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Meyer Type Wavelet Bases in $\mathbb{R}^2$

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It is shown that for any expansive, integer valued 2×2 matrix, there exists a (multi-)wavelet whose Fourier transform is compactly supported and smooth. A key step is showing that for almost every equivalence class of integrally similar matrices there is a representative A which is strictly expansive in the sense that there is a compact set K which tiles the plane by integer translations and such that $K \subset A(K^\circ)$, where K° is the interior of K.

Key Words: orthonormal wavelet; expansive matrix; low-pass filter.

1. INTRODUCTION AND PRELIMINARIES

A common thread in the theory of wavelets has been to ask for which dilations A do there exist (multi-)wavelets $\Psi = \{\psi^1, \ldots, \psi^l\}$, usually with some additional properties. Gröchenig *et al.* [18, 19], and Lagarias and Wang [25–27] studied, for example, which dilations A yield Haar type wavelets. This subject turns out to be intimately connected with the theory of self-affine lattice tilings which has been studied, among others, by Bandt and Gelbrich [3], Kirat and Lau [24], Lagarias and Wang [28–30], and Zhou [37]. Strichartz [34] was able to show that for each dilation which admits a Haar type wavelet, there is also an r-regular wavelet, a result that was extended to all integer valued, expansive matrices in [8]. On the

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Fourier transform side, Dai and Larson [12] and Hernández *et al.* [21, 22] initiated the study of minimally supported frequency wavelets, which were also studied in [6, 13, 14, 17, 33]. It is known that all expansive, integer valued matrices admit minimally supported frequency wavelets. Gu and Han [20] proved that all determinant two integer valued expansive matrices admit MRA wavelets, a result that was extended to arbitrary expansive, integer valued matrices in [2, 7]. More recently, the question of extending Daubechies [15, 16] construction to higher dimensions has been considered by Ayache [1] and independently by Belogay and Wang [5]. Calogero [10, 11] has studied the construction of Meyer type wavelets for the quinconx matrix in $\mathbb{R}^2$. The purpose of this paper is to solve the general existence problem for Meyer type wavelets in two dimensions. That is, we will show that for all expansive, integer valued 2×2 matrices, there exists a (multi-)wavelet Ψ such that for each i, $\hat{\psi}^i$ is smooth and compactly supported.

A matrix is said to be *expansive* if all of its eigenvalues have modulus bigger than one. Such a matrix is often referred to as a *dilation*. We restrict our attention to dilations A which preserve a lattice $\Gamma = P\mathbb{Z}^n$, i.e., $A\Gamma \subset \Gamma$, where P is some $n \times n$ non-degenerate matrix. By standard considerations, see [24], we will assume that $\Gamma = \mathbb{Z}^n$ and hence A has integer entries. We say that the matrices A and B are *integrally similar* if there is an integer matrix C of determinant ± 1 such that $A = CBC^{-1}$.

Given an expansive matrix A, a (multi-)wavelet (with respect to A) is a collection of square integrable functions $\Psi = \{\psi^1, \ldots, \psi^l\}$ such that $\{\psi^i_{j,k} : i = 1, \ldots, l, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Here, for $\psi \in L^2(\mathbb{R}^n)$ we let

$$\psi_{j,k}(x) = D_{A^j} T_k \psi(x) = |\det A|^{j/2}\, \psi(A^j x - k), \qquad j \in \mathbb{Z}, \qquad k \in \mathbb{Z}^n,$$

where $T_y f(x) = f(x - y)$ is a translation operator by the vector $y \in \mathbb{R}^n$, and $D_A f(x) = \sqrt{|\det A|}\, f(Ax)$ is a dilation by the matrix A.

To fix notation, the Fourier transform we will use in this paper is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \langle x, \xi \rangle}\, dx.$$

Given a subset $X \subset \mathbb{R}^n$, $\operatorname{conv} X$ denotes the convex hull of X, $\operatorname{sym\,conv} X = \operatorname{conv}(-X \cup X)$, and X° is the interior of X.

A *minimally supported frequency (MSF)* wavelet is a wavelet ψ such that $|\hat{\psi}| = \mathbf{1}_W$, for some measurable set W. For the purposes of this paper, we will say that the matrix A admits a Meyer type wavelet if there is a (multi-) wavelet Ψ such that each $\hat{\psi}^i$ is smooth and compactly supported.

It is easy to see that whenever $A=CBC^{-1}$ with C an integer matrix of determinant ± 1 and Ψ is a (multi-)wavelet with respect to B, then $\{\psi^1(Cx), ..., \psi^l(Cx)\}$ is a (multi-)wavelet with respect to A. Moreover, if A and B are integrally similar, then A admits a Meyer type wavelet if and only if B admits a Meyer type wavelet.

Finally, for any measurable set $W \subset \mathbb{R}^n$ that satisfies

$$\sum_{j\in\mathbb{Z}} \mathbf{1}_W(A^j\xi)=1 \qquad \text{for a.e.} \quad \xi\in\mathbb{R}^n \tag{1.1}$$

(in particular, for the support of any MSF wavelet associated to A^T), we define the *dilation projection* $d_A : \mathscr{P}(\mathbb{R}^n) \to \mathscr{P}(W)$ by

$$d_A(S)=\bigcup_{j=-\infty}^{\infty} (A^j(S)\cap W) \qquad \text{for} \quad S\in\mathscr{P}(\mathbb{R}^n), \tag{1.2}$$

where $\mathscr{P}(W)$ is the power set of W.

The remainder of the terminology in this paper is standard, as can be found in [23, 31, 36].

2. THE DETERMINANT TWO CASE

In this section, we prove that every expansive, 2×2 matrix of determinant ± 2 admits a Meyer type wavelet. Most of the work for this has been done by previous authors, so this section will consist of theorem quoting and one example.

The following follows from a theorem due to Latimer and MacDuffee (see [32]), or an elementary argument as presented in [24].

PROPOSITION 2.1. *Let A be an expansive matrix with determinant ± 2. Let $C_0=\left(\begin{smallmatrix}0&1\\-2&0\end{smallmatrix}\right)$, $C_1=\left(\begin{smallmatrix}0&1\\-2&1\end{smallmatrix}\right)$, $C_2=\left(\begin{smallmatrix}0&1\\-2&2\end{smallmatrix}\right)$, and $D=\left(\begin{smallmatrix}0&1\\2&0\end{smallmatrix}\right)$. Then A is integrally similar to one of the following six matrices: D, C_0, $\pm C_1$, $\pm C_2$.*

From Proposition 2.1, it follows that in order to show that all integral matrices of determinant ± 2 admit Meyer type wavelets, it suffices to show that Meyer type wavelets exist for the six matrices listed in Proposition 2.1. The matrices D and C_0 follow from taking tensor products of one dimensional wavelets as explained, for example, in [36]. The existence of Meyer type wavelets for the quinconx matrix $\left(\begin{smallmatrix}1&-1\\1&1\end{smallmatrix}\right)$ was shown by Calogero in [10, 11]. Since the quinconx matrix is integrally similar to C_2 and the same argument works for $-C_2$, it suffices to prove that Meyer type wavelets exist for $\pm C_1$. Since our construction is going to be symmetric with respect to the origin, we will focus solely on C_1 (actually, we will focus on the matrix $\begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$, which is integrally similar to C_1). We begin with an easy

PROPOSITION 2.2. *Let A be an $n\times n$ expansive matrix and $W\subset\mathbb{R}^n\setminus\{0\}$ be a compact set that is bounded away from the origin and satisfies* (1.1). *Let m_0 be a function with $L=\operatorname{supp}(m_0)=\{\xi\in\mathbb{R}^n : m_0(\xi)\neq 0\}$ and $K=\mathbb{R}^n\setminus L$. If there exist $\epsilon, N>0$ such that $\mathbf{B}(0,\epsilon)\subset L$ and $d_A(\mathbf{B}(0,N)\cap K)=W$, then $\prod_{j=1}^{\infty} m_0(A^{-j}\xi)$ converges and is compactly supported. Moreover, if m_0 is smooth and $m_0(0)=1$ then $\prod_{j=1}^{\infty} m_0(A^{-j}\xi)$ converges to a smooth, compactly supported function.*

Proof. Note that since W and $K\cap\mathbf{B}(0,N)$ are bounded and bounded away from the origin, there is an $R\in\mathbb{N}$ such that whenever $|j|>R$, $A^j(K\cap\mathbf{B}(0,N))\cap W=\emptyset$. For $|j|\leqslant R$ let $E_j=K\cap\mathbf{B}(0,N)\cap A^j(W)$. Consider the set $E:=A^R(\bigcup_{j=-R}^{R} A^{-j}E_j)$ and note that $d_A(E)=A^{-R}(E)=W$. Moreover, if $j>R$ and $\xi\in A^j(W)$, then $\xi\in A^j(A^{-R}(E))=\bigcup_{k=-R}^{R} A^{j-k}(E_k)$. Therefore, $\xi\in A^l(K)$ for some $l>0$ and $\prod_{j=1}^{\infty} m_0(A^{-j}\xi)=0$. Since W is compact and bounded away from zero and A is expansive, $\bigcup_{j\in\mathbb{Z}} A^j(W)\cup\{0\}$ is closed. By (1.1) this implies that $\bigcup_{j\in\mathbb{Z}} A^j(W)=\mathbb{R}^n\setminus\{0\}$ and thus $\operatorname{supp}(\prod_{j=1}^{\infty} m_0(A^{-j}\xi))\subset\bigcup_{k=-\infty}^{R} A^k(W)\cup\{0\}$. Since A is expansive, $\bigcup_{k=-\infty}^{R} A^k(W)$ is bounded.

