Inverse Volume Inequalities for Matrix Weights

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ABSTRACT. For weights in the matricial Muckenhoupt classes we investigate a number of properties analogous to properties which hold in the scalar Muckenhoupt classes. In contrast to the scalar case we exhibit for each p, $1 < p < \infty$, a matrix weight $W \in \mathcal{A}_{p,q} \setminus \bigcup_{p'<p} \mathcal{A}_{p',q'}$. We also give a necessary and sufficient condition on W in $\mathcal{A}_{p,q}$, a "reverse inverse volume inequality", to ensure that W is in $\mathcal{A}_{p',q'}$ for some $p' < p$.

1. INTRODUCTION

Let H be the Hilbert transform and M be the Hardy-Littlewood maximal function; H is initially defined for smooth compactly supported functions on $\mathbb{R}$. The classical theory of singular integral operators shows that these (and related) operators extend to bounded operators from $L^p(\mathbb{R})$ to itself, for $1 < p < \infty$. In the '70's the theory was expanded to include systematic consideration of weighted L^p spaces. In particular, it was shown by Richard Hunt, Benjamin Muckenhoupt, and Richard Wheeden [M], [HMW] that the necessary and sufficient condition for either H or M to be bounded on $L^p(w(t)\,dt)$ $(1 < p < \infty)$, i.e., that there is a constant C so that

$$\int_{\mathbb{R}} |Hf(t)|^p w(t)\,dt \le C \int_{\mathbb{R}} |f(t)|^p w(t)\,dt \quad \text{for all } f,$$

(and similarily for M) is that w be in the Muckenhoupt class A_p. We say that a positive scalar function w on $\mathbb{R}$ is in the Muckenhoupt A_p class $(1 < p < \infty)$ if and only if there is a constant $C > 0$ so that

$$\langle w^{-1/p} \rangle_{I,q} \le C \langle w^{1/p} \rangle_{I,p}^{-1} \quad \text{for all intervals } I \subset \mathbb{R}, \tag{1.1}$$

where $\langle \varphi \rangle_{I,p}$ is an abbreviation for $(1/|I| \int_I (\varphi(t))^p\,dt)^{1/p}$, and $\langle \varphi \rangle_I = \langle \varphi \rangle_{I,1} = 1/|I| \int_I \varphi(t)\,dt$. These classes are known to play a major role in singular integral theory. We merely mention [CF] and [C] as two milestones, and [GR] as a general, if slightly dated, reference.

There is little difficulty in extending the basic $L^p(\mathbb{R})$ boundedness results from scalar functions to functions valued in $\mathbb{C}^n$ (however, there are interesting obstacles for Banach space valued functions). However, the extension of the weighted norm inequalities to vector valued functions has proven difficult. Informally, the difficulty is this. If the matrix weight, which is a matrix valued function on $\mathbb{R}$ (full definitions will be given later), is diagonal, then the directions decouple, the questions reduce to n independent questions for scalar valued functions, and there are no new difficulties. However, even when $n = 2$, there is a fundamentally new issue which can arise; the frames which diagonalize the weight may, as a function of $t \in \mathbb{R}$, vary wildly. We will see in simple explicit examples that this can produce fundamentally new phenomena. Beyond the simple examples those issues are tightly intertwined, and the full story is complicated.

In spite of the difficulties, substantial progress has been made on these problems in recent years and the correct analog of A_p condition has been given in [NT] and [V].

