets of $\mathbb{R}^n$.

We use (5.2) with $\epsilon = \epsilon_0$, where ϵ_0 is the same as in (iii), and (5.3) with $S = E_1$ to define $\epsilon_1 = \delta(\epsilon_0)$,

$$F_1 = \emptyset, \quad G_1 = \Big(C \setminus \big(\tau(E) \cup B_{\epsilon_1}(0)\big)\Big) + k(E_1).$$

Given $\epsilon_1, \ldots, \epsilon_i$, $F_1, \ldots, F_i$, $G_1, \ldots, G_i$, we use (5.2) with $\epsilon = \epsilon_i$ and (5.3) with $S = E_{i+1}$ to define $\epsilon_{i+1} = \delta(\epsilon_i)$,

$$F_{i+1} = A^{*M(\epsilon_i)}(E_i \setminus d(G_i)),$$
$$G_{i+1} = \Big(C \setminus \tau\Big(E \cup \bigcup_{j=1}^{i+1} F_j \cup \bigcup_{j=1}^{i} G_j \cup B_{\epsilon_{i+1}}(0)\Big)\Big) + k(E_{i+1}).$$

A simple induction argument shows that the dilation projections $d(E)$, $d(F_1)$, $d(G_1)$, $d(F_2)$, $d(G_2)$, $\ldots$ are mutually disjoint. Moreover, for any $i \in \mathbb{N}$, $\bigcup_{j=1}^{i+1} E_j = d\big(\big(\bigcup_{j=1}^{i+1} F_j\big) \cup \big(\bigcup_{j=1}^{i} G_j\big)\big)$. Likewise, the translation projections $\tau(E)$, $\tau(F_1)$, $\tau(G_1)$, $\tau(F_2)$, $\tau(G_2)$, $\ldots$ are mutually disjoint and $C \setminus B_{\epsilon_i}(0) = \tau\big(E \cup \big(\bigcup_{j=1}^{i} F_j\big) \cup \big(\bigcup_{j=1}^{i} G_j\big)\big)$. Therefore, $E \cup \big(\bigcup_{i=1}^{\infty} F_i\big) \cup \big(\bigcup_{i=1}^{\infty} G_i\big)$ is a wavelet set containing E by (5.1). $\square$

We will also use the following two elementary facts.

Lemma 5.2. *Let $\mathcal{F}$ be a countable collection of continuous functions. Suppose that $\mathcal{F}=\{f_m : m \in \mathbb{N}\}$. For each $m \in \mathbb{N}$, let Y_m be the fixed points for f_m, and let $Y = \bigcup_{m\in\mathbb{N}} Y_m$. If (a) $y \notin Y$ and (b) there is a $\gamma > 0$ such that $f_m(B_\gamma(y)) \cap B_\gamma(y) = \emptyset$ for all but finitely many m's, then there is an $\epsilon > 0$ such that $f_m(B_\epsilon(y)) \cap B_\epsilon(y) = \emptyset$ for all $m \in \mathbb{N}$.*

Proof: Assume the contrary. Then, for every integer $k \geq 1$ there is an $m_k \in \mathbb{N}$ such that $f_{m_k}(B_{1/k}(y)) \cap B_{1/k}(y) \neq \emptyset$. By condition (b), m_k is bounded, so by passing to a subsequence, we can assume $m_k = m$ for all k. But, this implies by continuity that $f_m(y) = y$, contradicting condition (a). $\square$

Lemma 5.3. *Suppose A is an expansive matrix, $k_0 \in \mathbb{Z}^n \setminus \{0\}$, and $p \in \mathbb{Z} \setminus \{0\}$. Let Y be the collection of fixed points of $\mathcal{F} \cup \mathcal{G}$, where*

$$\mathcal{F} = \{f_c(x) = A^{*-p}x - c : c \in \mathbb{Z}^n\}, \quad \mathcal{G} = \{g_j(x) = A^{*-j}x - k_0 : j \in \mathbb{Z}\}.$$

Then for any $y \notin \overline{Y}$, there exists $\epsilon > 0$ such that

$$f_c(B_\epsilon(y)) \cap B_\epsilon(y) = g_j(B_\epsilon(y)) \cap B_\epsilon(y) = \emptyset \qquad \text{for all } c \in \mathbb{Z}^n, j \in \mathbb{Z}. \tag{5.4}$$

