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Combined MSF Multiwavelets

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ABSTRACT. The properties of (bi)orthogonal multiwavelets associated with general expansive dilation matrices are studied. It is shown that for almost every dilation (bi)orthogonal multiwavelets must be of the very special form, i. e., the union of their supports has minimal measure in the Fourier domain. This extends a one dimensional, single wavelet result of Chui and Shi.

1. Introduction

In this note we investigate properties of (bi)orthogonal multiwavelets associated with fairly general groups of dilations. This is a companion article to the work by Chui, Czaja, Maggioni, and Weiss [9], where tight frame wavelets with arbitrary dilation matrix are studied. For simplicity we only consider the standard lattice $\mathbb{Z}^n$; results corresponding to any other choice of lattice are obtained by a simple change of variables.

Definition 1. Suppose A is an *expansive dilation matrix,* i. e., $n \times n$ real matrix all of whose eigenvalues λ satisfy $|\lambda| > 1$. Let Ψ be a finite family of functions $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$. The *affine system* generated by Ψ is defined as

$$X(\Psi) = \left\{ \psi^l_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, \dots, L \right\} .$$

Here, for $\psi \in L^2(\mathbb{R}^n)$ we let

$$\psi_{j,k}(x) = D_{A^j} T_k \psi(x) = |\det A|^{j/2} \psi\left(A^j x - k\right) \qquad j \in \mathbb{Z}, k \in \mathbb{Z}^n ,$$

where $T_y f(x) = f(x-y)$ is a translation operator by the vector $y \in \mathbb{R}^n$, and $D_A f(x) = \sqrt{|\det A|} f(Ax)$ is a dilation by the matrix A. We say that Ψ is a *multiwavelet* if $X(\Psi)$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.

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In the wavelet literature, the notion of multiwavelet is sometimes referred to as a vector-valued function defined by a matrix two-scale relationship in terms of vector-valued scaling function. However, here we follow Definition 1.

One of the main results of [9] is the following characterization of pairs of dual frame wavelets.

Theorem 1.

Suppose that $\Psi = \{\psi^1, \dots, \psi^L\}$, $\Phi = \{\phi^1, \dots, \phi^L\} \subset L^2(\mathbb{R}^n)$. If $X(\Psi)$ and $X(\Phi)$ are Bessel families which satisfy

$$\sum_{l=1}^{L} \sum_{\substack{(j,m)\in\mathbb{Z}\times\mathbb{Z}^n,\\ \alpha=(A^*)^{-j}m}} \hat{\phi}^l\left(\left(A^*\right)^j \xi\right) \overline{\hat{\psi}^l\left((A^*)^j(\xi+\alpha)\right)} = \delta_{\alpha,0} \qquad \text{for a. e. } \xi \in \mathbb{R}^n\,, \tag{1.1}$$

and for all $\alpha \in \Lambda$ then $X(\Psi)$ and $X(\Phi)$ form a pair of dual frames. Here Λ denotes the set of all A-adic vectors, i. e.,

$$\Lambda = \left\{\alpha \in \mathbb{R}^n : \alpha = \left(A^*\right)^{-j} m \quad \text{for some } (j,m) \in \mathbb{Z}\times\mathbb{Z}^n\right\}\,. \tag{1.2}$$

Conversely, if $X(\Psi)$ and $X(\Phi)$ is a pair of dual frames then (1.1) *holds.*

This theorem is a generalization of a series of results, for a history see [9] and references therein. The hard [the necessity of (1.1)] direction of Theorem 1 was first shown with certain mild decay conditions on $\hat{\psi}^l$'s and $\hat{\phi}^l$'s. Subsequently, these conditions were removed by the author of this note and the complete proof is included in [9].

The goal of this article is to study only these pairs of dual frames in the above setting which form biorthogonal bases. We are going to show that for "most" dilations (bi)orthogonal wavelets must have very specific form in the Fourier domain, i. e., their supports must have minimal support in frequency (MSF). This phenomenon was first observed by Chui and Shi [8] in one dimensional single wavelet case.

Theorem 2.