To introduce matricial weighted norm inequalities consider the Hilbert transform H acting on a finite dimensional complex space, which we identify with $\mathbb{C}^n$, given by

$$Hf(t) = \text{p.v.}\,\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t-s)}{s}\,ds,$$

when $f : \mathbb{R} \to \mathbb{C}^n$ is smooth and compactly supported. Clearly H extends to a bounded operator acting on L^p for $1 < p < \infty$ and even an isometry on L^2, where

$$L^p = \left\{ f : \mathbb{R} \to \mathbb{C}^n \text{ measurable, } \|f\|_{L^p} = \left(\int_{\mathbb{R}} \|f(t)\|^p\,dt \right)^{1/p} < \infty \right\}.$$

A weight is defined as a measurable mapping $\rho : \mathbb{R} \to \{\text{all norms on } \mathbb{C}^n\}$, i.e., for all $x \in \mathbb{C}^n$, $t \mapsto \rho_t(x)$ is measurable. For $1 < p < \infty$ define the weighted space by

$$L^p(\rho) = \left\{ f : \mathbb{R} \to \mathbb{C}^n \text{ measurable, } \|f\|_{L^p(\rho)} = \left(\int_{\mathbb{R}} \rho_t(f(t))^p\,dt \right)^{1/p} < \infty \right\}.$$

If the dimension $n = 1$, we are in the realm of the usual scalar weights. The Hilbert transform H is bounded on $L^p(\rho)$ precisely when $w(t) = \rho_t(1)^p$ belongs to the Muckenhoupt A_p class.

For simplicity we will consider matrix weights, that is measurable maps $W : \mathbb{R} \to \{\text{positive (self-adjoint) matrices on } \mathbb{C}^n\}$, for which we define $L^p(W)$ to be $L^p(\rho)$, where $\rho_t(\cdot) = \|W^{1/p}(t)\cdot\|$. The choice of exponent $1/p$ is not accidental, we want our definition to overlap with the usual scalar weights in the dimension

$n = 1$. Moreover, the restriction of attention to matrix weights doesn't actually limit our problem, because John's theorem asserts that every norm ρ on $\mathbb{C}^n$ is at most $\sqrt{n}$ in the Banach-Mazur distance from the standard Euclidean norm $\|\cdot\|$ (see [P] or [TJ]). Hence there is a positive matrix A s.t. $1/\sqrt{n}\|Ax\| \le \rho(x) \le \|Ax\|$ for all $x \in \mathbb{C}^n$. This reduction significantly simplifies the analysis.

The first result giving a characterization of matrix weights for which the Hilbert transform is bounded on $L^2(W)$ was obtained by Sergei Treil and Alexander Volberg in [TV1]. They proved that H is bounded on $L^2(W)$ iff there exists $C > 0$ so that

$$\langle W^{-1}\rangle_I \le C\langle W\rangle_I^{-1} \quad \text{for all intervals } I \subset \mathbb{R},$$

or equivalently that $\sup_{I\subset\mathbb{R}} \|\langle W^{-1}\rangle_I^{1/2}\langle W\rangle_I^{1/2}\| < \infty$.

For a general $1 < p < \infty$, the following condition was introduced by Fedor Nazarov and Sergei Treil in [NT].

Definition 1.1. We say that a matrix weight W satisfies the $\mathcal{A}_{p,q}$ condition (where $1 < p < \infty$, $1/p + 1/q = 1$) if there is a constant $C > 0$ so that

$$\langle \rho^\star\rangle_{I,q} \le C\langle \rho\rangle_{I,p}^\star \quad \text{for all intervals } I \subset \mathbb{R}, \tag{1.2}$$

where $\rho_t(\cdot) = \|W^{1/p}(t)\cdot\|$, and $\rho^\star$ denotes the dual of the norm ρ.

Naturally, in the dimension $n = 1$ (1.2) is equivalent to (1.1), where $w(t) = \rho_t(1)^p$. Note that if $p \ge 1$, then $\langle\rho\rangle_{I,p}$ (if not infinite for some $x \in \mathbb{C}^n$) is a norm by Minkowski's inequality.

Fedor Nazarov, Sergei Treil in [NT] and Alexander Volberg in [V], using different approaches, have given the complete characterization of matrix weights for which the Hilbert transform is bounded.

