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ON CHARACTERIZATIONS OF MULTIWAVELETS IN $L^2(\mathbb{R}^n)$

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ABSTRACT. We present a new approach to characterizing (multi)wavelets by means of basic equations in the Fourier domain. Our method yields an uncomplicated proof of the two basic equations and a new characterization of orthonormality and completeness of (multi)wavelets.

1. INTRODUCTION

An *orthonormal wavelet* is a function $\psi \in L^2(\mathbb{R})$ such that the system $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, where

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \qquad j,k \in \mathbb{Z}.$$

The following result characterizes wavelets in terms of two basic equations; see section 7.1 in [HW].

Theorem 1.1. *A function $\psi \in L^2(\mathbb{R})$ with $||\psi||_2 = 1$ is an orthonormal wavelet if and only if*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}, \tag{1.1}$$

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi+s))} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R},\ s \in 2\mathbb{Z}+1. \tag{1.2}$$

In fact, equations (1.1) and (1.2) alone characterize the system $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ being a tight frame with constant 1 for $L^2(\mathbb{R})$. On the other hand, the orthonormality of the system $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is equivalent to

$$\sum_{k\in\mathbb{Z}} \hat{\psi}(\xi+k)\overline{\hat{\psi}(2^j(\xi+k))} = \delta_{j,0} \qquad \text{for a.e. } \xi \in \mathbb{R},\ j \geq 0. \tag{1.3}$$

For the proof see section 3.1 in [HW] or [HKLS]. In [HKLS] it is also shown that a function ψ is an orthonormal wavelet if and only if (1.1), (1.2) and (1.3) hold. This

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is an easy consequence of Theorem 1.1 and the orthonormality of $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ $\iff$ (1.3). Equation (1.3) is often written as two separate equations:

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+k)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}, \tag{1.3a}$$

$$\sum_{k\in\mathbb{Z}} \hat{\psi}(\xi+k)\overline{\hat{\psi}(2^j(\xi+k))} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R},\ j \geq 1. \tag{1.3b}$$

In this note we present a new approach based on general results about shift invariant systems in [RS1] and [B2], and quasi affine systems in [CSS]. As a result we give an alternative characterization of wavelets in which the orthonormality condition is explicit.

Theorem 1.2. *Suppose $\psi \in L^2(\mathbb{R})$. Then the following are equivalent:*

(i) *ψ is an orthonormal wavelet,*
(ii) *ψ satisfies* (1.1) *and* (1.3),
(iii) *ψ satisfies* (1.3) *and*

$$\int_{\mathbb{R}} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} = 2\ln 2. \tag{1.4}$$

Paraphrasing Theorem 1.2 we can say that a necessary and sufficient condition for the orthonormal system $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ to be complete is (1.4). Implications (i) $\implies$ (ii) $\implies$ (iii) are clear. The hard part is to show (iii) $\implies$ (i). The equivalence of (i) and (ii) was conjectured by Guido Weiss and was first shown by the author. An elementary proof was later found by Ziemowit Rzeszotnik; see [Rz]. Notice that conditions (1.2) and (1.3) are too weak to characterize wavelets, e.g. consider function ψ given by $\hat{\psi} = \mathbf{1}_{[1,2]}$.

Since our results hold in $\mathbb{R}^n$ with general dilation matrices we first establish necessary terminology.

Assume we have a *dilation* matrix A preserving $\mathbb{Z}^n$, i.e., A is an n by n integer matrix and all eigenvalues λ of A satisfy $|\lambda| > 1$. Let Ψ be a finite family of functions $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$. The *affine system* (resp. *quasi affine system*) generated by Ψ and associated with A is the collection

$$X(\Psi) = \{\psi^l_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, \dots, L\}, \tag{1.5}$$

$$X^q(\Psi) = \{\tilde{\psi}^l_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, \dots, L\}, \tag{1.6}$$

where for $\psi \in L^2(\mathbb{R}^n)$ we use the convention

$$\psi_{j,k}(x) = D_{A^j}T_k\psi(x) = |\det A|^{j/2}\psi(A^jx - k), \qquad j \in \mathbb{Z}, k \in \mathbb{Z}^n,$$

$$\tilde{\psi}_{j,k}(x) = \begin{cases} D_{A^j}T_k\psi(x) = |\det A|^{j/2}\psi(A^jx-k), & j \geq 0, k \in \mathbb{Z}^n, \\ |\det A|^{j/2}T_kD_{A^j}\psi(x) = |\det A|^{j}\psi(A^j(x-k)), & j < 0, k \in \mathbb{Z}^n, \end{cases}$$

where $T_yf(x) = f(x-y)$ is translation by the vector $y \in \mathbb{R}^n$ and $D_Af(x) = \sqrt{|\det A|}f(Ax)$ is dilation by the matrix A.