Now, assume that m_0 is smooth and $m_0(0)=1$. Since $m(\xi)=1+O(\xi)$ as $\xi\to 0$ the product $\prod_{j=1}^{\infty} m_0(A^{-j}\xi)$ converges pointwise. Furthermore, by standard considerations involving the infinite product rule, see the proof of [7, Theorem 3], this product defines a smooth function.

EXAMPLE 2.3. There exists a Meyer type wavelet for the matrix $A=\begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$.

Proof. We note here that the set

$$W=\operatorname{conv}\{(0,\tfrac{1}{2}),(-1,\tfrac{1}{2}),(-1,-\tfrac{1}{2})\}\cup\operatorname{conv}\{(0,-\tfrac{1}{2}),(1,\tfrac{1}{2}),(1,-\tfrac{1}{2})\}\tag{2.1}$$

is the support of an MSF wavelet with low pass filter the $\mathbb{Z}^2$ periodization of $\mathbf{1}_T$, where $T=\operatorname{sym}\ \operatorname{conv}\{(0,\frac{1}{2}),(\frac{1}{2},0)\}$. The scaling set for W is $\operatorname{sym}\ \operatorname{conv}\{(-1,-\frac{1}{2}),(0,-\frac{1}{2})\}$.

We will now smooth the low pass filter associated with W. Let $\mathscr{D}=\{(0,0),(1,0)\}$ be the set of representatives of different cosets of $\mathbb{Z}^2/A\mathbb{Z}^2$. Hence the Smith–Barnwell equation for A is $|m_0(\xi)|^2+|m_0(\xi+(\frac{1}{2},\frac{1}{2}))|^2=1$. We smooth the low pass filter in such a way that the support of the new filter is the interior of the $\mathbb{Z}^2$ periodization of $\operatorname{sym}\ \operatorname{conv}\{(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{8}),(\frac{1}{2},-\frac{1}{8})\})$.

Let $g\colon\mathbb{R}\to[0,\infty)$ be a C^∞ function such that

$$\operatorname{supp} g:=\{\eta\in\mathbb{R} : g(\eta)\neq 0\}=(-\infty,1/4).$$

Let $v_1 = (3/8, 1/2)$, $v_2 = (3/8, -1/2)$, $v_3 = (-3/8, 1/2)$, $v_4 = (-3/8, -1/2)$. Define a C^∞ function $h: \mathbb{R}^2 \to [0, \infty)$ by

$$h(\xi) = \prod_{j=1}^{4} g(\langle \xi, v_j \rangle).$$

Clearly

$$\operatorname{supp} h = (\operatorname{conv}\{(\pm 2/3, 0), (0, \pm 1/2)\})^\circ.$$

Let f be $\mathbb{Z}^2$ periodization of h, i.e., $f(\xi) = \sum_{k \in \mathbb{Z}^2} f(\xi + k)$. Finally define the function m_0 by

$$m_0(\xi) = \sqrt{f(\xi)/(f(\xi) + f(\xi + (1/2, 1/2)))}\,.$$

Since the denominator is always positive and f "vanishes strongly" (see the proof of Claim 3.3), m_0 is C^∞, $\mathbb{Z}^2$ periodic function satisfying $|m_0(\xi)|^2 + |m_0(\xi + (1/2, 1/2))|^2 = 1$ for all $\xi \in \mathbb{R}^2$, and $m_0(0) = 1$.

Now, we show that Cohen's condition is satisfied for this set. The natural first guess for K is the scaling set $K_1 = \operatorname{sym} \operatorname{conv}\{(-1, -\frac{1}{2}), (0, -\frac{1}{2})\}$. This set almost works; the points $\pm(1, \frac{1}{2})$ get mapped under A^{-1} to the points $\pm(0, \frac{1}{2})$, which are not in the support of m_0. However, letting B_1 be a small neighborhood of $(1, \frac{1}{2})$ intersected with K_1, the set $K = K_1 \setminus (B_1 \cup -B_1) \cup (\overline{B_1} + (-1, 0)) \cup (-\overline{B_1} + (1, 0))$ satisfies Cohen's condition. Therefore $\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(A^{-j}\xi)$ is the scaling function for the multiresolution analysis $(V_j)_{j \in \mathbb{Z}}$ associated to the dilation A^T defined by

$$V_j = \overline{\operatorname{span}}\{D_{(A^T)^j} T_l \varphi : l \in \mathbb{Z}^n\} \qquad \text{for} \quad j \in \mathbb{Z}.$$

Finally, it suffices to show that $\hat{\varphi}$ is compactly supported by verifying the hypotheses of Proposition 2.2 with W given by (2.1). A direct computation shows that the images of the zero set in Fig. 1 under various powers A^j line up as pictured in Fig. 2.

Clearly, all of W is covered by the union of these sets. Furthermore, ψ given by

$$\hat{\psi}(\xi) = m_0(A^{-1}\xi + (1/2, 1/2))\, e^{\pi i(\xi_1 + \xi_2)} \hat{\varphi}(A^{-1}\xi) \qquad \text{for} \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

is a Meyer type wavelet associated to the dilation A^T. By Proposition 2.1, A is integrally similar to A^T and thus there exists a Meyer type wavelet associated to A.

We have thus proven.

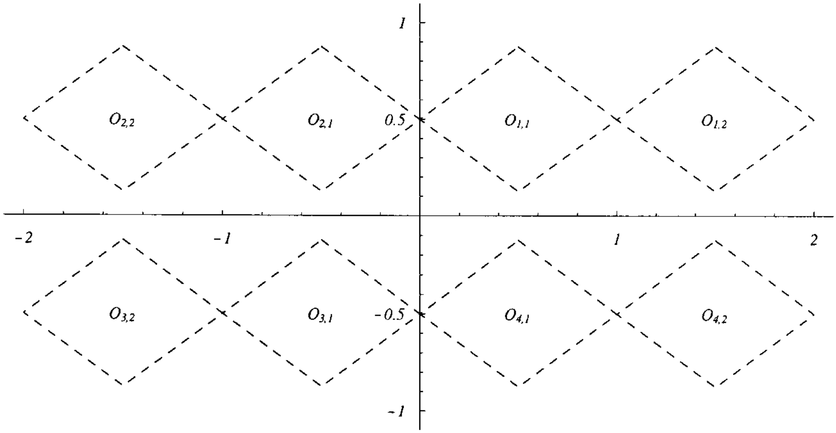

FIG. 1. Zero set of m_0.