Theorem 1.2 (Nazarov, Treil, Volberg)**.** *The Hilbert transform is bounded on $L^p(W)$ for $1 < p < \infty$ if and only if W satisfies the $\mathcal{A}_{p,q}$ condition.*

In the scalar case it is known that the A_p classes have substantial structure. However, the situation for the matrix $\mathcal{A}_{p,q}$ classes is much more complicated and that is the subject of the present paper. We will see that some of the scalar results extend, some do not, and some do but only under additional hypotheses. In particular we will consider matricial analogs of the following properties of scalar weights:

(1) $w \in A_p \;\Rightarrow\; \forall s, s > p \quad \exists c_s > 0$ s.t. $\forall I \subset \mathbb{R} \quad \langle |w^{1/s}|^s\rangle_I \le c_s\langle |w^{1/p}|^p\rangle_I$,
(2) $w \in A_p \;\Rightarrow\; \exists s, s < p \quad \exists c_s > 0$ s.t. $\forall I \subset \mathbb{R} \quad \langle |w^{1/s}|^s\rangle_I \le c_s\langle |w^{1/p}|^p\rangle_I$,
(3) $w \in A_p \;\Rightarrow\; \forall \alpha, 0 < \alpha < 1 \quad w^\alpha \in A_p$,
(4) $w \in A_p \;\Rightarrow\; \forall s, s > p \quad w \in A_s$,
(5) $w \in A_p \;\Rightarrow\; \log w \in \mathrm{BMO}$,
(6) $w \in A_p \;\Rightarrow\; \exists \varepsilon > 0$ s.t. $w \in A_{p-\varepsilon}$ (open ended property),
(7) $w \in A_p \;\Rightarrow\; \exists r > 1$ s.t. $w^r \in A_p$,

(8) $w \in A_p \Rightarrow \exists r > 1$ s.t. $w^r \in A_{pr}$,
(9) $w \in A_p \Rightarrow \exists r > 1,\ c > 0$ s.t. $\forall I \subset \mathbb{R} \quad \langle w \rangle_{I,r} \le c \langle w \rangle_I$ (reverse Hölder inequality),
(10) $\exists c_p, b \in \mathrm{BMO}$ s.t. $\|b\| < c_p \Rightarrow e^b \in A_p$,
(11) $b \in \mathrm{VMO} \Rightarrow \forall p > 1 \quad e^b \in A_p$.

In the scalar case properties (1) and (2) are trivial equalities (i.e., $c_s = 1$ and one does not even need $w \in A_p$) but we will see that their matricial analogs are not. Properties (3), (4), and (5) are elementary; they follow from the definitions and convexity inequalities (Hölder, Jensen). To prove any of the next three properties is more delicate, although now quite well understood. All these three properties follow from the reverse Hölder inequality (9). The last pair also requires more than just convexity considerations.

We will be interested in the analogs of these statements for $\mathcal{A}_{p,q}$ classes. Here we give a very informal summary of our conclusions, we will give the precise definitions later. Versions of (3), (4), and (5) are again true. This is shown using convexity, John's theorem which allows reduction of the case of general norms to that of matricial norms, and some basic matrix inequalities. On the other hand (6), (7), (10), and (11) all fail. We present explicit counterexamples in two dimensions. (The failure of (10) had been known earlier by an indirect argument.)

Interestingly, matrix analogs of (8) and (9) still hold and their proofs do require some of the depth of the scalar theory, the reverse Hölder inequality for A_p weights.

The analogs of (1) and (2) in the matricial case involve integrals of the form $\langle \|W^{1/p} v\|^p \rangle_I$, where v is a constant vector and W is a positive matrix valued function. In contrast to the scalar case, these integrals depend effectively on p and the statements are no longer trivial. We will show that an analog of (1) continues to hold for convexity reasons. (2) however, can fail; again, we will see a counterexample.