Suppose a dilation factor $|a| > 1$ is such that a^j is irrational for all integers $j \geq 1$. If ψ is an orthogonal wavelet associated to the dilation a then ψ is an MSF wavelet.

The main consequence of Theorem 2 is the fact that wavelets associated to dilation factors as above can not be well localized in the time domain as was pointed out in [8] in answering a question of Daubechies [13]. However, it is not immediate if the same conclusion holds in the multiwavelet case. Auscher [1] has shown that if a is an integer, $|a| \geq 2$, and L is not divisible by $|a|-1$ then any multiwavelet $\Psi = \{\psi^1, \dots, \psi^L\}$ has to be poorly localized. Hence, if a is an integer and $|a| \geq 3$ then there are no single well localized wavelets, but there are nice well localized multiwavelets Ψ with $L = |a|-1$.

Nevertheless, we will show that all multiwavelets associated to dilation factors as in Theorem 2 must be poorly localized in the time domain. Theorem 5 gives an extension of this fact to the several dimensional setting.

In order to work effectively in the multiwavelet case we need to use the concept of a range function, see [3, 4, 16]. For $f \in L^2(\mathbb{R}^n)$, define

$$\mathcal{T}f : \mathbb{T}^n \to l^2\left(\mathbb{Z}^n\right), \quad \mathcal{T}f(\xi) = \left(\hat{f}(\xi+k)\right)_{k\in\mathbb{Z}^n}\,,$$

where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is identified with its fundamental domain $\mathbb{T}^n = [-1/2, 1/2)^n$. Let $\{e_k : k \in \mathbb{Z}^n\}$ be the standard basis of $l^2(\mathbb{Z}^n)$. A *range function* is a mapping

$$J : \mathbb{T}^n \to \left\{\text{closed subspaces of } l^2\left(\mathbb{Z}^n\right)\right\} .$$

Proposition 1.

A closed subspace $V \subset L^2(\mathbb{R}^n)$ is shift invariant ($f \in V \implies T_k f \in V$ for $k \in \mathbb{Z}^n$) if and only if

$$V = \left\{ f \in L^2\left(\mathbb{R}^n\right) : \mathcal{T} f(\xi) \in J(\xi) \text{ for a. e. } \xi \in \mathbb{T}^n \right\} ,$$

where J is a measurable range function. The correspondence between V and J is one-to-one under the convention that the range functions are identified if they are equal a. e. Furthermore, if $V = \overline{\text{span}}\{T_k\varphi : \varphi \in \mathcal{A}, k \in \mathbb{Z}^n\}$ for some countable $\mathcal{A} \subset L^2(\mathbb{R}^n)$, then

$$J(\xi) = \overline{\text{span}}\, \{\mathcal{T}\varphi(\xi) : \varphi \in \mathcal{A}\} .$$

Given a shift invariant space $V \subset L^2(\mathbb{R}^n)$ define its *dimension function* $\dim_V : \mathbb{T}^n \to \{0, 1, 2, \ldots, \infty\}$ as $\dim_V(\xi) = \dim J(\xi)$. Note that for $V = \check{L}^2(K) := \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset K\}$ we have $\dim_V(\xi) = \sum_{k\in\mathbb{Z}^n} \mathbf{1}_K(\xi + k)$.

Recall that a subset X of a Hilbert space $\mathcal{H}$ is a *Riesz family* with constants $0 < a \leq b < \infty$, if

$$a \sum_{\eta\in X} \left|c_\eta\right|^2 \leq \left\|\sum_{\eta\in X} c_\eta \eta\right\|_{\mathcal{H}}^2 \leq b \sum_{\eta\in X} \left|c_\eta\right|^2 ,$$

for all (finitely supported) sequences $(c_\eta)_{\eta\in X}$. If a Riesz family X is complete in $\mathcal{H}$, we say X forms a *Riesz basis.* We will also use the following simple fact, see [3, 4, 20], which is a part of a more general result.

Theorem 3.

Suppose $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(\mathbb{R}^n)$. The system $\{T_k\psi^l : k \in \mathbb{Z}^n, l = 1, \ldots, L\}$ is a Riesz family with constants a, b if and only if $\{\mathcal{T}\psi^l(\xi) : l = 1, \ldots, L\}$ is a Riesz family with constants a, b for a. e. $\xi \in \mathbb{T}^n$.