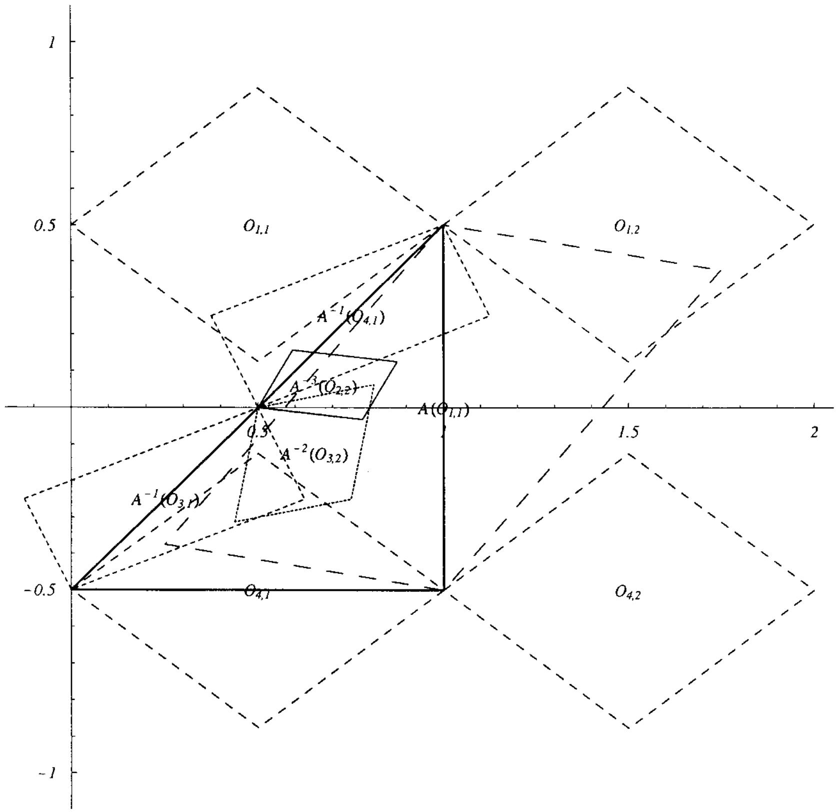

FIG. 2. Verification of Proposition 2.2.

THEOREM 2.4. *Let A be a 2×2 expansive integer matrix of determinant ± 2. Then, A admits a Meyer type wavelet.*

3. GENERAL FACTS

The main goal of this section is to give a simple sufficient condition on a dilation A which guarantees the existence of Meyer type wavelets associated with A. This condition says that A is strictly expansive; i.e., there exists a lattice tiling of $\mathbb{R}^n$ such that the dilate of a tile contains some neighborhood of this tile. Even though we do not require that our tile is self-affine, the existence of strictly expansive tillings appears to be connected with the existence of self-affine tilings [3, 24–30, 37]. Since the strict expansiveness condition is meaningful regardless of the dimension, we will work on $\mathbb{R}^n$. An application of this condition to $\mathbb{R}^2$ is given by Corollary 3.6.

DEFINITION 3.1. We say that an $n \times n$ integral dilation B is *strictly expansive* if there exists a compact set $K \subset \mathbb{R}^n$ such that

$$\sum_{k \in \mathbb{Z}^n} \mathbf{1}_K(\xi + k) = 1 \qquad \text{for a.e.} \quad \xi \in \mathbb{R}^n, \tag{3.1}$$

$$K \subset BK^\circ, \qquad \text{where } K^\circ \text{ is the interior of } K. \tag{3.2}$$

When (3.1) and (3.2) hold, we say that B is strictly expansive with respect to K.

We shall prove the following existence theorem.

THEOREM 3.2. *Suppose A is a $n \times n$ integral dilation matrix. If $B = A^T$ is strictly expansive then there exists a multiresolution analysis with a scaling function and an associated wavelet family of* $(|\det A| - 1)$ *functions in the Schwartz class.*

Proof. Suppose the compact set K satisfies (3.1) and (3.2). Given $\varepsilon > 0$ we define

$$K^{-\varepsilon} = \{\xi \in \mathbb{R}^n : \mathbf{B}(\xi, \varepsilon) \subset K\}$$
$$K^{+\varepsilon} = \{\xi \in \mathbb{R}^n : \mathbf{B}(\xi, \varepsilon) \cap K \neq \varnothing\}.$$

Note that $K^{-\varepsilon}$ is closed, $K^{+\varepsilon}$ is open, and the interior of K satisfies $K^\circ = \bigcup_{\varepsilon > 0} K^{-\varepsilon}$. Hence there exists $\varepsilon > 0$ such that

$$K^{+\varepsilon} \subset B(K^{-\varepsilon}). \tag{3.3}$$

Pick a function $g\colon \mathbb{R}^n \to [0, \infty)$ in the class C^∞ such that $\int_{\mathbb{R}^n} g = 1$ and

$$\operatorname{supp} g := \{\xi \in \mathbb{R}^n : g(\xi) \neq 0\} = \mathbf{B}(0, \varepsilon).$$

Define a function f by

$$f(\xi) = (\mathbf{1}_K * g)(\xi).$$

Clearly f is in the class C^∞, $0 \leqslant f(\xi) \leqslant 1$, and

$$\operatorname{supp} f = \{\xi \in \mathbb{R}^n : f(\xi) \neq 0\} \subset K^{+\varepsilon}, \tag{3.4}$$

$$\{\xi \in \mathbb{R}^n : f(\xi) = 1\} = K^{-\varepsilon}. \tag{3.5}$$

Moreover, by (3.1)

$$\sum_{k \in \mathbb{Z}^n} f(\xi + k) = \sum_{k \in \mathbb{Z}^n} (\mathbf{1}_K * g)(\xi + k) = 1 \qquad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{3.6}$$

Finally define a function $m\colon \mathbb{R}^n \to [0, 1]$ by

$$m(\xi) = \sqrt{\sum_{k \in \mathbb{Z}^n} f(B(\xi + k))}\,. \tag{3.7}$$

CLAIM 3.3. *The function m given by* (3.7) *is C^∞, $\mathbb{Z}^n$-periodic, and*

$$\sum_{d \in \mathscr{D}} |m(\xi + B^{-1}d)|^2 = 1 \qquad \textit{for all} \quad \xi \in \mathbb{R}^n, \tag{3.8}$$

$$m(\xi) > 0 \Rightarrow \xi \in \mathbb{Z}^n + B^{-1}(K^{+\varepsilon}), \tag{3.9}$$

$$m(\xi) = 0 \qquad \textit{for} \quad \xi \in (B^{-1}\mathbb{Z}^n \setminus \mathbb{Z}^n) + B^{-1}(K^{-\varepsilon}), \tag{3.10}$$

where $B = A^T$, and $\mathscr{D} = \{d_1, \ldots, d_b\}$ is the set of representatives of different cosets of $\mathbb{Z}^n / B\mathbb{Z}^n$, where $b = |\det A|$.

Proof of Claim 3.3. To guarantee that m is C^∞, the function f must "vanish strongly," i.e., if $f(\xi_0) = 0$ for some ξ_0 then $\partial^\alpha f(\xi_0) = 0$ for any multi-index α. It is clear that if nonnegative function f in C^∞ "vanishes strongly" then $\sqrt{f}$ is also C^∞.

The condition (3.8) is a consequence of

$$\begin{aligned} \sum_{d \in \mathscr{D}} |m(\xi + B^{-1}d)|^2 &= \sum_{k \in \mathbb{Z}^n} \sum_{d \in \mathscr{D}} f(B(\xi + B^{-1}d + k)) \\ &= \sum_{k \in \mathbb{Z}^n} \sum_{d \in \mathscr{D}} f(B\xi + d + Bk) = 1, \end{aligned}$$

by (3.6).

To see (3.9), take ξ such that $m(\xi)>0$. By (3.4) and (3.7), $B(\xi+k)\in K^{+\varepsilon}$ for some $k\in\mathbb{Z}^n$, and hence (3.9) holds.

We claim that (3.10) follows from (3.8) and

$$m(\xi)=1 \qquad \text{for} \quad \xi\in\mathbb{Z}^n+B^{-1}(K^{-\varepsilon}). \tag{3.11}$$

Indeed, if $\xi\in B^{-1}d+k+B^{-1}(K^{-\varepsilon})$ for some $d\in\mathcal{D}\setminus B\mathbb{Z}^n$ and $k\in\mathbb{Z}^n$, then by (3.11) we have $m(\xi-B^{-1}d)=1$. Hence by (3.8) $m(\xi)=0$ and (3.10) holds. Finally, (3.11) is the immediate consequence of (3.5) and (3.7). This ends the proof of the claim.