Although (2), (6), and (7) do not hold for all $W \in \mathcal{A}_{p,q}$ their conclusions can certainly hold for individual W. Interestingly, these conclusions are not independent of each other. We will show that conclusions (6) or (7) hold for W if and only if W and its dual $W^{-q/p}$ satisfy statement (2).

We would like to call particular attention to the conclusion to (2), called a reverse inverse volume estimate in Section 3. It may be that that statement can be helpful in understanding how the matricial theory differs from the scalar theory. It is a truly matricial statement which has no analog in one dimension. In fact it is also true for elementary reasons if the function W takes only values which are diagonal matrices. Also the statement does not imply any non-trivial improvement in the size estimates for the entries of W.

We recall some further definitions and properties of matrix weights, see [NT] and [V]. For a homogenous $\sigma : \mathbb{C}^n \to [0, \infty)$, i.e., $\sigma(cx) = |c|\sigma(x)$, $c \in \mathbb{C}$ (in particular a norm) define the *inverse volume* $v^{-1}(\sigma)$ by

$$v^{-1}(\sigma) = \frac{\operatorname{vol}(\{x \in \mathbb{C}^n : \|x\| \le 1\})}{\operatorname{vol}(\{x \in \mathbb{C}^n : \sigma(x) \le 1\})}.$$

Note that $v^{-1}(\|A\cdot\|) = (\det A)^2$ for a positive matrix A, the exponent is 2 because we work in the complex space. v^{-1} automatically satisfies the following convexity estimate.

Lemma 1.3. *Let ρ be a weight such that $t \mapsto \log v^{-1}(\rho_t)$ is locally integrable. Then for any $p \ge 0$ and $I \subset \mathbb{R}$*

$$v^{-1}(\langle\rho\rangle_{I,p}) \ge \exp(\langle\log v^{-1}(\rho_t)\rangle_I). \tag{1.3}$$

Proof. Note that for $p < 1$, $\langle\rho\rangle_{I,p}$ in general is not a norm but still satisfies the homogeneity property $\langle\rho\rangle_{I,p}(cx) = |c|\langle\rho\rangle_{I,p}(x)$, $c \in \mathbb{C}$. We define $\langle\rho\rangle_{I,p}$ for $p = 0$ by $\langle\rho\rangle_{I,0}(x) = \exp(\langle\log\rho(x)\rangle_I)$. By monotonicity of $\langle\rho\rangle_{I,p}$ with respect to p, it suffices to show (1.3) for $p = 0$. For any homogenous $\sigma : \mathbb{C}^n \to [0,\infty)$, i.e., $\sigma(cx) = |c|\sigma(x)$, $c \in \mathbb{C}$,

$$\operatorname{vol}(\{x \in \mathbb{C}^n : \sigma(x) \le 1\}) = \frac{1}{2n}\int_S \sigma(e)^{-2n}\,ds(e), \tag{1.4}$$

where $S = \{e \in \mathbb{C}^n : \|e\| = 1\}$ and ds is $(2n-1)$ dimensional Lebesgue measure on S.

Therefore

$$\begin{aligned}
\exp(\langle\log\operatorname{vol}\{\rho_t(\cdot) \le 1\}\rangle_I) &= \exp\left(\frac{1}{|I|}\int_I \log\left(\frac{1}{2n}\int_S \rho_t(e)^{-2n}\,ds(e)\right)dt\right)\\
&\ge \frac{1}{2n}\int_S \exp\left(\frac{1}{|I|}\int_I \log(\rho_t(e)^{-2n})\,dt\right)ds(e)\\
&= \frac{1}{2n}\int_S \left(\exp\left(\frac{1}{|I|}\int_I \log\rho_t(e)\,dt\right)\right)^{-2n} ds(e)\\
&= \operatorname{vol}\{\langle\rho\rangle_{I,0}(\cdot) \le 1\},
\end{aligned}$$

by the exp-log Minkowski inequality [DS, Ex. 36, p. 535], see also (7.3). For the convenience we include the proof in the Appendix. This immediately yields (1.3). ∎