Finally, let $\tau : \mathbb{R}^n \to \mathbb{T}^n$ be a *translation projection* given by $\tau(\xi) = \xi - k$, where k is a unique element of $\mathbb{Z}^n$ satisfying $\xi - k \in \mathbb{T}^n$.

2. Combined MSF Multiwavelets

There are several articles (in varying degree of generality) devoted to the study of MSF (minimally supported frequency) wavelets, e. g., [11, 12, 14, 15, 18, 19, 21, 22]. A good introduction to the subject is [10, 17]. Here we would like to mention only the theorem of Dai, Larson, and Speegle [11] which asserts that every expansive dilation matrix has an MSF wavelet. The concept of a multiple MSF wavelet (or MSF multiwavelet) is relatively new, see [2, 6]. In fact, there appear to be two possible definitions of MSF multiwavelets.

Definition 2. We say that a multiwavelet $\Psi = \{\psi^1, \ldots, \psi^L\}$ associated with the dilation A is *MSF* (minimally supported frequency), if the support of each $\hat{\psi}^l$ has minimal (Lebesgue) measure.

By the orthogonality of $\{T_k\psi^l : k \in \mathbb{Z}^n\}$ this implies that $|\hat{\psi}_l| = \mathbf{1}_{K_l}$ for some measurable sets K_l. By [6, Theorem 2.4] these sets are characterized by the following equations

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^n} \mathbf{1}_{K_l}(\xi+k)\mathbf{1}_{K_{l'}}(\xi+k) = \delta_{l,l'} \qquad \text{a. e. } \xi \in \mathbb{R}^n, \quad l, l' = 1, \dots, L\,, \\ &\sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} \mathbf{1}_{K_l}\left(\left(A^*\right)^j \xi\right) = 1 \qquad \text{a. e. } \xi \in \mathbb{R}^n\,. \end{aligned} \tag{2.1}$$

Even though Definition 2 may look as the most natural extension of a class of single MSF wavelets in the multiwavelet setting, a larger class of combined MSF multiwavelets plays a much more important role as we will see in Section 3.

Definition 3. We say that a multiwavelet $\Psi = \{\psi^1, \dots, \psi^L\}$ associated with the dilation A is *combined MSF* if $\bigcup_{l=1}^L \operatorname{supp} \hat{\psi}^l$ has minimal (Lebesgue) measure.

Suppose $\Psi = \{\psi^1, \dots, \psi^L\}$ is a combined MSF multiwavelet, and let $K = \bigcup_{l=1}^L \operatorname{supp} \hat{\psi}^l$. For $j \in \mathbb{Z}$ define

$$W_j = \overline{\operatorname{span}} \left\{\psi^l_{j,k} : k \in \mathbb{Z}^n,\ l = 1, \dots, L\right\}\,. \tag{2.2}$$

Clearly, $W_0 \subset \check{L}^2(K)$, where $\check{L}^2(K) := \{f \in L^2(\mathbb{R}^n) : \operatorname{supp} \hat{f} \subset K\}$. Note that both W_0 and $L^2(K)$ are shift invariant subspaces of $L^2(\mathbb{R}^n)$. Since the translates of ψ_l's by $\mathbb{Z}^n$ form an orthonormal basis for W_0, its dimension function $\dim_{W_0}(\xi) = L$ for a. e. ξ, see [3, 4]. Therefore,

$$L = \dim_{W_0}(\xi) \le \dim_{\check{L}^2(K)}(\xi) = \sum_{k\in\mathbb{Z}^n} \mathbf{1}_K(\xi+k) \qquad \text{a. e. } \xi \in \mathbb{T}^n\,,$$

and hence $|K| \ge L$. Since K has minimal measure $|K| = L$ thus

$$\sum_{k\in\mathbb{Z}^n} \mathbf{1}_K(\xi+k) = L \qquad \text{a. e. } \xi \in \mathbb{R}^n\,. \tag{2.3}$$