We say $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$ is a *multiwavelet* if $X(\Psi)$ is an orthonormal basis of $L^2(\mathbb{R}^n)$.

Definition 1.3. $X \subset L^2(\mathbb{R}^n)$ is a *Bessel family* if there exists $b > 0$ so that

$$\sum_{\eta\in X} |\langle f,\eta\rangle|^2 \leq b||f||^2 \qquad \text{for } f \in L^2(\mathbb{R}^n). \tag{1.7}$$

If in addition there exist $0 < a \leq b$ so that

$$a||f||^2 \leq \sum_{\eta \in X} |\langle f, \eta \rangle|^2 \leq b||f||^2 \qquad \text{for } f \in L^2(\mathbb{R}^n), \tag{1.8}$$

then X is a *frame*. The frame is *tight* if a, b can be chosen so that $a = b$. The (quasi) affine system $X(\Psi)$ (resp. $X^q(\Psi)$) is a *(quasi) affine frame* if (1.8) holds for $X = X(\Psi)$ ($X = X^q(\Psi)$).

The concepts of affine and quasi affine frames are closely related. This was observed by Ron and Shen in [RS2] under some decay assumptions and proved by Chui, Shi and Stöckler in full generality in [CSS].

Theorem 1.4. *Suppose $\Psi \subset L^2(\mathbb{R}^n)$. Then:*

(i) *$X(\Psi)$ is a Bessel family if and only if $X^q(\Psi)$ is a Bessel family. Furthermore, their exact upper bounds are equal.*
(ii) *$X(\Psi)$ is an affine frame if and only if $X^q(\Psi)$ is a quasi affine frame. Furthermore, their lower and upper exact bounds are equal.*

Definition 1.5. For a given family of vectors $\{t_i : i \in \mathbb{N}\} \subset l^2(\mathbb{Z}^n)$ consider the operator $H : l^2(\mathbb{Z}^n) \to l^2(\mathbb{N})$ defined by

$$H(v) = (\langle v, t_i \rangle)_{i \in \mathbb{N}} \qquad \text{for } v = (v(k))_{k \in \mathbb{Z}^n} \in l^2(\mathbb{Z}^n). \tag{1.9}$$

If H is bounded, then the *dual Gramian* of $\{t_i : i \in \mathbb{N}\}$ is the operator $\tilde{G} : l^2(\mathbb{Z}^n) \to l^2(\mathbb{Z}^n)$ given by $\tilde{G} = H^\star H$.

Note that $\tilde{G}$ is a non-negative definite operator on $l^2(\mathbb{Z}^n)$. Moreover, for $k, p \in \mathbb{Z}^n$

$$\langle \tilde{G} e_k, e_p \rangle = \langle H e_k, H e_p \rangle = \sum_{i \in \mathbb{N}} t_i(k) \overline{t_i(p)}, \tag{1.10}$$

where $\{e_k : k \in \mathbb{Z}^n\}$ is the standard basis of $l^2(\mathbb{Z}^n)$. By the Cauchy-Schwarz inequality the entries of the matrix $\tilde{G}$ in (1.10) are meaningfully defined if the series $\sum_{i \in \mathbb{N}} |t_i(k)|^2 < \infty$ for all $k \in \mathbb{Z}^n$. If the matrix $(\sum_{i \in \mathbb{N}} t_i(k)\overline{t_i(p)})_{k \in \mathbb{N}, p \in \mathbb{N}}$ represents a bounded operator on $l^2(\mathbb{Z}^n)$, then the operator H given by (1.9) is bounded. Therefore, H is bounded if and only if $\tilde{G}$ is bounded.