Proof: Let T_0, T_1 be the collection of points in $\mathbb{R}^n$ that are fixed by some function in $\mathcal{F}, \mathcal{G}$, respectively. Clearly, $T_0 = (A^{*-p} - I)^{-1}\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$ and $T_1 = \{(A^{*-j} - I)^{-1}k_0 : j \in \mathbb{Z}\}$ is a set with accumulation points 0 and $-k_0$. Suppose $y \notin \overline{Y} = T_0 \cup T_1 \cup \{0, -k_0\}$. Let $\gamma = \min(||y||, ||y + k_0||)/2$. Clearly, $f_c(B_\gamma(y)) \cap B_\gamma(y) = \emptyset$ for all but finitely many $c \in \mathbb{Z}^n$. We claim that $g_j(B_\gamma(y)) \cap B_\gamma(y) = \emptyset$ for all but finitely many $j \in \mathbb{Z}$. Indeed, $g_j(B_\gamma(y)) \cap B_\gamma(y) = (A^{*-j}(B_\gamma(y)) - k_0) \cap B_\gamma(y) = \emptyset$ for sufficiently large $j > 0$. Likewise $B_\gamma(y) \cap g_j^{-1}(B_\gamma(y)) = B_\gamma(y) \cap A^{*j}(B_\gamma(y) + k_0) = \emptyset$ for sufficiently small $j < 0$. Therefore, by Lemma 5.2, there exists $\epsilon > 0$ such that (5.4) holds. □

We are now ready to present our main result.

Theorem 5.4. *Let A be an expansive $n \times n$ matrix. Then there is an A-wavelet ψ which is not MSF if and only if there exists a $p \in \mathbb{Z} \setminus \{0\}$ such that $A^{*p}(\mathbb{Z}^n) \cap \mathbb{Z}^n \neq \{0\}$.*

Proof: The forward direction was proven in [10] in the one-dimensional case and in [6] for the n-dimensional case. To see the reverse direction, let $k_0 \neq 0$ be an element of $A^{*p}(\mathbb{Z}^n) \cap \mathbb{Z}^n$, and denote $k_1 = A^{*-p}k_0$, which is also in $\mathbb{Z}^n$. We show that there exists a measurable set $I \subset \mathbb{R}^n$ such that the following four conditions hold:

$$\tau(I) \cap \tau(A^{*-p}(I)) = \emptyset, \tag{5.5}$$

$$d(I) \cap d(I + k_0) = \emptyset, \tag{5.6}$$

$$I \cup (A^{*-p}(I) + k_1) \text{ is contained in a wavelet set, and} \tag{5.7}$$

$$|I| > 0. \tag{5.8}$$

Indeed, let Y be the collection of fixed points of $\mathcal{F} \cup \mathcal{G}$ as in Lemma 5.3. Choose any $y \notin \overline{Y} \cup \mathbb{Z}^n \cup A^{*p}\mathbb{Z}^n$. By Lemma 5.3, there is an $\epsilon > 0$ such that (5.4) holds, i.e., (5.5) and (5.6) hold for the set $I = B_\epsilon(y)$. It is now clear that if we take $\epsilon' < \epsilon$ to be small enough and $I = B_{\epsilon'}(y)$, the hypotheses in Theorem 5.1 are satisfied for the set $I \cup (A^{*-p}(I) + k_1)$. Indeed, (i) and (ii) follow from (5.6) and (5.7), whereas (iii) and (iv) are a consequence of $y \notin \mathbb{Z}^n \cap A^{*p}\mathbb{Z}^n$. Hence, equations (5.5)–(5.8) above are satisfied.

Let W be a wavelet set containing $I \cup (A^{*-p}(I) + k_1)$, and define $T = W \setminus \big(I \cup (A^{*-p}(I) + k_1)\big)$. Now, we define ψ by

$$\hat{\psi}(\xi) = \begin{cases} 1/\sqrt{2} & \text{for } \ \xi \in I \cup A^{*-p}I \cup (I + k_0) \\ -1/\sqrt{2} & \text{for } \xi \in A^{*-p}I + k_1 \\ 1 & \text{for } \ \xi \in T \\ 0 & \text{otherwise.} \end{cases}$$

To see that $\|\hat{\psi}\| = 1$, note that for almost every ξ exactly one of the following conditions holds:

(a) $\xi + w \in T$ for precisely one $w \in \mathbb{Z}^n$,

(b) $\xi + w \in I$ and $\xi + w + k_0 \in I + k_0$ for precisely one $w \in \mathbb{Z}^n$,

(c) $\xi + w \in A^{*-p}I$ and $\xi + w + k_1 \in A^{*-p}I + k_1$ for precisely one $w \in \mathbb{Z}^n$.

Thus, for a.e. ξ, $\sum_{w \in \mathbb{Z}^n} |\hat{\psi}(\xi + w)|^2 = 1$ from which it follows that $\|\hat{\psi}\| = 1$.

We turn now to showing that (1.2) is satisfied by $\hat{\psi}$, i.e., ψ is an A-wavelet. For $\alpha = 0$, an argument similar to the above paragraph works and shows $\sum_{j \in \mathbb{Z}} |\hat{\psi}(A^{*j}\xi)|^2 = 1$ for a.e. ξ. Indeed, note that for almost every ξ, exactly one of the following conditions holds:

(a') $A^{*j}\xi \in T$ for precisely one $j \in \mathbb{Z}$,

(b') $A^{*j}\xi \in I$ and $A^{*j-p}\xi \in A^{*-p}I$ for precisely one $j \in \mathbb{Z}$,

(c') $A^{*j}\xi \in I + k_0$ and $A^{*j-p}\xi \in A^{*-p}I + k_1$ for precisely one $j \in \mathbb{Z}$.