Since $W_0 \subset \check{L}^2(K)$ have equal dimension functions we must have $W_0 = \check{L}^2(K)$. Hence $W_j = \check{L}^2((A^*)^j K)$ and $\bigoplus_{j\in\mathbb{Z}} W_j = \check{L}^2(\mathbb{R}^n)$, and we have

$$\sum_{j\in\mathbb{Z}^n} \mathbf{1}_K\left(\left(A^*\right)^j \xi\right) = 1 \qquad \text{a. e. } \xi \in \mathbb{R}^n\,. \tag{2.4}$$

By [3, Theorem 2.6] (2.3) and (2.4) characterize K being a multiwavelet set of order L. Recall from [3] that K is a *multiwavelet set* of order L if $K = \bigcup_{l=1}^L K_l$ for some $K_1, \dots, K_L$ satisfying (2.1). We are now ready to give a description of all combined MSF multiwavelets.

Theorem 4.

Suppose A is a dilation, $L \in \mathbb{N}$, and $K = \bigcup_{l=1}^L K_l$ is a multiwavelet set of order L, where K_l's satisfy (2.1). *$\Psi = \{\psi^1, \dots, \psi^L\}$ is a combined MSF multiwavelet supported (in the Fourier domain) on K if and only if*

$$\hat{\psi}^m(\xi) = \begin{cases} (U(\tau(\xi)))_{l,m} & \text{for } \xi \in K_l, \\ 0 & \text{otherwise}\,, \end{cases} \qquad \text{for } m = 1, \dots, L\,, \tag{2.5}$$

for some measurable $U : \mathbb{T}^n \to U_L(\mathbb{C})$, where $U_L(\mathbb{C})$ denotes the group of $L \times L$ unitary matrices over $\mathbb{C}$. The correspondence between such multiwavelets Ψ and U is one-to-one under the convention that U's are identified if they are equal a. e.

Naturally this correspondence is not canonical and depends on the choice of the partition $K = K_1 \cup \cdots \cup K_L$.

Proof. Suppose $U : \mathbb{T}^n \to U_L(\mathbb{C})$ is any measurable mapping. Define ψ^l's by (2.5). We claim that the system $\{\psi^l_{0,k} : k \in \mathbb{Z}^n, l = 1, \ldots, L\}$ is an orthonormal basis of W_0. Indeed, for $\xi \in \mathbb{T}^n$, $\mathcal{T}\psi^l(\xi) = (\hat{\psi}^l(\xi+k))_{k\in\mathbb{Z}^n}$ is a unit vector in $l^2(\mathbb{Z}^n)$ and $\langle \mathcal{T}\psi^l(\xi), \mathcal{T}\psi^{l'}(\xi)\rangle = \delta_{l,l'}$ by (2.5). Theorem 3 shows then the claim. Furthermore, by Proposition 1, W_0 is a shift invariant space with the range function $J(\xi) = \text{span}\{e_l : \xi + l \in K\}$. Since the same J is the range function of $\check{L}^2(K)$ we have $W_0 = \check{L}^2(K)$, and Ψ is a combined MSF multiwavelet.

Conversely, suppose that Ψ is combined MSF. Define $U : \mathbb{T}^n \to U_L(\mathbb{C})$ by

$$(U(\xi))_{l,m} = \hat{\psi}^m\left(\left(\tau|_{K_l}\right)^{-1}\xi\right) \qquad \text{for } l, m = 1, \ldots, L\ .$$

Indeed, $U(\xi) \in U_L(\mathbb{C})$ since

$$\begin{aligned}\sum_{l=1}^{L} \hat{\psi}^m\left(\left(\tau|_{K_l}\right)^{-1}\xi\right)\overline{\hat{\psi}^{m'}\left(\left(\tau|_{K_l}\right)^{-1}\xi\right)} &= \sum_{k\in\mathbb{Z}^n,\ \xi+k\in K} \hat{\psi}^m(\xi+k)\overline{\hat{\psi}^{m'}(\xi+k)} \\ &= \sum_{k\in\mathbb{Z}^n} \hat{\psi}^m(\xi+k)\overline{\hat{\psi}^{m'}(\xi+k)} = \delta_{m,m'}\ . \quad \square\end{aligned}$$