The following result due to Ron and Shen [RS1] (see also Theorem 2.5 in [B2]) characterizes when the system of translates of a given family of functions is a frame in terms of the dual Gramian. Here we identify $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ with its fundamental domain, that is $\mathbb{T}^n = [-1/2, 1/2)^n$. The Fourier transform is given by

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, y \rangle} dx.$$

Theorem 1.6. *Suppose $\{\varphi_i : i \in \mathbb{N}\} \subset L^2(\mathbb{R}^n)$. Then for a.e. $\xi \in \mathbb{T}^n$, let $\tilde{G}(\xi)$ denote the dual Gramian of $\{t_i = (\hat{\varphi}_i(\xi + k))_{k \in \mathbb{Z}^n} : i \in \mathbb{N}\} \subset l^2(\mathbb{Z}^n)$. The system of translates $\{T_k \varphi_i : k \in \mathbb{Z}^n, i \in \mathbb{N}\}$ is a frame for $L^2(\mathbb{R}^n)$ with constants a, b if and only if $\tilde{G}(\xi)$ is bounded for a.e. $\xi \in \mathbb{T}^n$ and*

$$a||v||^2 \leq \langle \tilde{G}(\xi) v, v \rangle \leq b||v||^2 \qquad \text{for } v \in l^2(\mathbb{Z}^n), \text{ a.e. } \xi \in \mathbb{T}^n, \tag{1.11}$$

i.e., the spectrum of $\tilde{G}(\xi)$ is contained in $[a, b]$ for a.e. $\xi \in \mathbb{T}^n$.

Theorem 1.6 still holds if $a = 0$, and then characterizes the system of translates $\{T_k \varphi_i : k \in \mathbb{Z}^n, i \in \mathbb{N}\}$ being a Bessel family with constant b.

Finally we need the notion of a quasi-norm associated with a dilation B.

Definition 1.7. A *quasi-norm* associated with a dilation B is a measurable mapping $\rho : \mathbb{R}^n \to [0,\infty)$, so that

(i) $\rho(\xi) = 0 \iff \xi = 0$,
(ii) $\rho(B\xi) = |\det B|\rho(\xi)$ for all $\xi \in \mathbb{R}^n$,
(iii) there is $c > 0$ so that $\rho(\xi + \zeta) \le c(\rho(\xi) + \rho(\zeta))$ for all $\xi, \zeta \in \mathbb{R}^n$.

Lemarié-Rieusset [LR] shows how to construct ρ which is C^∞ on $\mathbb{R}^n \setminus \{0\}$. An elementary argument in [B3] shows that for every $r > 1$ we have

$$0 < \inf_{1/r<|\xi|<r} \rho(\xi) \le \sup_{1/r<|\xi|<r} \rho(\xi) < \infty. \tag{1.12}$$

Given a quasi-norm ρ we define its *characteristic number* $\kappa(\rho)$ by

$$\kappa(\rho) = \int_{\mathbb{R}^n} \frac{\mathbf{1}_D(\xi)}{\rho(\xi)} d\xi, \tag{1.13}$$

where $D \subset \mathbb{R}^n$ is a measurable set such that $\{B^j D : j \in \mathbb{Z}\}$ partitions $\mathbb{R}^n$ (modulo sets of measure zero), i.e., $\bigcup_{j\in\mathbb{Z}} B^j D = \mathbb{R}^n$ and $B^i D \cap B^j D = \emptyset$ for $i \ne j \in \mathbb{Z}$. The number $\kappa(\rho)$ does not depend on the choice of D. Indeed, if D' is another set such that $\{B^j D' : j \in \mathbb{Z}\}$ partitions $\mathbb{R}^n$, then $\{D' \cap B^i D : i \in \mathbb{Z}\}$ partitions D' and $\{D \cap B^i D' : i \in \mathbb{Z}\}$ partitions D. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\mathbf{1}_{D'}(\xi)}{\rho(\xi)} d\xi &= \sum_{i\in\mathbb{Z}} \int_{\mathbb{R}^n} \frac{\mathbf{1}_{D'\cap B^i D}(\xi)}{\rho(\xi)} d\xi = \sum_{i\in\mathbb{Z}} \int_{\mathbb{R}^n} \frac{\mathbf{1}_{D'\cap B^i D}(B^i\xi)}{\rho(B^i\xi)} |\det B|^i d\xi \\ &= \sum_{i\in\mathbb{Z}} \int_{\mathbb{R}^n} \frac{\mathbf{1}_{(B^{-i}D')\cap D}(\xi)}{\rho(\xi)} d\xi = \int_{\mathbb{R}^n} \frac{\mathbf{1}_D(\xi)}{\rho(\xi)} d\xi = \kappa(\rho).\end{aligned}$$

Furthermore, by choosing D such that $D \subset \{\xi : 1/r < |\xi| < r\}$ for some $r > 1$ we conclude that the characteristic number $\kappa(\rho)$ is always finite by (1.12). For example, if $B = 2Id$ in dimension n, then $\rho(\xi) = |\xi|^n$ is its quasi-norm and by choosing $D = \{\xi : 1 < |\xi| < 2\}$,

$$\kappa(|\cdot|^n) = \int_{\mathbb{R}^n} \frac{\mathbf{1}_D(\xi)}{|\xi|^n} d\xi = \int_0^\infty \int_{S^{n-1}} \frac{\mathbf{1}_{(1,2)}(r)}{r^n} d\sigma r^{n-1} dr = \sigma(S^{n-1}) \ln 2,$$

where σ is the induced Lebesgue measure on $S^{n-1} = \{\xi \in \mathbb{R}^n : |\xi| = 1\}$.