Definition 1.4. A matrix weight W satisfies the $\mathcal{A}_{p,0}$ condition if $t \mapsto \log v^{-1}(\rho_t)$ is locally integrable and there is $C > 0$ so that

$$v^{-1}(\langle\rho\rangle_{I,p}) \le C\exp(\langle\log v^{-1}(\rho)\rangle_I) \quad \text{for all } I \subset \mathbb{R},$$

where $\rho_t(\cdot) = \|W^{1/p}(t)\cdot\|$, or equivalently

$$v^{-1}(\langle\|W^{1/p}\cdot\|\rangle_{I,p}) \le C\exp\left(\frac{2}{p}\langle\log\det W\rangle_I\right) \quad \text{for all } I \subset \mathbb{R}. \tag{1.5}$$

Now the $\mathcal{A}_{p,q}$ condition can be rewritten in the following very useful way, see [NT].

Theorem 1.5. *A weight ρ belongs to the class $\mathcal{A}_{p,q}$ iff ρ and $\rho^\star$ satisfy the $\mathcal{A}_{p,0}$ and $\mathcal{A}_{q,0}$ conditions respectively. That is, $W \in \mathcal{A}_{p,q}$ iff W satisfies both* (1.5) *and*

$$v^{-1}(\langle \|W^{-1/p}\cdot\|\rangle_{I,q}) \le C\exp\left(-\frac{2}{p}\langle\log\det W\rangle_I\right) \quad \text{for all } I\subset\mathbb{R}. \tag{1.6}$$

The paper is organized as follows. In Section 2 we prove basic facts about the class $\mathcal{A}_{p,q}$, among them $\mathcal{A}_{p,q}\subset\mathcal{A}_{p',q'}$ for $1<p<p'<\infty$. In the next section we answer a question posed by Alexander Volberg by giving a necessary and sufficient condition, for a matrix weight $W\in\mathcal{A}_{p,q}$ to be in a smaller class $\mathcal{A}_{p',q'}$ for some $p'<p$. In Section 4 we give an example of a matrix weight in $\mathbb{C}^2$ for $p=2$ which fails the open ended property satisfied by scalar weights, and in the following section we describe a family of such weights for any $1<p<\infty$. In Section 6 we discuss the relation between logarithms of weights and the space BMO. Finally, in the last section we present the proof of the exp-log Minkowski inequality.

2. Basic properties

Lemma 2.1. *Suppose W is a* (*self-adjoint*) *positive matrix on $\mathbb{C}^n$. Then for $0<\alpha<1$ we have*

$$\|W^\alpha x\|^{1/\alpha}\le\|Wx\| \quad \text{for } \|x\|=1. \tag{2.1}$$

Proof. Let $\{v_1,\dots,v_n\}$ be an orthonormal basis of eigenvectors of W with eigenvalues $\lambda_1,\dots,\lambda_n>0$. Take any $x\in\mathbb{C}^n$ with $\|x\|=1$ and write it as $x=\sum_{i=1}^n x_iv_i$. Then $Wx=\sum_{i=1}^n\lambda_ix_iv_i$ and $W^\alpha x=\sum_{i=1}^n\lambda_i^\alpha x_iv_i$. Inequality (2.1) reads now

$$\Big(\sum_{i=1}^n\lambda_i^{2\alpha}|x_i|^2\Big)^{1/2\alpha}\le\Big(\sum_{i=1}^n\lambda_i^2|x_i|^2\Big)^{1/2},$$

which can be rewritten as

$$\Big(\sum_{i=1}^n\lambda_i^{2\alpha}|x_i|^2\Big)\le\Big(\sum_{i=1}^n\lambda_i^2|x_i|^2\Big)^{\alpha},$$

which follows by Jensen inequality. □

Lemma 2.2. *There is a universal constant $C>0$, depending only on the dimension n, such that if ρ is a norm on $\mathbb{C}^n$, then*

$$\frac{1}{C}v^{-1}(\rho)\le\inf(\rho(v_1)\cdots\rho(v_n))^2\le Cv^{-1}(\rho), \tag{2.2}$$

where the infimum is taken over all orthonormal bases $\{v_1, \dots, v_n\}$ *of* $\mathbb{C}^n$.