As an immediate consequence of Theorem 4 we conclude that if a multiwavelet $\Psi = \{\psi^1, \ldots, \psi^L\}$ is combined MSF then

$$\sum_{l=1}^{L} \left|\hat{\psi}^l(\xi)\right|^2 = \mathbf{1}_K(\xi) \qquad \text{for a. e. } \xi \in \mathbb{R}^n\ , \tag{2.6}$$

for some multiwavelet set K of order L. Conversely, if a multiwavelet Ψ satisfies (2.6) then $L = \sum_{l=1}^{L} ||\psi^l||_2^2 = |K|$ and $K = \bigcup_{l=1}^{L} \text{supp}\,\hat{\psi}^l$, and thus Ψ is combined MSF.

As a corollary we conclude that combined MSF multiwavelets can not be well localized in the space domain. Indeed, if $\psi^l(x)$ were merely integrable for all $l = 1, \ldots, L$ then $\hat{\psi}^l(\xi)$ would be continuous and (2.6) could never hold. Hence if $\Psi = \{\psi^1, \ldots, \psi^L\}$ is a combined MSF multiwavelet then at least one of ψ^l's is poorly localized in the space domain.

3. Biorthogonal Multiwavelets

In this section we concentrate our attention only to dilations A satisfying

$$\mathbb{Z}^n \cap \left(A^*\right)^j \mathbb{Z}^n = \{0\} \qquad \text{for all } j \in \mathbb{Z}\setminus\{0\}\ . \tag{3.1}$$

Condition (3.1) means that A is very far from preserving the standard lattice $\mathbb{Z}^n$, i. e., $A\mathbb{Z}^n \subset \mathbb{Z}^n$, which is a common assumption in the wavelet literature. Nevertheless, almost every expansive dilation matrix A satisfies (3.1). More precisely, we claim that the set of all expansive matrices A failing (3.1) has a null (n^2-dimensional Lebesgue) measure as a subset of all expansive matrices which is an open subset of $GL_n(\mathbb{R}) \subset \mathbb{R}^{n\times n}$. Indeed, first note that (3.1) is equivalent to the same condition only for $j \geq 1$. Given $j \geq 1$, $k, l \in \mathbb{Z}^n \setminus \{0\}$, the set $\{A \in GL_n(\mathbb{R}^n) : (A^*)^j k = l\}$ has measure zero as a consequence of the fact that the zero set of a non-zero polynomial (in n^2 variables) has measure zero. Finally, matrices failing (3.1) comprise the countable union of the above sets.

It can be shown by following the arguments in [8] that if the dilation A satisfies (3.1) and ψ is an orthonormal wavelet ($L = 1$), then necessarily ψ is an MSF wavelet. However, this is not the case when $L > 1$. It turns out that all multiwavelets associated with dilations A satisfying (3.1) are **combined** MSF, and thus not well localized in the space domain. Moreover, this phenomenon also occurs in the larger class of biorthogonal wavelets.

Definition 4. We say that a pair $\Psi = \{\psi^1, \dots, \psi^L\}$ and $\Phi = \{\phi^1, \dots, \phi^L\}$ is a *biorthogonal* (multi)wavelet if $X(\Psi)$ and $X(\Phi)$ are Riesz bases in $L^2(\mathbb{R}^n)$ and

$$\left\langle \phi^l_{j,k}, \psi^{l'}_{j',k'} \right\rangle = \delta_{l,l'}\delta_{j,j'}\delta_{k,k'} \qquad \text{for all } j \in \mathbb{Z},\ k \in \mathbb{Z}^n,\ l, l' = 1, \dots, L\,. \tag{3.2}$$

The main result, Theorem 5, states that all biorthogonal multiwavelets associated to dilations satisfying (3.1) have to be combined MSF. In analogy to Theorem 4, Theorem 6 gives a description of the class of biorthogonal combined MSF multiwavelets. In particular, Theorem 6 describes all biorthogonal multiwavelets associated with dilations A satisfying (3.1).

Theorem 5.