2. Main result

In this general setting Theorem 1.1 takes the following form.

Theorem 2.1. *Suppose $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$. The affine system $X(\Psi)$ associated with a dilation A is a tight frame with constant 1 for $L^2(\mathbb{R}^n)$, i.e.,*

$$||f||^2 = \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\langle f, \psi^l_{j,k}\rangle|^2 \qquad \textit{for all } f \in L^2(\mathbb{R}^n)$$

if and only if

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^l(B^j\xi)|^2 = 1 \qquad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{2.1}$$

and

(2.2)
$$t_s(\xi) := \sum_{l=1}^{L}\sum_{j=0}^{\infty} \hat{\psi}^l(B^j\xi)\overline{\hat{\psi}^l(B^j(\xi+s))} = 0 \qquad \text{for a.e. } \xi \in \mathbb{R}^n,\ s \in \mathbb{Z}^n \setminus B\mathbb{Z}^n,$$

where $B = A^T$. In particular, Ψ is a multiwavelet if and only if (2.1), (2.2), *and* $||\psi^l||_2 = 1$ *for* $l = 1, \dots, L$.

The direct, but long proofs of this result are in [C2] and [B1]. This theorem is also shown in [RS2] under some decay conditions on Ψ and for special dilations in [FGWW] and [HW]. In this note we will present the new characterization of multiwavelets in Theorem 2.4. Our methods will yield a quick proof of Theorem 2.1 based on Theorems 1.4, 1.6 and Lemma 2.3. Before that, note that the orthonormality of affine systems can be characterized using the following lemma.

Lemma 2.2. *Suppose $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$. The affine system $X(\Psi)$ associated with a dilation A is orthonormal in $L^2(\mathbb{R}^n)$ if and only if*

(2.3)
$$\sum_{k\in\mathbb{Z}^n} \hat{\psi}^l(\xi+k)\overline{\hat{\psi}^{l'}(B^j(\xi+k))} = \delta_{l,l'}\delta_{j,0} \qquad \text{a.e. } \xi \in \mathbb{R}^n \text{ for } j \geq 0,\ l,l' = 1,\dots,L,$$

where $B = A^T$.

Proof. By a simple change of variables

$$\langle \psi^l_{j,k}, \psi^{l'}_{j',k'}\rangle = \delta_{l,l'}\delta_{j,j'}\delta_{k,k'} \qquad \text{for } j,j' \in \mathbb{Z}, k,k' \in \mathbb{Z}^n, l,l' = 1,\dots,L,$$

is equivalent to

$$\langle \psi^l_{j,k}, \psi^{l'}_{0,0}\rangle = \delta_{l,l'}\delta_{j,0}\delta_{k,0} \qquad \text{for } j \geq 0, k \in \mathbb{Z}^n, l,l' = 1,\dots,L.$$

Take any $j \geq 0, k \in \mathbb{Z}^n, l, l' = 1, \dots, L$. By Plancherel's formula

$$\begin{aligned}\delta_{l,l'}\delta_{j,0}\delta_{k,0} = \langle \hat{\psi}^l_{j,k}, \hat{\psi}^{l'}_{0,0}\rangle &= \int_{\mathbb{R}^n} q^{-j/2}\hat{\psi}^l(B^{-j}\xi)e^{-2\pi i\langle k,B^{-j}\xi\rangle}\overline{\hat{\psi}^{l'}(\xi)}d\xi \\ = \int_{\mathbb{R}^n} q^{j/2}\hat{\psi}^l(\xi)e^{-2\pi i\langle k,\xi\rangle}\overline{\hat{\psi}^{l'}(B^j\xi)}d\xi &= \sum_{l\in\mathbb{Z}^n} q^{j/2}\int_{l+\mathbb{T}^n} \hat{\psi}^l(\xi)\overline{\hat{\psi}^{l'}(B^j\xi)}e^{-2\pi i\langle k,\xi\rangle}d\xi \\ = q^{j/2}\int_{\mathbb{T}^n}\left[\sum_{l\in\mathbb{Z}^n}\hat{\psi}^l(\xi+l)\overline{\hat{\psi}^{l'}(B^j(\xi+l))}\right]e^{-2\pi i\langle k,\xi\rangle}d\xi &= q^{j/2}\int_{\mathbb{T}^n} K(\xi)e^{-2\pi i\langle k,\xi\rangle},\end{aligned}$$