We can write m in the Fourier expansion as

$$m(\xi)=\frac{1}{\sqrt{|\det A|}}\sum_{k\in\mathbb{Z}^n} h_k e^{-2\pi i\langle k,\,\xi\rangle}, \tag{3.12}$$

where we include the factor $|\det A|^{-1/2}$ outside the summation as in [7]. Since m is C^∞, the coefficients h_k decay polynomially at infinity, that is, for all $N>0$ there is $C_N>0$ so that

$$|h_k|\leqslant C_N\,|k|^{-N} \qquad \text{for} \quad k\in\mathbb{Z}^n\setminus\{0\}.$$

Since m satisfies (3.8) and $m(0)=1$, m is a low-pass filter which is regular in the sense of the definition following [7, Theorem 1]. By [7, Theorem 5] $\varphi\in L^2(\mathbb{R}^n)$ defined by

$$\hat{\varphi}(\xi)=\prod_{j=1}^{\infty} m(B^{-j}\xi), \tag{3.13}$$

has orthogonal translates, i.e.,

$$\langle\varphi, T_l\varphi\rangle=\delta_{l,0} \qquad \text{for} \quad l\in\mathbb{Z}^n,$$

if and only if m satisfies the Cohen condition, that is there exists a compact set $\tilde{K}\subset\mathbb{R}^n$ such that

- $\tilde{K}$ contains a neighborhood of zero,
- $\bigcup_{l\in\mathbb{Z}^n}(l+\tilde{K})=\mathbb{R}^n$,
- $m(B^{-j}\xi)\neq 0$ for $\xi\in\tilde{K}$, $j\geqslant 1$.

The first guess for $\tilde{K}$ to be K is in general incorrect, e.g., if K has isolated points. Instead we claim that there is $0<\delta<1$ so that

$$\tilde{K}=\{\xi\in K:|\mathbf{B}(\xi,\varepsilon)\cap K|\geqslant\delta\,|\mathbf{B}(\xi,\varepsilon)|\} \tag{3.14}$$

does the job. Clearly, if $\xi \in \tilde{K}$ then $f(\xi) \neq 0$ and thus $m(B^{-1}\xi) \neq 0$. By (3.3), $B^{-1}\tilde{K} \subset B^{-1}K^{+\varepsilon} \subset K^{-\varepsilon} \subset \tilde{K}$ and thus $m(B^{-j}\xi) \neq 0$ for all $j \geqslant 1$. Since $0 \in K^{\circ}$ by (3.2) thus $0 \in (K^{-\varepsilon})^{\circ}$ by (3.3) and hence $0 \in \tilde{K}^{\circ}$. Finally, it suffices to check that

$$\sum_{k \in \mathbb{Z}^n} \mathbf{1}_{\tilde{K}}(\xi + k) \geqslant 1 \qquad \text{for all} \quad \xi \in \mathbb{R}^n. \tag{3.15}$$

By the compactness of K there is a finite index set $I \subset \mathbb{Z}^n$ such that

$$\sum_{k \in I} \mathbf{1}_{K}(\xi + k) \geqslant 1 \qquad \text{for all} \quad \xi \in [-1, 1]^n. \tag{3.16}$$

Take any $\xi \in [-1/2, 1/2]^n$ and integrate (3.16) over $\mathbf{B}(\xi, \varepsilon)$ to obtain

$$\sum_{k \in I} |\mathbf{B}(\xi + k, \varepsilon) \cap K| \geqslant |\mathbf{B}(\xi, \varepsilon)|.$$

Therefore, if we take $\delta = 1/\#I$ then there is $k \in I$ such that $|\mathbf{B}(\xi + k, \varepsilon) \cap K| \geqslant \delta\, |\mathbf{B}(\xi, \varepsilon)|$ and hence $\xi + k \in \tilde{K}$. Thus (3.15) holds and $\tilde{K}$ given by (3.14) satisfies the Cohen condition. Therefore φ is a scaling function for the multiresolution analysis $(V_j)_{j \in \mathbb{Z}}$ defined by

$$V_j = \overline{\operatorname{span}}\{D_{A^j} T_l \varphi : l \in \mathbb{Z}^n\} \qquad \text{for} \quad j \in \mathbb{Z}.$$

It remains to show that φ is in the Schwartz class. We are going to prove that φ is band-limited, i.e., $\hat{\varphi}$ is compactly supported. By (3.9) and (3.13)

$$\hat{\varphi}(\xi) \neq 0 \;\Rightarrow \xi \in B\mathbb{Z}^n + K^{+\varepsilon}. \tag{3.17}$$

On the other hand, by (3.10), $m(B^{-j}\xi) = 0$ for $\xi \in B^{j-1}\mathbb{Z}^n \setminus B^j\mathbb{Z}^n + B^{j-1}(K^{-\varepsilon})$. Since

$$\bigcup_{j=2}^{\infty} (B^{j-1}\mathbb{Z}^n \setminus B^j\mathbb{Z}^n) = B\mathbb{Z}^n \setminus \{0\},$$

and

$$K^{+\varepsilon} \subset B(K^{-\varepsilon}) \subset B^{j-1}(K^{-\varepsilon}) \qquad \text{for} \quad j \geqslant 2,$$

we have

(3.18) $$\hat{\varphi}(\xi)=0 \qquad \text{for} \quad \xi \in B\mathbb{Z}^n \setminus \{0\} + K^{+\varepsilon}.$$

Combining (3.17) and (3.18) we have $\hat{\varphi}(\xi)=0$ for $\xi \in (K^{+\varepsilon})^c$. Therefore $\operatorname{supp} \hat{\varphi} \subset K^{+\varepsilon}$ and therefore φ is in the Schwartz class. To conclude the proof of Theorem 3.2 it suffices to use Proposition 3.4 due to Wojtaszczyk, see [36, Corollary 5.17] which also holds for $r=\infty$.

Remark. It is widely known (see [35]) that for every dilation B there is an ellipsoid Δ and an $s>1$ such that $\Delta \subset s\Delta \subset B\Delta$. If one uses this fact to construct wavelets, the wavelets obtained do not have compactly supported Fourier transforms, see [8] for details. It is thus crucial to our considerations that the set K in the definition of strictly expansive matrices be compact and tile the plane by translations.

PROPOSITION 3.4. *Assume that we have a multiresolution analysis on* $\mathbb{R}^n$ *associated with an integral dilation A. Assume that this MRA has a scaling function* $\varphi(x)$ *in the Schwartz class such that* $\hat{\varphi}(\xi)$ *is real. Then there exists a wavelet family associated with this MRA consisting of* $(|\det A|-1)$ *Schwartz class functions.*

The remainder of this section consists of finding a computationally convenient form of Theorem 3.2 in the two-dimensional case. The following proposition can be found in [3].

PROPOSITION 3.5. *A* 2×2 *integer matrix A is expansive if and only if* (a) $|\det(A)| \geqslant 2$, (b) $|\operatorname{tr}(A)| \leqslant \det(A)$ *when the determinant is positive, and* (c) $|\operatorname{tr}(A)| \leqslant -\det(A)-2$ *when the determinant of A is negative.*

COROLLARY 3.6. *Let* $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an expansive, integer valued matrix. If there exists* $u \in \mathbb{R}$ *such that*

$$\eta_u(A) := \max\{|-uc+a|, |-(u+2)\,c+a|, |ud-b+(-1-u)(-uc+a)|, \\ |(u+2)\,d-b+(-1-u)(-(u+2)\,c+a)|\} < |\det(A)|,$$

then A is strictly expansive.

Proof. Let X be the (2-dimensional) Banach space with unit ball

$$B_X = \operatorname{conv}\{\pm(u,1),\ \pm(u+2,1)\}.$$

Under this norm, $\|(x, y)\|_X = \max(|x-(u+1)\,y|, |y|)$. Consider $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ as a linear map on X. Then

$$
\begin{aligned}
|\det(A)|\,\|A^{-1}\| &= \sup_{(x,y)\in B_X} \left\| \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|_X \\
&= \max\left(\left\| \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} u \\ 1 \end{pmatrix} \right\|_X, \left\| \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} u+2 \\ 1 \end{pmatrix} \right\|_X \right) \\
&= \eta_u(A).
\end{aligned}
$$

Thus, under the hypotheses of the corollary, $\|A^{-1}\| < 1$ and $B_X \subset A(B_X)^\circ$. Hence A is strictly expansive with respect to the compact set $K = \frac{1}{2} B_X$ which satisfies (3.1) and (3.2).

Remarks. (i) If we set $u = -1$ in Corollary 3.6, then we obtain that the matrix A is strictly expansive if the ℓ_1 norm of each column is less than the absolute value of the determinant of A.

(ii) When the u in $\eta_u(A)$ is understood, we simply write $\eta(A)$.

4. A REDUCTION

The goal of this section is to prove

THEOREM 4.1. *Let A be a 2×2 dilation matrix with integer entries that is not integrally similar to $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$. Then, A is strictly expansive.*

We will investigate in Section 5 which of the matrices of the form $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$ are strictly expansive.