Suppose A is an expansive dilation satisfying (3.1). *If Ψ and Φ is a biorthogonal multiwavelet associated to A then both Ψ and Φ are combined MSF in the sense that (modulo null sets)*

$$K := \bigcup_{l=1}^{L} \operatorname{supp} \hat{\psi}^l = \bigcup_{l=1}^{L} \operatorname{supp} \hat{\phi}^l \tag{3.3}$$

is a multiwavelet set of order L, i. e., K satisfies (2.3) *and* (2.4).

Theorem 6.

Suppose A is an expansive dilation and K is a multiwavelet set of order L. Suppose $K = \bigcup_{l=1}^{L} K_l$, where K_l's satisfy (2.1). *A pair Ψ, Φ is a biorthogonal combined MSF multiwavelet supported (in the Fourier domain) on K, i. e.,* (3.3) *holds, if and only if*

$$\hat{\psi}^m(\xi) = \begin{cases} (P(\tau(\xi)))_{l,m} & \text{for } \xi \in K_l\,, \\ 0 & \text{otherwise}\,, \end{cases} \quad \hat{\phi}^m(\xi) = \begin{cases} \left((P^*)^{-1}(\tau(\xi))\right)_{l,m} & \text{for } \xi \in K_l\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.4}$$

for $m = 1, \dots, L$, and for some measurable $P : \mathbb{T}^n \to GL_L(\mathbb{C})$ satisfying

$$0 < \operatorname{ess\,inf}_{\xi\in\mathbb{T}^n} \left\| P(\xi)^{-1} \right\|^{-1} \leq \operatorname{ess\,sup}_{\xi\in\mathbb{T}^n} ||P(\xi)|| < \infty\,. \tag{3.5}$$

The correspondence between such pairs Ψ, Φ and P is one-to-one under the convention that P's are identified if they are equal a. e.

Naturally, as in Theorem 4, this correspondence is not canonical and depends on the choice of the partition $K = K_1 \cup \cdots \cup K_L$.

Proof of Theorem 5. (3.1) means that every non-zero A-adic vector $\alpha \in \Lambda$, where Λ is given by (1.2) has a unique representation as $\alpha = (A^*)^{-j}m$ for some $(j, m) \in \mathbb{Z}\times(\mathbb{Z}^n\setminus\{0\})$. By the necessity part of Theorem 1 the Equation (1.1) implies

$$\sum_{l=1}^{L} \hat{\phi}^l(\xi)\overline{\hat{\psi}^l(\xi+m)} = 0 \qquad \text{for all } m \in \mathbb{Z}^n \setminus \{0\} . \tag{3.6}$$

On the other hand, the biorthonormality of $\{\psi^l_{0,k}\}$ and $\{\phi^l_{0,k}\}$ implies that

$$\sum_{k\in\mathbb{Z}^n} \hat{\phi}^l(\xi+k)\overline{\hat{\psi}^{l'}(\xi+k)} = \delta_{l,l'} \qquad \text{for } l, l' = 1, \ldots, L , \tag{3.7}$$

and for a. e. $\xi \in \mathbb{R}^n$.

Given $\xi \in \mathbb{T}^n$ define vectors

$$v(\xi) = \begin{pmatrix} \hat{\psi}^1(\xi) \\ \vdots \\ \hat{\psi}^L(\xi) \end{pmatrix} \in \mathbb{C}^L, \qquad w(\xi) = \begin{pmatrix} \hat{\phi}^1(\xi) \\ \vdots \\ \hat{\phi}^L(\xi) \end{pmatrix} \in \mathbb{C}^L .$$

Since for a. e. ξ, the vectors $\{\mathcal{T}\psi^l(\xi) : l = 1, \ldots, L\}$ (and $\{\mathcal{T}\phi^l(\xi) : l = 1, \ldots, L\}$) are linearly independent by Theorem 3,

$$\operatorname{span}\left\{v(\xi+m) : m \in \mathbb{Z}^n\right\} = \operatorname{span}\left\{w(\xi+m) : m \in \mathbb{Z}^n\right\} = \mathbb{C}^L \qquad \text{for a. e. } \xi .$$