where $q = |\det A|$, and K denotes the expression in the bracket. The interchange of summation and integration is justified by

$$\begin{aligned}\int_{\mathbb{T}^n}\sum_{l\in\mathbb{Z}^n}|\hat{\psi}^l(\xi+l)||\hat{\psi}^{l'}(B^j(\xi+l))|d\xi &= \int_{\mathbb{R}^n}|\hat{\psi}^l(\xi)||\hat{\psi}^{l'}(B^j\xi)|d\xi \\ &\leq q^{-j/2}||\psi^l||^2||\psi^{l'}||^2 < \infty.\end{aligned}$$

The above computation shows that all Fourier coefficients of $K(\xi) \in L^1(\mathbb{T}^n)$ are zero except for the coefficient corresponding to $k = 0$ which is 1 if $j = 0$ and $l = l'$. Therefore, $K(\xi) = \delta_{j,0}\delta_{l,l'}$ for a.e. $\xi \in \mathbb{T}^n$. ∎

Suppose we have $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$. For $j \geq 0$, let $\mathcal{D}_j$ denote a set of q^j representatives of distinct cosets of $\mathbb{Z}^n / A^j \mathbb{Z}^n$, where $q = |\det A|$. For $j < 0$ we define $\mathcal{D}_j = \{0\}$. Since the quasi affine system $X^q(\Psi)$ is invariant under shifts by $k \in \mathbb{Z}^n$, we have

$$X^q(\Psi) = \{T_k \varphi : k \in \mathbb{Z}^n, \varphi \in \mathcal{A}\}, \quad \mathcal{A} := \{\tilde{\psi}^l_{j,d} : j \in \mathbb{Z}, d \in \mathcal{D}_j, l = 1, \dots, L\}. \tag{2.4}$$

The dual Gramian $\tilde{G}(\xi)$ of the quasi affine system $X^q(\Psi)$ at $\xi \in \mathbb{T}^n$ is defined as the dual Gramian of $\{(\hat{\varphi}(\xi + k))_{k \in \mathbb{Z}^n} : \varphi \in \mathcal{A}\} \subset l^2(\mathbb{Z}^n)$, where $\mathcal{A}$ is given in (2.4). We now show that $\tilde{G}(\xi)$ does not depend on a choice of representatives $\mathcal{D}_j$ and can be computed explicitly in terms of the Fourier transforms of functions in Ψ.

Lemma 2.3. *Suppose $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$. The dual Gramian $\tilde{G}(\xi)$ of $X^q(\Psi)$ at $\xi \in \mathbb{T}^n$ is equal to*

$$\langle \tilde{G}(\xi) e_k, e_k \rangle = \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^l(B^j(\xi + k))|^2 \qquad \text{for } k \in \mathbb{Z}^n, \tag{2.5}$$

$$\langle \tilde{G}(\xi) e_k, e_p \rangle = t_{B^{-m}(p-k)}(B^{-m}\xi + B^{-m}k) \qquad \text{for } k \neq p \in \mathbb{Z}^n, \tag{2.6}$$

where $B = A^T$, $m = \max\{j \in \mathbb{Z} : B^{-j}(p-k) \in \mathbb{Z}^n\}$, and functions t_s, $s \in \mathbb{Z}^n \setminus B\mathbb{Z}^n$, are given by (2.2).