For $0 \neq \alpha \in \Lambda$, where Λ represents the set of A-adic vectors and is given by (1.3), the only possibility that the sum in (1.2) would not be zero is when there exist $(j, m) \in \mathbb{Z} \times \mathbb{Z}^n$, $\alpha = A^{*-j}m$ such that both $A^{*j}\xi$ and $A^{*j}\xi + m$ are in the support of $\hat{\psi}$. In this case, it is clear by (5.5), (5.7) and the definition of ψ, that either $m = k_0$ or $m = k_1$. Now, in the first case we have that $A^{*j}\xi \in I$ and $A^{*j}\xi + k_0 \in I + k_0$ if and only if $A^{*j-p}\xi \in A^{*-p}I$ and $A^{*j-p}\xi + k_1 \in A^{*-p}I + k_1$, where $(j - p, k_1) \in \mathbb{Z} \times \mathbb{Z}^n$, $\alpha = A^{*-j+p}k_1$. Hence, the sum in (1.2) is $(1/\sqrt{2})(1/\sqrt{2}) + (1/\sqrt{2})(-1/\sqrt{2}) = 0$. The second case follows similarly showing (1.2). By Theorem 1.4, ψ is an A-wavelet. □

We note here that Theorem 5.4 does not provide a characterization of all matrices which yield non-MSF wavelets. Indeed, there is not yet even a full characterization of matrices which yield wavelets. Dai, Larson and the second author [12] showed that each expansive matrix admits an MSF wavelet, but that is not a necessary condition.

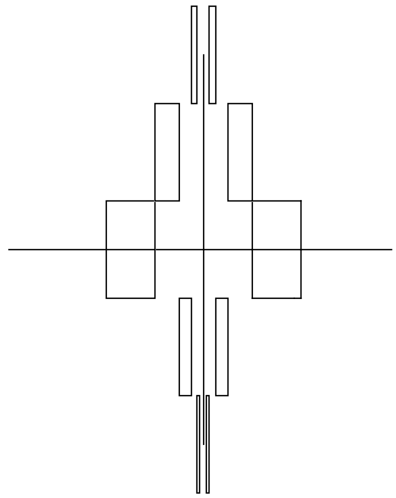

Fig. 1. Support of MSF wavelet for non-expansive dilation.

Example 5.5. *Let A be the (non-expansive) matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then there exists an A-wavelet.*

Proof: A function of the form $\hat{\psi} = \mathbf{1}_E$ is an A-wavelet if and only if (5.1) holds. In other words, we need that $\{A^{*j}(E) : j \in \mathbb{Z}\}$ and $\{E+k : k \in \mathbb{Z}\}$ form partitions of $\mathbb{R}^2$. Let $k : \{0, 1, 2, 3, \ldots\} \to \mathbb{Z}$ be a bijection, and let

$$E_j = \big([-1/2^j, -1/2^{j+1}] \cup [1/2^{j+1}, 1/2^j]\big) \times \big([-1/2, 1/2] + k(j)\big).$$

Then ψ given by $\hat{\psi} = \mathbf{1}_E$, where $E = \cup_{j \geq 0} E_j$, is a wavelet. $\square$

For example, taking $k(2m+1) = m+1$ and $k(2m) = -m$ yields Figure 1, which is easily verified as the support of an MSF wavelet.

Finally, we turn to providing an explicit example of the type mentioned in Section 3. More examples of this type for other dilation factors can be easily constructed with the help of Theorem 5.1.

Example 5.6. *A $\sqrt{2}$-wavelet exists for which $D_\Psi(\xi)$ given by (3.4) is not integer valued, and the core space V_0 given by (3.2) is not shift invariant.*

Proof: Let $I = [\frac{1}{3\sqrt{2}}, 1/3]$. Then, $I \cup (2I - 2)$ is contained in a wavelet set T by Theorem 5.1. So, as in the proof of Theorem 5.4,

$$\hat{\psi}(\xi) = \begin{cases} 1/\sqrt{2} & \text{for } \ \xi \in I \cup 2I \cup (I-1) \\ -1/\sqrt{2} & \text{for } \xi \in 2I - 2 \\ 1 & \text{for } \ \xi \in T \\ 0 & \text{otherwise} \end{cases}$$

is the Fourier transform of a $\sqrt{2}$-wavelet. By a straightforward calculation $D_\Psi(\xi) = \sum_{k\in\mathbb{Z}} \sum_{j\geq 1} |\hat{\psi}(2^{j/2}(\xi+k))|^2 \in \mathbb{Z} + 1/2$ for $\xi \in (\sqrt{2}I \cup \sqrt{2}(I-1)) + \mathbb{Z}$, and $D_\Psi(\xi) \in \mathbb{Z}$ otherwise. As in the proof of Theorem 3.3, one can show that the core space V_0 is not shift invariant, since $j_0(\sqrt{2},\psi)$ given by (3.7) is 2.

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