Hence we can find $m_1, \ldots, m_L \in \mathbb{Z}^n$ such that $\operatorname{span}\{v(\xi+m_l) : l = 1, \ldots, L\} = \mathbb{C}^L$. Since by (3.6)

$$\langle v(\xi+k), w(\xi+m)\rangle = 0 \qquad \text{for any } k \neq m \in \mathbb{Z}^n ,$$

hence

$$w(\xi+m) = 0 \qquad \text{for all } m \in \mathbb{Z}^n \setminus \{m_1, \ldots, m_L\} . \tag{3.8}$$

Since the vectors $w(\xi+m_l)$ are linearly independent by Theorem 3, hence by (3.6)

$$v(\xi+m) = 0 \qquad \text{for all } m \in \mathbb{Z}^n \setminus \{m_1, \ldots, m_L\} . \tag{3.9}$$

Therefore, (3.3) holds and the set K satisfies (2.3). For $j \in \mathbb{Z}$ define spaces W_j by (2.2), and $\tilde{W}_j$ by

$$\tilde{W}_j = \overline{\operatorname{span}}\left\{\phi^l_{j,k} : k \in \mathbb{Z}^n,\ l = 1, \ldots, L\right\} .$$

By (3.8), (3.9), and

$$\operatorname{span}\{v(\xi+m_l) : l = 1, \ldots, L\} = \operatorname{span}\{w(\xi+m_l) : l = 1, \ldots, L\} = \mathbb{C}^L ,$$

we have

$$\begin{aligned} J(\xi) &= \operatorname{span}\left\{\mathcal{T}\psi^l(\xi) : l = 1, \ldots, L\right\} \\ &= \operatorname{span}\left\{\mathcal{T}\phi^l(\xi) : l = 1, \ldots, L\right\} = \operatorname{span}\{e_l : \xi + l \in K\} . \end{aligned} \tag{3.10}$$

Since J is the range function of W_0, $\tilde{W}_0$, and $\check{L}^2(K)$ we must have $W_0 = \tilde{W}_0 = \check{L}^2(K)$. Since $X(\Psi)$ is Riesz basis, $W_0 \cap W_j = \{0\}$ for $j \in \mathbb{Z} \setminus \{0\}$, hence $K \cap (A^*)^j K$ is a null set, and hence $W_0 \perp W_j$ for $j \in \mathbb{Z} \setminus \{0\}$. Since $\oplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{R}^n)$ we have (2.4), and therefore K given by (3.3) is a multiwavelet set of order L. This completes the proof of Theorem 5. $\square$

Proof of Theorem 6. Let $K = \bigcup_{l=1}^L K_l$, where K_l's satisfy (2.1) and Ψ, Φ is a biorthogonal multiwavelet supported (in the Fourier domain) on K. Note that by Theorem 3, (3.10) holds. Define $P : \mathbb{T}^n \to GL_L(\mathbb{C})$ by

$$(P(\xi))_{l,m} = \hat{\psi}^m \left(\left(\tau|_{K_l}\right)^{-1} \xi \right) \qquad \text{for } l, m = 1, \dots, L\ .$$

Since $\{\psi^l_{0,k} : k \in \mathbb{Z}^n, l = 1, \dots, L\}$ is a Riesz family with constants a, b then by Theorem 3, (2.1), and (3.10), $\sqrt{a} \le ||(P(\xi)^*)^{-1}||^{-1} \le ||P(\xi)^*|| \le \sqrt{b}$ for a. e. $\xi \in \mathbb{T}^n$. By taking conjugates we have

$$\sqrt{a} \le \left\| P(\xi)^{-1} \right\|^{-1} \le ||P(\xi)|| \le \sqrt{b} \qquad \text{for a. e. } \xi \in \mathbb{T}^n\ , \tag{3.11}$$

which shows (3.5). To show (3.4) we define $Q : \mathbb{T}^n \to GL_L(\mathbb{C})$ by

$$(Q(\xi))_{l,m} = \hat{\phi}^m \left(\left(\tau|_{K_l}\right)^{-1} \xi \right) \qquad \text{for } l, m = 1, \dots, L\ .$$