Proof. By (1.10) and (2.4) we have for $k, p \in \mathbb{Z}^n$

$$\langle \tilde{G}(\xi) e_k, e_p \rangle = \sum_{\varphi \in \mathcal{A}} \hat{\varphi}(\xi + k) \overline{\hat{\varphi}(\xi + p)} = \sum_{l=1}^{L} \sum_{j<0} \hat{\psi}^l(B^{-j}(\xi + k)) \overline{\hat{\psi}^l(B^{-j}(\xi + p))}$$

$$+ \sum_{l=1}^{L} \sum_{j \geq 0} \hat{\psi}^l(B^{-j}(\xi + k)) \overline{\hat{\psi}^l(B^{-j}(\xi + p))} \Bigg[\sum_{d \in \mathcal{D}_j} |\det A|^{-j} e^{-2\pi i \langle d, B^{-j}(k-p) \rangle} \Bigg]$$

$$= \sum_{l=1}^{L} \sum_{j=-\infty}^{m} \hat{\psi}^l(B^{-j}(\xi + k)) \overline{\hat{\psi}^l(B^{-j}(\xi + p))},$$

where $m = \max\{j \in \mathbb{Z} : k - p \in B^j \mathbb{Z}^n\}$, i.e., m is the unique integer so that $B^{-m}(k-p) \in \mathbb{Z}^n \setminus B\mathbb{Z}^n$, and $m = \infty$ when $k = p$. Indeed, by Lemma 1 in [M] (see also [GH]) the expression in the bracket equals 1 if $k - p \in B^j \mathbb{Z}^n$ and 0 otherwise. Therefore, if $k = p$, then

$$\langle \tilde{G}(\xi) e_k, e_k \rangle = \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} |\hat{\psi}^l(B^j(\xi + k))|^2.$$

If $k \neq p$, then

$$\langle \tilde{G}(\xi) e_k, e_p \rangle = \sum_{l=1}^{L} \sum_{j \geq 0} \hat{\psi}^l(B^{j-m}(\xi + k)) \overline{\hat{\psi}^l(B^{j-m}(\xi + p))}$$

$$= \sum_{l=1}^{L} \sum_{j \geq 0} \hat{\psi}^l(B^j(B^{-m}\xi + B^{-m}k)) \overline{\hat{\psi}^l(B^j(B^{-m}\xi + B^{-m}k + B^{-m}(p-k)))}$$

$$= t_{B^{-m}(p-k)}(B^{-m}\xi + B^{-m}k),$$

where t_s is defined by (2.2). $\square$

Proof of Theorem 2.1. By Theorem 1.4, $X(\Psi)$ is a tight frame with constant 1 if and only if $X^q(\Psi)$ is. By Theorem 1.6, this is equivalent to the spectrum of $\tilde{G}(\xi)$ consisting of single point 1, i.e., $\tilde{G}(\xi)$ is the identity on $l^2(\mathbb{Z}^n)$ for a.e. $\xi \in \mathbb{T}^n$. This in turn is equivalent to (2.1) and (2.2) by Lemma 2.3. By Theorem 1.8, section 7.1 in [HW], a tight frame $X(\Psi)$ is an orthonormal basis if and only if $||\psi^l||_2 = 1$ for $l = 1, \dots, L$. □

The main result of this note is a new characterization of multiwavelets.

Theorem 2.4. *Suppose* $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$. *Then the following are equivalent:*

(i) Ψ *is a multiwavelet associated with a dilation* A.

(ii) Ψ *satisfies*

$$\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^l(B^j\xi)|^2 = 1 \qquad \textit{for a.e. } \xi \in \mathbb{R}^n, \tag{2.1}$$

$$\sum_{k\in\mathbb{Z}^n} \hat{\psi}^l(\xi+k)\overline{\hat{\psi}^{l'}(B^j(\xi+k))} = \delta_{l,l'}\delta_{j,0} \qquad \textit{for } j \geq 0,\ l, l' = 1, \dots, L, \textit{a.e. } \xi \in \mathbb{R}^n, \tag{2.3}$$

where $B = A^T$.

(iii) Ψ *satisfies* (2.3) *and*

$$\sum_{l=1}^{L} \int_{\mathbb{R}^n} |\hat{\psi}^l(\xi)|^2 \frac{d\xi}{\rho(\xi)} = \kappa(\rho), \tag{2.7}$$

for some (or any) quasi-norm ρ *associated with* B.

Recall that $X(\Psi)$ is an orthonormal system (not necessarily complete) if and only if (2.3) holds. Condition (2.1), or even weaker (2.7), guarantees that this system is complete. This result is a consequence of a more general result.