Suppose we are given an arbitrary dilation matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Although there are algorithms [4] for determining when A is integrally similar to $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$ based on the theorem of Latimer and MacDuffee (see [32]), these techniques do not seem to be well-suited for our purposes. Therefore, we will devise an *ad hoc* procedure of reducing a dilation matrix to a matrix which is either strictly expansive or integrally similar to a matrix of the form $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$. (Note that by the theorem of Latimer and MacDuffee, there are (many) 2×2 dilation matrices which are not integrally similar to matrices of the form $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$.)

The proof of Theorem 4.1 will consist of a breaking down into many cases, but before we begin with the listing of cases, we note the following elementary transformations:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \tag{4.1}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} d & c \\ b & a \end{pmatrix}, \tag{4.2}$$

and

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} a+\lambda c & -\lambda^2 c+\lambda(d-a)+b \\ c & -\lambda c+d \end{pmatrix}. \tag{4.3}$$

For the purposes of this paper, we will call a 2×2 integer matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of type:

(I) if $|b|, |c| \geqslant |a-d|$,
(II) if $b=0$ and $|c| \geqslant |a-d|$,
(III) if $b=c=0$.

Clearly, these types are not mutually exclusive.

LEMMA 4.2. *Every integer matrix is integrally similar to a matrix of type I, type II, or type III.*

Proof. Suppose A is a given dilation matrix. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an integrally similar matrix to A with the minimal sum of the diagonal entries, i.e., a^2+d^2. Since the sum of the diagonal entries of a matrix obtained by the transformation (4.3) is equal to $a^2+d^2+2\lambda c(\lambda c+a-d)$ we must have

$$\lambda c(\lambda c+a-d) \geqslant 0 \qquad \text{for all} \quad \lambda\in\mathbb{Z}. \tag{4.4}$$

By taking $\lambda=\pm 1$ we see that $0\neq |c|<|a-d|$ would contradict (4.4). Therefore, either $|c|\geqslant|a-d|$ or $c=0$. Likewise, using transformations (4.2) and (4.3) we have that either $|b|\geqslant|a-d|$ or $b=0$. This finishes the proof of Lemma 4.2.

The following lemma (stated without proof) will be used to reduce the general case to the case $\operatorname{tr} A\geqslant 0$.

LEMMA 4.3. (a) *A dilation matrix A is integrally similar to a matrix that is strictly expansive with respect to a centrally symmetric set if and only* *if $-A$ is integrally similar to a matrix that is strictly expansive with respect to a centrally symmetric set.*

(b) *A dilation matrix A is integrally similar to $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$ if and only if $-A$ is integrally similar to $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$.*

We turn now to studying strict expansiveness for the three types of matrices introduced in this section. Clearly, expansive matrices of type III are strictly expansive with respect to the unit square. Lemma 4.4 guarantees that expansive matrices of type II are strictly expansive with respect to some centrally symmetric set. Finally, Lemmas 4.5 and 4.6 cover the case of expansive matrices of type I which are not integrally similar to $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$.

LEMMA 4.4. *Let A be a type II dilation matrix with $\operatorname{tr} A \geqslant 0$. Then A is strictly expansive.*

Proof. Case 1. $a \neq d$. Let $u = -1 + b/(d-a)$. It is easy to check that Corollary 3.6 is satisfied with this u.

Case 2. $a = d$. Let A be a matrix of the form

$$A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

where $a \geqslant 0$ and $b \in \mathbb{Z}$. By Proposition 3.5, A is expansive if and only if $a \geqslant 2$. We will show that for any $a \geqslant 2$, $a, b \in \mathbb{Z}$, A is strictly expansive with respect to a centrally symmetric set K_h of the form (4.5).

By elementary transformation (4.1), it suffices to consider the case $b \geqslant 0$.

Let K_h be a subset of $\mathbb{R}^2$ given by

$$K_h = \{(x, y) \in \mathbb{R}^2 : h(y) - 1/2 \leqslant x \leqslant h(y) + 1/2, |y| \leqslant 1/2\}, \tag{4.5}$$

where $h\colon [-1/2, 1/2] \to \mathbb{R}$ is an odd function. Clearly, if h is continuous then K_h is closed and

$$\sum_{k \in \mathbb{Z}^2} \mathbf{1}_{K_h}(\xi + k) = 1 \qquad \text{for a.e.} \quad \xi \in \mathbb{R}^2.$$

Moreover, since h is odd, K_h is centrally symmetric, i.e., $K_h = -K_h$. Our goal now is to choose a continuous odd function h such that $K_h \subset A(K_h^\circ)$.

First, notice that

$$A(K_h) = \{(x, y) \in \mathbb{R}^2 : a(h(y/a) - 1/2) + by/a \leqslant x \leqslant a(h(y/a) + 1/2) \\ + by/a, |y| \leqslant a/2\}.$$

Therefore, $K_h \subset A(K_h)$ if and only if

$$|ah(y/a)+by/a-h(y)| \leqslant (a-1)/2 \qquad \text{for all} \quad |y| \leqslant 1/2.$$

Suppose that h is odd and continuous, and

$$|ah(y/a)+by/a-h(y)| < (a-1)/2 \qquad \text{for all} \quad |y| \leqslant 1/2. \tag{4.6}$$

We claim that this implies that $K_h \subset A(K_h^\circ)$. Indeed, by (4.6) and uniform continuity of h there exists $\delta > 0$ such that

$$|ah(y/a)+by/a-h(y_0)| < (a-1)/2-\delta$$
$$\text{for} \quad |y-y_0| < \delta, \qquad |y|, |y_0| \leqslant 1/2.$$

Consequently, $K_h + (-\delta, \delta)^2 \subset A(K_h)$ and thus $K_h \subset A(K_h^\circ)$. It now remains to construct an odd continuous h satisfying (4.6).

Let $\varepsilon > 0$ be sufficiently small, say $\varepsilon = 1/(2b)$ if $b \neq 0$ and $\varepsilon = 1$ otherwise. Define a piecewise linear h initially on the interval $[0, \varepsilon]$ by

$$h(y) = \begin{cases} 0 & \text{for} \quad 0 \leqslant y \leqslant \varepsilon/a, \\ b(y-\varepsilon/a)/(a-1) & \text{for} \quad \varepsilon/a \leqslant y \leqslant \varepsilon. \end{cases}$$

Clearly,

$$|ah(y/a)+by/a-h(y)| \leqslant b\varepsilon/a, \qquad \text{for} \quad 0 \leqslant y \leqslant \varepsilon,$$

since $h(\varepsilon) = b\varepsilon/a$. Hence, $h(y) = ah(y/a)+by/a$ for $y = \varepsilon$. We extend h to $[0, \infty)$ by the iteration. If h is defined on $[0, a^n\varepsilon]$ for some $n = 0, 1, \ldots$ we extend h by

$$h(y) = ah(y/a)+by/a \qquad \text{for} \quad y \in (a^n\varepsilon, a^{n+1}\varepsilon].$$

It is easy to see that h is continuous everywhere except possibly points of the form $a^n\varepsilon$, $n = 0, 1, \ldots$. However, h is continuous at ε and thus h is also continuous at every point $a^n\varepsilon$. Finally, we can extend h on $\mathbb{R}$ by defining $h(y) = -h(-y)$ for $y < 0$.

Hence we have constructed an odd continuous and piecewise linear function h such that

$$|ah(y/a)+by/a-h(y)| \leqslant b\varepsilon/a \qquad \text{for all} \quad y \in \mathbb{R}.$$

Therefore, (4.6) holds and by the above argument, A is strictly expansive with respect to the centrally symmetric set K_h given by (4.5).

LEMMA 4.5. *Let A be a type I dilation matrix with* $\operatorname{tr} A \geqslant 0$ *which is not integrally similar to* $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$. *If*

$$|a| < |b| \quad \text{and} \quad |d| < |c|, \tag{4.7}$$

then A is strictly expansive. In particular, if $a < 0$ or $d < 0$, then A is strictly expansive.

Proof. Geometrically, (4.7) says that the vertices of the square $[-1, 1]^2$, i.e., $(\pm 1, \pm 1)$ are mapped by the dilation A into different quadrants of the plane. Furthermore, the mapped vertices $A(\pm 1, \pm 1)$ can not lie on the axes.