By (2.1), (3.7), and (3.10) we have that for $m, m' = 1, \dots, L$,

$$\begin{aligned}\sum_{l=1}^L \hat{\psi}^m \left(\left(\tau|_{K_l}\right)^{-1} \xi \right) \overline{\hat{\phi}^{m'}((\tau|_{K_l})^{-1}\xi)} &= \sum_{k\in\mathbb{Z}^n, \xi+k\in K} \hat{\psi}^m(\xi+k)\overline{\hat{\phi}^{m'}(\xi+k)} \\ &= \sum_{k\in\mathbb{Z}^n} \hat{\psi}^m(\xi+k)\overline{\hat{\phi}^{m'}(\xi+k)} = \delta_{m,m'} \text{ for a. e. } \xi\ ,\end{aligned}$$

and hence, $P(\xi)Q(\xi)^* = Id$ for a. e. ξ. Therefore, $Q(\xi) = (P(\xi)^*)^{-1}$, and we obtain (3.4).

Conversely, if $P : \mathbb{T}^n \to GL_L(\mathbb{C})$ is any measurable mapping satisfying (3.11) define Ψ, Φ by (3.4). By Theorem 3, $\{\psi^l_{0,k} : k \in \mathbb{Z}^n, l = 1, \dots, L\}$ ($\{\phi^l_{0,k} : k \in \mathbb{Z}^n, l = 1, \dots, L\}$) is a Riesz family with constants a, b (b^{-1}, a^{-1}). Since $W_j = \tilde{W}_j = \check{L}^2((A^*)^j K)$ for all $j \in \mathbb{Z}$ and (2.4), $X(\Psi)$ and $X(\Phi)$ are Riesz bases for $L^2(\mathbb{R}^n)$ with the constants a, b (b^{-1}, a^{-1}), respectively. By a direct calculation we have (3.7) and therefore (3.2). This shows that Ψ, Φ is a biorthogonal multiwavelet, and completes the proof of Theorem 6. $\square$

As an immediate consequence of Theorems 5 and 6 we conclude that a biorthogonal multiwavelet Ψ, Φ associated to a dilation (3.1) must satisfy

$$a\mathbf{1}_K(\xi) \le \sum_{l=1}^L \left|\hat{\psi}^l(\xi)\right|^2 \le b\mathbf{1}_K(\xi) \qquad \text{for a. e. } \xi \in \mathbb{R}^n\ ,$$

$$b^{-1}\mathbf{1}_K(\xi) \le \sum_{l=1}^L \left|\hat{\phi}^l(\xi)\right|^2 \le a^{-1}\mathbf{1}_K(\xi) \qquad \text{for a. e. } \xi \in \mathbb{R}^n\ ,$$

for some $0 < a \le b < \infty$ and a multiwavelet set K of order L. As a corollary, such biorthogonal multiwavelets can not be well localized in the space domain.

A special case of Theorem 5 is the following.

Corollary 1.

If Ψ is an orthonormal multiwavelet associated with a dilation A satisfying (3.1) *then necessarily Ψ is a combined MSF multiwavelet.*

Remarks.

1) The presented proof of Theorem 5 developed the necessity of the characterization equation of Theorem 1. It is worth noting that there is a direct (but longer) proof which uses only orthogonality condition and density of $\tau((A^*)^j\mathbb{Z}^n)$ in $\mathbb{T}^n$ for all $j \geq 1$, see [5, Theorem 3.6 in Chapter 2]. The density condition is equivalent (via the Weyl Criterion of uniform distribution mod 1) to (3.1), which in turn says that rows of A^j (treated as vectors in $\mathbb{R}^n$) together with the standard basis vectors $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ are linearly independent over $\mathbb{Q}$, see [5]. However, the use of characterization Equation (1.2) gives a possibility to investigate dilations A which do not satisfy (3.1).

2) A natural question is whether Corollary 1 has a converse. Namely, is it true that if all wavelets associated with a certain expansive dilation A are MSF then A has to satisfy (3.1)? Speegle and the author [7] have given a positive answer to this question by constructing non-MSF wavelets for any dilation A such that $\mathbb{Z}^n \cap (A^*)^j\mathbb{Z}^n \neq \{0\}$ for some $j \in \mathbb{Z} \setminus \{0\}$. This shows that Theorem 5 and Corollary 1 are sharp, i.e., the condition (3.1) can not be replaced by a weaker assumption.

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