Theorem 2.5. *Suppose* $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$. *Assume that* $X(\Psi)$ *is a Bessel family with constant* 1. *Then the following are equivalent:*

(i) $X(\Psi)$ *is a tight frame with constant* 1,

(ii) (2.1) *holds,*

(iii) (2.7) *holds.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are immediate. If $X(\Psi)$ is a tight frame with constant 1, then by Theorem 2.1 (2.1) holds. If in turn (2.1) holds, then

$$\begin{aligned}\sum_{l=1}^{L} \int_{\mathbb{R}^n} |\hat{\psi}^l(\xi)|^2 \frac{d\xi}{\rho(\xi)} &= \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \int_{B^jD} |\hat{\psi}^l(\xi)|^2 \frac{d\xi}{\rho(\xi)} \\ &= \sum_{l=1}^{L} \int_{D} \sum_{j\in\mathbb{Z}} |\hat{\psi}^l(B^j\xi)|^2 \frac{d\xi}{\rho(\xi)} = \kappa(\rho),\end{aligned}$$

where $D \subset \mathbb{R}^n$ is such that $\{B^jD : j \in \mathbb{Z}\}$ partitions $\mathbb{R}^n$ (modulo sets of measure zero).

To prove (iii) $\Longrightarrow$ (i) we assume (2.7). Since $X(\Psi)$ is a Bessel family with constant 1 then $X^q(\Psi)$ is also a Bessel family with constant 1 by Theorem 1.4(i). Let $\tilde{G}(\xi)$ be the dual Gramian of $X^q(\Psi)$ at $\xi \in \mathbb{T}^n$. Since $X^q(\Psi)$ is a Bessel family with constant 1 we have $||\tilde{G}(\xi)|| \le 1$ for a.e. $\xi \in \mathbb{T}^n$ by Theorem 1.6. In particular, $||\tilde{G}(\xi)e_k|| \le 1$. Hence

(2.8)
$$1 \ge ||\tilde{G}(\xi)e_k||^2 = \sum_{p\in\mathbb{Z}^n} |\langle \tilde{G}(\xi)e_k, e_p\rangle|^2 = |\langle \tilde{G}(\xi)e_k, e_k\rangle|^2 + \sum_{p\in\mathbb{Z}^n, p\neq k} |\langle \tilde{G}(\xi)e_k, e_p\rangle|^2.$$

By Lemma 2.3

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(B^j(\xi+k))|^2 \le 1 \qquad \text{for } k\in\mathbb{Z}^n, \text{ a.e. } \xi\in\mathbb{T}^n.$$

Since

$$\kappa(\rho) = \int_D \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(B^j\xi)|^2 \frac{d\xi}{\rho(\xi)} \le \int_D \frac{d\xi}{\rho(\xi)} = \kappa(\rho),$$

we have $\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(B^j\xi)|^2 = 1$ for a.e. $\xi \in D$ and hence for a.e. $\xi \in \mathbb{R}^n$, i.e., (2.1) holds. By Lemma 2.3, $|\langle \tilde{G}(\xi)e_k, e_k\rangle|^2 = 1$ for all $k \in \mathbb{Z}^n$, a.e. $\xi \in \mathbb{T}^n$. By (2.8) $\langle \tilde{G}(\xi)e_k, e_p\rangle = 0$ for $k \neq p$, and $\tilde{G}(\xi)$ is an identity on $l^2(\mathbb{Z}^n)$ for a.e. $\xi \in \mathbb{T}^n$. Hence by Theorem 1.6, $X^q(\Psi)$ is a tight frame with constant 1. So is $X(\Psi)$ by Theorem 1.4. □

Proof of Theorem 2.4. The implications (i) $\implies$ (ii) $\implies$ (iii) are immediate by Lemma 2.2 and Theorem 2.5.

To prove (iii) $\implies$ (i) assume (2.3) and (2.7). (2.3) implies $X(\Psi)$ is a Bessel family with constant 1. By Theorem 2.5, this and (2.7) implies $X(\Psi)$ is a tight frame with constant 1. Since $||\psi^l||_2 = 1$ for $l = 1, \dots, L$, we have $X(\Psi)$ is an orthonormal basis for $L^2(\mathbb{R}^n)$, i.e., Ψ is a multiwavelet. □

3. Final remarks

In some circumstances the new characterization of wavelets is advantageous over the classical characterization. In this section we present an interesting application of Theorem 2.4 to provide a quick proof of the completeness theorem due to Garrigós and Speegle [GS]. We are going to show this result in greater generality.