We need to consider several subcases. Assume that exactly two of the mapped vertices $A(\pm 1, \pm 1)$ are of the form $(\pm 1, \pm 1)$. By the symmetry of the vertices $(\pm 1, \pm 1)$ and $|\det A| \geqslant 2$, the only remaining possibility is that none of the mapped vertices $A(\pm 1, \pm 1)$ is of the form $(\pm 1, \pm 1)$. Furthermore, if the dilation $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ maps exactly two of the vertices $(\pm 1, \pm 1)$ into each other, so does a dilation obtained by the transformation (4.1) or (4.2). Hence by applying (4.1) and (4.2) we can also assume that $a, b \geqslant 0$. By (4.7) this uniquely determines then the first row, and A must be one of the matrices below

$$\begin{array}{llll} \begin{pmatrix} a & a+1 \\ d+1 & d \end{pmatrix} & \text{or} & \begin{pmatrix} a & a+1 \\ -d-1 & d \end{pmatrix} & \text{if} \quad d \geqslant 0, \\ \begin{pmatrix} a & a+1 \\ d-1 & d \end{pmatrix} & \text{or} & \begin{pmatrix} a & a+1 \\ -d+1 & d \end{pmatrix} & \text{if} \quad d \leqslant 0. \end{array} \tag{4.8}$$

Note that we must have $a, |d| \geqslant 1$, since our dilation is not integrally similar to $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$. The first matrix in (4.8) can not be even expansive by Proposition 3.5. The second and the fourth matrix are strictly expansive by the remark following Corollary 3.6, since their determinants are $2ad + a + d + 1$ and $2ad - a + d - 1$, respectively. Finally, using that the third dilation in (4.8) is type I, we must necessarily have that $a = 1$ and $d = -1$. It is then easy to show that $\left(\begin{smallmatrix} 1 & 2 \\ -2 & -1 \end{smallmatrix}\right)$ is strictly expansive by Corollary 3.6 with $u = -1.1$.

Therefore, we can assume that none of the vertices $(\pm 1, \pm 1)$ is mapped by A into $(\pm 1, \pm 1)$. By (4.7) we have that $[-1, 1]^2 \subset A[-1, 1]^2$. If $[-1, 1]^2 \subset A(-1, 1)^2$ then A is strictly expansive. By a simple geometry, the last inclusion may fail only if at least one (and thus two by the symmetry) sides of the square $[-1, 1]^2$ are contained in the boundary of the parallelogram $A[-1, 1]^2$. This means that the vertices of $A[-1, 1]^2$ must be of the form $\pm(x_1, 1)$, $\pm(x_2, 1)$ or $\pm(1, y_1)$, $\pm(1, y_2)$. But then the matrix A must be of the form $\left(\begin{smallmatrix} * & * \\ \pm 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & \pm 1 \\ * & * \end{smallmatrix}\right)$, respectively. This would mean that A is integrally similar to $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$—a contradiction.

Now, suppose that a or d is negative. Then, the other must be positive (since $\operatorname{tr} A \geqslant 0$) and we automatically have (4.7) since A is type I. This ends the proof of Lemma 4.5.

LEMMA 4.6. *Let A be a dilation of type I with* $\operatorname{tr} A \geqslant 0$ *that is not integrally similar to* $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$. *If*

$$a, c, d \geqslant 0, \qquad \textit{and} \qquad 2 \leqslant c \leqslant d, \tag{4.9}$$

then A is integrally similar to a strictly expansive matrix.

Proof. Case 1. $\det A \geqslant 2$. First, if A is of the form $\left(\begin{smallmatrix} 0 & b \\ c & c \end{smallmatrix}\right)$, then $b \leqslant -2$. If $b = -2$ and $c = 2$, then A is strictly expansive by Corollary 3.6 with $u = -1.1$, and if either $b < -2$ or $c > 2$, then A is expansive by the remark following Corollary 3.6.

Now, if A is not of the form $\left(\begin{smallmatrix} 0 & b \\ c & c \end{smallmatrix}\right)$, we claim that A is strictly expansive by virtue of Corollary 3.6 with $u = -d/c + \delta$ for sufficiently small $\delta > 0$.

First take $u = -d/c$. We claim that the last three expressions appearing in the definition of $\eta_u(A)$ are (strictly) less than $|\det A|$. If this is the case then for sufficiently small $\delta > 0$ these three inequalities will continue to hold with $u = -d/c + \delta$, and since

$$|-uc + a| = |-\delta c + a + d| < \operatorname{tr} A \leqslant \det A,$$

A is strictly expansive by Corollary 3.6 with $u = -d/c + \delta$.

To prove the claim, set $u = -d/c$. The second expression appearing in $\eta(A)$ satisfies by (4.9)

$$|-(u+2)\, c + a| = |-2c + a + d| < \operatorname{tr} A \leqslant \det A,$$

where we have strict inequality since either $a \neq 0$ or $c \neq d$. The third expression in $\eta(A)$ satisfies

$$|ud - b + (-1-u)(-uc + a)| = |-b - d - a + ad/c| = |\det A/c - \operatorname{tr} A| < \det A.$$

Finally, we need to show that the fourth expression in $\eta(A)$ satisfies

$$\begin{aligned} |(u+2)\, d - b + (-1-u)(-(u+2)\, c + a)| &= |-b - d - a + 2c + ad/c| \\ &= |\det A/c - \operatorname{tr} A + 2c| < \det A. \end{aligned}$$

Indeed,

$$\det A(1-1/c)+\operatorname{tr} A-2c \geqslant \operatorname{tr} A(2-1/c)-2c=(2-1/c)(a+d-c)-1>0,$$

unless $a+d-c=0$; that is, $a=0$ and $c=d$. This shows the claim and ends the proof of case 1.

Case 2. $\det A \leqslant -2$. We claim that A is strictly expansive by virtue of Corollary 3.6 with

$$u=\frac{(a-d)-\sqrt{(a-d)^2+4bc}}{2c}.$$

Let $\lambda_1<0$ be the negative eigenvalue of A and $\lambda_2>0$ be the positive eigenvalue of A. Note that by expansiveness $\lambda_1<-1$ and $\lambda_2>1$. We need to check that the four expressions appearing in the definition of $\eta_u(A)$ are (strictly) less than $|\det A|$.

$$\begin{aligned} -uc+a &= \frac{(d+a)+\sqrt{(a+d)^2-4(ad-bc)}}{2} \\ &= \frac{\operatorname{tr} A+\sqrt{(\operatorname{tr} A)^2-4\det A}}{2} \\ &= \lambda_2 < |\det A|, \end{aligned}$$

since $|\lambda_1|>1$. Note that the above implies that

$$cu=-\lambda_2+a. \tag{4.10}$$

Hence we also need to show that

$$|-(u+2)\,c+a|=|\lambda_2-2c|<|\det A|.$$

Since $c>0$ and $1<\lambda_2<|\det A|$, it suffices to show that

$$\lambda_2-2c>\det A=\lambda_1\lambda_2.$$

Since $c\leqslant d$

$$\lambda_2-2c-\lambda_1\lambda_2 \geqslant \lambda_2-2d-\lambda_1\lambda_2 \geqslant \lambda_2-2\operatorname{tr} A-\lambda_1\lambda_2=-2\lambda_1(-\lambda_1-1)\,\lambda_2>0.$$

The third inequality

$$|ud-b+(-1-u)(-uc+a)|<|\det A|$$

is a consequence of

$$ud - b + (-1-u)(-uc + a) = uc - a = -\lambda_2. \tag{4.11}$$

Indeed, if we solve the quadratic equation above we have

$$u = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c}.$$

Finally we need to show

$$|(u+2)\, d - b + (-1-u)(-(u+2)\, c + a)| < |\det A|.$$

Note that by (4.10) and (4.11)

$$\begin{aligned}(u+2)\, d - b + (-1-u)(-(u+2)\, c + a) &= -\lambda_2 + 2d + 2c + 2cu \\ &= -\lambda_2 + 2c + 2d + 2a - 2\lambda_2 \\ &= -\lambda_2 + 2\lambda_1 + 2c.\end{aligned}$$

To see that $-\lambda_2 + 2\lambda_1 + 2c < -\det A$ note that since $c \leqslant d \leqslant \operatorname{tr} A$,

$$-\lambda_2 + 2\lambda_1 + 2c + \lambda_1\lambda_2 \leqslant -\lambda_2 + 2\lambda_1 + 2(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = 4\lambda_1 + \lambda_2(1+\lambda_1) < 0.$$