Given a dilation A and an integer $L \ge 1$ consider

$$\mathcal{W}_{A,L} = \mathcal{W} = \{(\psi^1, \dots, \psi^L) \in L^2(\mathbb{R}^n)^{\oplus L} : \{\psi^1, \dots, \psi^L\} \text{ is a multiwavelet}\}.$$

There are two natural metrics d_1, d_2 on the set $\mathcal{W}$,

$$d_1(\Psi,\Phi) = \left(\sum_{l=1}^{L} ||\hat{\psi}^l - \hat{\phi}^l||^2_{L^2(\mathbb{R}^n, d\xi)}\right)^{1/2},$$
$$d_2(\Psi,\Phi) = \left(\sum_{l=1}^{L} ||\hat{\psi}^l - \hat{\phi}^l||^2_{L^2(\mathbb{R}^n, \rho(\xi)^{-1}d\xi)}\right)^{1/2},$$

where $\Psi = (\psi^1, \dots, \psi^L), \Phi = (\phi^1, \dots, \phi^L) \in \mathcal{W}$, and ρ is a quasi-norm associated with the dilation $B = A^T$. Finally define the metric d on $\mathcal{W}$ by

$$d(\Psi,\Phi) = d_1(\Psi,\Phi) + d_2(\Psi,\Phi).$$

Theorem 3.1. *The metrics d_1, d_2 and d are topologically equivalent on $\mathcal{W}$. Moreover, $(\mathcal{W}, d)$ is a complete metric space.*

Proof. Suppose we have a sequence $(\Psi_i)_{i\in\mathbb{N}} \subset \mathcal{W}$ converging to Ψ in d_1. In particular, for $l = 1, \dots, L$

$$\langle \hat{\psi}_i^l, g\rangle_{L^2(\mathbb{R}^n, d\xi)} \to \langle \hat{\psi}^l, g\rangle_{L^2(\mathbb{R}^n, d\xi)} \quad \text{as } i \to \infty,$$

for all $g \in L^\infty$ with bounded and bounded away from zero support, i.e., there is $r > 1$ such that $g(\xi) \neq 0 \implies 1/r < |\xi| < r$. Since $||\hat{\psi}_i^l||_{L^2(\mathbb{R}^n, \rho(\xi)^{-1}d\xi)} \leq \kappa(\rho)^{1/2}$, $\hat{\psi}_i^l \to \hat{\psi}^l$ weakly in $L^2(\mathbb{R}^n, \rho(\xi)^{-1}d\xi)$ as $i \to \infty$. Since

$$||\hat{\Psi}_i||^2_{L^2(\mathbb{R}^n, \rho(\xi)^{-1}d\xi)^{\oplus L}} = ||\hat{\Psi}||^2_{L^2(\mathbb{R}^n, \rho(\xi)^{-1}d\xi)^{\oplus L}} = \sum_{l=1}^{L} ||\hat{\psi}^l||^2_{L^2(\mathbb{R}^n, \rho(\xi)^{-1}d\xi)} = \kappa(\rho),$$

we also have $\hat{\Psi}_i \to \hat{\Psi}$ in $L^2(\mathbb{R}^n, \rho(\xi)^{-1}d\xi)^{\oplus L}$, hence in the metric d_2. Conversely, we can show that convergence in d_2 implies convergence in d_1.

To see that the metric d is complete, take a Cauchy sequence $(\Psi_i)_{i\in\mathbb{N}} \subset \mathcal{W}$. By the completeness of L^2 there are G_1 and G_2 such that $\hat{\Psi}_i \to G_1$ in $L^2(\mathbb{R}^n, d\xi)^{\oplus L}$, and $\hat{\Psi}_i \to G_2$ in $L^2(\mathbb{R}^n, \rho(\xi)^{-1}d\xi)^{\oplus L}$ as $i \to \infty$. By the argument with the weak convergence as above we must have $G_1 = G_2$. Since the affine system $X(\Psi_i)$ is an orthonormal system for every $i \in \mathbb{N}$ so is $X(\Psi)$, where $\hat{\Psi} = G_1 = G_2$. Since $||\hat{\Psi}||^2_{L^2(\mathbb{R}^n, \rho(\xi)^{-1}d\xi)^{\oplus L}} = \kappa(\rho)$, $X(\Psi)$ is complete by Theorem 2.4. Therefore Ψ is a multiwavelet. □

Remark 3.2. Although the presented results work nominally for the standard lattice $\mathbb{Z}^n$ they can be easily extended to the general lattice $\Gamma = P\mathbb{Z}^n$, where P is an n by n non-degenerate real matrix. In this setup the dilation is an n by n real matrix A such that all eigenvalues λ satisfy $|\lambda| > 1$ and $A\Gamma \subset \Gamma$; see [C1]. The standard considerations involving the unitary operator D_P given by $D_P f(x) = \sqrt{|\det P|}f(Px)$ yield the corresponding results for general lattices; see [B1] for more details.

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