It remains to show

$$2\lambda_1 - \lambda_2 + 2c - \det A > 0.$$

If $\lambda_1 \leqslant -3$ then

$$2\lambda_1 - \lambda_2 + 2c - \lambda_1\lambda_2 = 2\lambda_1 - (1+\lambda_1)\, \lambda_2 + 2c \geqslant 2\lambda_1 + 2\lambda_2 + 2c = 2(\operatorname{tr} A + c) > 0.$$

If $-3 < \lambda_1 < -1$ then using

$$\sqrt{(\operatorname{tr} A)^2 - 4 \det A} < 1 - \det A, \tag{4.12}$$

we have

$$\begin{aligned}2\lambda_1 - \lambda_2 + 2c - \det A &= \lambda_1 - \sqrt{(\operatorname{tr} A)^2 - 4 \det A} + 2c - \det A \\ &> \lambda_1 - 1 + 2c \geqslant 3 + \lambda_1 > 0,\end{aligned}$$

since $c \geqslant 2$. Finally, (4.12) is equivalent to $(\operatorname{tr} A)^2 < (1 + \det A)^2$ which is a consequence of $|\operatorname{tr} A| \leqslant |2 + \det A|$ by Proposition 3.5. This ends the proof of Case 2 and Lemma 4.6.

We are now ready to give the proof of Theorem 4.1.

Proof of Theorem 4.1. Let A be an expansive, integer matrix. By Lemma 4.3, it suffices to prove Theorem 4.1 for matrices with nonnegative trace. By Lemma 4.2, there are three types of matrices to consider: type I ($|b|, |c| \geqslant |a-d|$), type II ($c=0$ and $b \geqslant |a-d|$), and type III ($b=c=0$). Expansive type III matrices are strictly expansive with respect to the unit square. Expansive type II matrices are strictly expansive by Lemma 4.4.

For type I matrices, by Lemma 4.5 it suffices to show that every A satisfying

$$|a| \geqslant |b| \quad \text{or} \quad |d| \geqslant |c|, \qquad b, c \neq 0, \qquad a, d \geqslant 0 \tag{4.13}$$

that is not integrally similar to $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$ is strictly expansive. So, assume (4.13) holds. Without loss of generality, $|d| \geqslant |c|$ and $c \geqslant 1$ by (4.1) and (4.2). If $c=1$, then applying (4.3) with $\lambda = -d$ implies A is integrally similar to $\left(\begin{smallmatrix} 0 & 1 \\ * & * \end{smallmatrix}\right)$. If $c \geqslant 2$, then Lemma 4.6 implies that A is strictly expansive. This completes the proof of Theorem 4.1.

5. SPECIAL CASES

In this section, we prove that there exist Meyer type wavelets for all expansive matrices of the form $\left(\begin{smallmatrix} 0 & 1 \\ -d & t \end{smallmatrix}\right)$. The determinant two case was completed in section 2, our main concern in this section is to prove that every 2×2 dilation A is integrally similar to a strictly expansive matrix with respect to a centrally symmetric set unless $|\det(A)| = 2$ or $\det(A) = 3$ and $\operatorname{tr}(A) = 0$.

We proceed again by breaking the theorem down into cases.

LEMMA 5.1. *Let A be an integer valued, expansive matrix of the form $A = \left(\begin{smallmatrix} 0 & 1 \\ -d & t \end{smallmatrix}\right)$ with $t \geqslant 0$. Then A is integrally similar to a strictly expansive matrix with respect to a centrally symmetric set if either* (a) $d > 3$ *and* $t > 2$ *or* (b) $d < -3$ *and* $t \neq -d-2$.

Proof. Case 1. $d > 3$ and $t > 3$. (Since A is expansive, we can rewrite as $3 < t \leqslant d$.) Then

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -d & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -4-d+2t & -2+t \end{pmatrix}.$$

This matrix is strictly expansive by Corollary 3.6 with $u = -1$, since $1 + |-2+t| < d$ and $2 + |-4-d+2t| < d$.

Case 2. $d>3$ and $t=3$. Note that

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -d & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ -1-d+t & 1+t \end{pmatrix}.$$

That the resulting matrix is strictly expansive follows from Corollary 3.6 with $u=-1$ by noting that $1+|t-1|=3<d$ and $1+|-1-d+t|=1+|2-d|<d$.

Case 3. $d<0$, $t\neq -d-2$, and $t\neq 0$. Then

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -d & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}=\begin{pmatrix} -1 & 1 \\ -1-d-t & 1+t \end{pmatrix}.$$

We proceed with checking the ℓ_1 norms of the columns. First, $1+|1+t|=t+2<|d|$ by assumptions. Second, note that $1+|-1-d-t|\leqslant \max\{1+|-1-d-1|, 1+|-1-d-(-d-2)|\}=\max\{-d-1, 2\}<-d$.

Case 4. $d<0$ and $t=0$. We can see that $\left(\begin{smallmatrix} 0 & 1 \\ -d & 0 \end{smallmatrix}\right)$ is integrally similar to $\left(\begin{smallmatrix} 2 & 1 \\ -4-d & -2 \end{smallmatrix}\right)$. Then, $1+|-2|=3<|d|$ and $2+|-4-d|=-d-2<|d|$, as desired.

The remainder of this section focuses on the special cases which are not covered in Lemma 5.1; namely,

- $d>3$ and $0\leqslant t\leqslant 2$,
- $d<-3$ and $t=|d|-2$,
- $|d|=3$.

LEMMA 5.2. *All matrices of the form* $A=\left(\begin{smallmatrix} 0 & 1 \\ -d & 2 \end{smallmatrix}\right)$ *are integrally similar to strictly expansive matrices with respect to centrally symmetric sets, where* $d\geqslant 3$.

Proof. Apply the transformation

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -d & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ -d+1 & 1 \end{pmatrix}.$$

We wish to show that there is a u Corollary 3.6 that makes $\eta_u(A)<\det(A)$. We claim that there is an $\epsilon=\epsilon(d)>0$ such that $-1<u<-1+\epsilon$ works.

For the first expression in $\eta(A)$, note that if $u<0$, then $|u(-d+1)+1|=u(-d+1)+1$, which is less than d if and only if $u>-1$. If we evaluate the second through the fourth expression at $u=-1$, we obtain $\max(2, d-2)<d$. The lemma then follows from the continuity of the expressions in η.

LEMMA 5.3. *The matrices* $\begin{pmatrix} 0 & 1 \\ -d & 1 \end{pmatrix}$ *are integrally similar to strictly expansive matrices with respect to centrally symmetric sets for* $d \geqslant 3$.

Proof. Consider

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -d & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -d & 0 \end{pmatrix}.$$

Let

$$B = \text{sym conv}\{(1, 1), (1, -1)\}.$$

Then

$$A(B) = \text{sym conv}\{(2, -d), (0, -d)\},$$

as in Fig. 3.

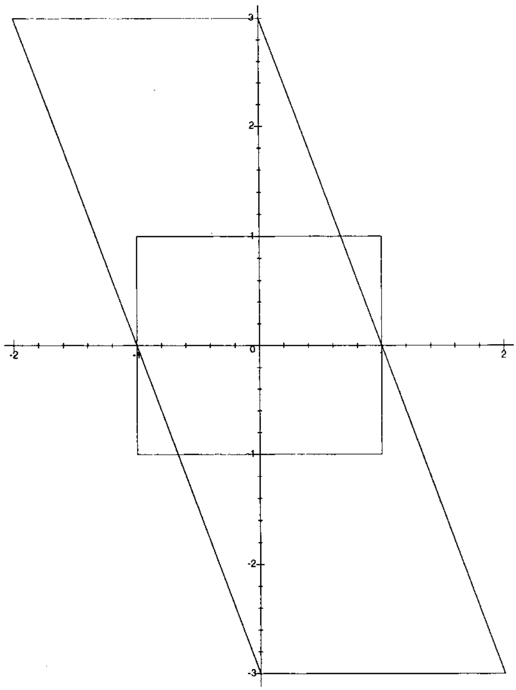

FIG. 3. First try for strict expansiveness.

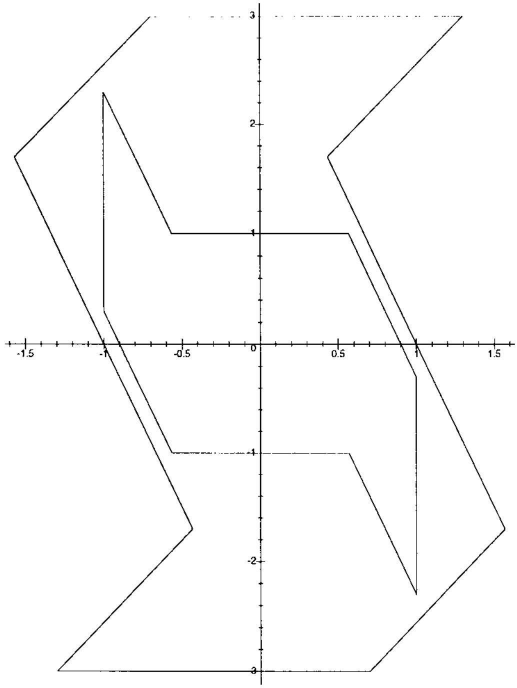

FIG. 4. Strict expansiveness for trace 1.

We need to (carefully) move the pieces of B which are not also in $A(B)$. Let $T = \text{conv}\{(1, 0), (1, 1), (1-1/d, 1)\}$ and $T_\epsilon = \text{conv}\{(1, -\epsilon), (1, 1), (1-1/d-\epsilon, 1)\}$. Then, $A(T_\epsilon) = \text{conv}\{(1-\epsilon, -d), (2, -d), (2-1/d-\epsilon, -d+1+d\epsilon)\}$. So, for $\epsilon > 0$ and small enough, if we let $B' = B \setminus (T_\epsilon \cup (-T_\epsilon)) \cup (T_\epsilon - (0, 2)) \cup (-T_\epsilon + (0, 2))$, then $\overline{B'} \subset (A(B'))^\circ$ as pictured in Fig. 4.

LEMMA 5.4. *For $d \geqslant 4$, the matrices $\begin{pmatrix} 0 & 1 \\ -d & 0 \end{pmatrix}$ are integrally similar to strictly expansive matrices with respect to centrally symmetric sets.*

Proof. Consider

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -d & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1-d & -1 \end{pmatrix}.$$

Let $B = \text{sym conv}\{(1, 1), (1, -1)\}$. Then, $A(B) = \text{sym conv}\{(2, -d-2), (0, -d)\}$.

Now, in the case that d is positive, let $T = \text{conv}\{(1, -1), (1, 1), (1-2/(d+1), 1)\}$ and let

$$K_0 = \text{sym conv}\{(1-2/(d+1), 1), (1, -1)\} \cup (T + (0, -2)) \cup (-T + (0, 2)).$$

Since $A(T)=\operatorname{conv}\{(0,-d),(2,-d-2),(2-2/(d+1),-d)\}$, we have $K_0\subset A(K_0)$. Moreover, for $d\geqslant 4$, $K_0\setminus A(K_0^\circ)$ consists of two line segments connecting $\pm(1-2/(d+1),1)$ and $\pm(1,-1)$. Finally, to show that A is strictly expansive it suffices to slightly modify the set K_0 into a compact set K satisfying $K\subset A(K^\circ)$ and $\sum_{k\in\mathbb{Z}^2}\mathbf{1}_K(\xi+2k)=1$ for a.e. $\xi\in\mathbb{R}^2$. Given $\varepsilon\geqslant 0$ define points $v_1=(1-2/(d+1)-\varepsilon,1)$, $v_2=(1,-1-(d+1)\,\varepsilon)$, $v_3=(1,-3+(d+1)\,\varepsilon)$, $v_4=(1+\varepsilon,-3)$, $v_5=(1-\varepsilon,-3)$, $v_6=(1-2/(d+1)-\varepsilon,-1)$, and $v_j=-v_{j-6}$ for $7\leqslant j\leqslant 12$. Let K_ε be a polygon whose boundary consists of line segments connected by vertices $v_1, v_2, \ldots, v_{12}$. Note that for $\varepsilon=0$, K_ε is just K_0 given as above. Finally, it follows by simple (but long) calculations, that $K=K_\varepsilon$ satisfies strict expansiveness condition for sufficiently small $\varepsilon=\varepsilon(d)>0$. Figure 5 shows polygons K_ε and $A(K_\varepsilon)$ when $d=4$ and $\varepsilon=0.1$.

In the case that d is negative, note that by (4.1) A is integrally similar to $\left(\begin{smallmatrix}0 & -d\\ 1 & 0\end{smallmatrix}\right)$. One easily checks that this matrix satisfies Corollary 3.6 with $u=-2-\delta$, for $\delta>0$ small enough.

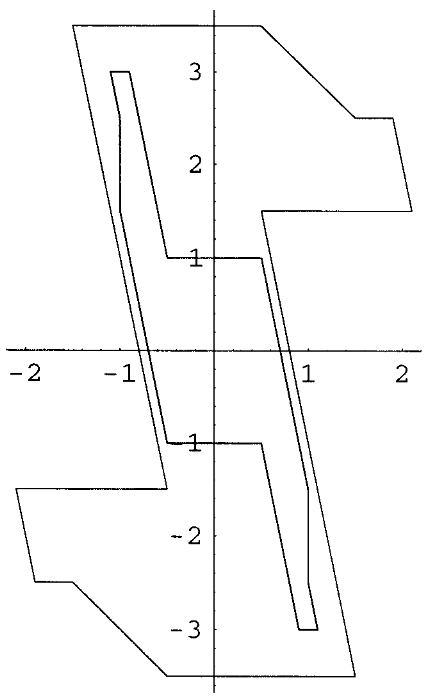

FIG. 5. Strict expansiveness for trace 0, Det $\geqslant 4$.

LEMMA 5.5. *Every matrix of the form* $\begin{pmatrix} 0 & 1 \\ -d & -d-2 \end{pmatrix}$ *is integrally similar to a strictly expansive matrix with respect to a centrally symmetric set when* $d \leqslant -3$.

Proof. Note that A is integrally similar to $\begin{pmatrix} -1 & 1 \\ 1 & -d-1 \end{pmatrix}$, which satisfies Corollary 3.6 with $u = -1.1$. The easy details of verification are omitted.

THEOREM 5.6. *Every* 2×2 *dilation is integrally similar to a strictly expansive matrix with respect to a centrally symmetric set unless* $|\det(A)| = 2$ *or* $\det(A) = 3$ *and* $\operatorname{tr}(A) = 0$.

Proof. By Theorem 4.1, it suffices to prove Theorem 5.6 for matrices of the form $\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix}$. By Lemma 4.3 and Eq. (4.1), we may assume that $t \geqslant 0$. Lemmas 5.1 through 5.5 cover the following cases:

Lemma	Cases covered
5.1	$d < -3$ and $t \neq -d-2$
5.5	$d \leqslant -3$ and $t = -d-2$
5.1	$d > 3$ and $\|t\| > 2$
5.2	$d \geqslant 3$ and $t = 2$
5.3	$d \geqslant 3$ and $t = 1$
5.4	$d \geqslant 4$ and $t = 0$

Thus, the only dilation matrices with $|d| \geqslant 3$ that are not covered by the above lemmas are the matrices $\begin{pmatrix} 0 & 1 \\ -3 & 3 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ \pm 3 & 0 \end{pmatrix}$. Notice that $\begin{pmatrix} 0 & 1 \\ -3 & 3 \end{pmatrix}$ is similar to $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, which has $\eta(A) = 2.9$ when $u = -0.9$.

Finally, $\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$ is integrally similar to $\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$, which is strictly expansive by Corollary 3.6 with $u = -1.1$. This completes the proof of Theorem 5.6.

The authors do not know whether the remaining matrix $\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$ is integrally similar to a strictly expansive matrix, nor whether any dilation of determinant ± 2 can be strictly expansive. However, it is easy to see that there is a Meyer type wavelet for the matrix $A = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$. Indeed, by Corollary 3.6, there are Meyer type wavelets ψ_1, ψ_2 with scaling function ϕ for dilation by -3 on the line. One can easily check that the functions $\psi_1 \otimes \phi$ and $\psi_2 \otimes \phi$ are wavelets for the dilation A. Thus, we have proven the following

THEOREM 5.7. *Let* A *be an expansive,* 2×2 *integer matrix. Then, there exist Meyer type wavelets* $\{\psi^1, ..., \psi^l\}$, *where* $l = |\det A| - 1$.

Proof. Combine Theorem 5.6 with Theorem 2.4, Theorem 4.1, and the remark immediately preceding this theorem statement.

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