Proof. By John's Theorem, there exists a positive matrix A, so that

$$\frac{1}{\sqrt{n}}\|Ax\| \le \rho(x) \le \|Ax\| \quad \text{for all } x \in \mathbb{C}^n. \tag{2.3}$$

Fix any orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbb{C}^n$.

$$\begin{aligned}\prod_{i=1}^n \|Av_i\| &\ge \prod_{i=1}^n \langle Av_i, v_i\rangle = \prod_{i=1}^n (a_1(v_i^1)^2 + \cdots + a_n(v_i^n)^2) \\ &\ge \prod_{i=1}^n a_1^{(v_i^1)^2} \cdots a_n^{(v_i^n)^2} = \prod_{i=1}^n a_i^{(v_1^i)^2 + \cdots + (v_n^i)^2} = \prod_{i=1}^n a_i = \det A,\end{aligned} \tag{2.4}$$

where $\{a_1, \dots, a_n\}$ are eigenvalues of A with eigenvectors $\{w_1, \dots w_n\}$, and $v_i = v_i^1 w_1 + \cdots + v_i^n w_n$. We have equality in (2.4) if $\{v_1, \dots, v_n\}$ are the eigenvectors of A. Therefore by (2.3)

$$n^{-n}\prod_{i=1}^n \|Av_i\|^2 \le \prod_{i=1}^n \rho(v_i)^2 \le \prod_{i=1}^n \|Av_i\|^2,$$

and by taking the infimum over all orthonormal basis of $\mathbb{C}^n$

$$n^{-n}v^{-1}(\rho) \le n^{-n}(\det A)^2 \le \inf(\rho(v_1)\cdots\rho(v_n))^2 \le (\det A)^2 \le n^n v^{-1}(\rho),$$

where the extreme inequalities follow from (2.3). Hence $C = n^n$ works in (2.2). $\square$

Lemma 2.3 (inverse volume inequality)**.** *There exists a universal constant* $C_n > 0$ *so that if* W *is a matrix weight and* $0 < p < p' < \infty$, *then*

$$v^{-1}(\langle\|W^{1/p'}\cdot\|\rangle_{I,p'}) \le C_n(v^{-1}(\langle\|W^{1/p}\cdot\|\rangle_{I,p}))^{p/p'}. \tag{2.5}$$

Proof. Take any $x \in \mathbb{C}^n$ with $\|x\| = 1$; then

$$\left(\frac{1}{|I|}\int_I \|W^{1/p}(t)x\|^p\,dt\right)^{1/p} \ge \left(\frac{1}{|I|}\int_I \|W^{1/p'}(t)x\|^{p'}\,dt\right)^{1/p}$$

by applying (2.1) with exponent $\alpha = p/p'$. Applying this inequality for unit vectors $\{v_1, \dots, v_n\}$ gives us

$$\begin{aligned}&\langle\|W^{1/p}v_1\|\rangle_{I,p}\cdot \cdots \cdot\langle\|W^{1/p}v_n\|\rangle_{I,p} \\ &\qquad\ge \left(\langle\|W^{1/p'}v_1\|\rangle_{I,p'}\cdot \cdots \cdot\langle\|W^{1/p'}v_n\|\rangle_{I,p'}\right)^{p'/p}.\end{aligned}$$

If we square this and take the infimum over all orthonormal bases $\{v_1,\dots,v_n\}$, we obtain by Lemma 2.2

$$Cv^{-1}(\langle \|W^{1/p}\cdot\|\rangle_{I,p}) \geq \left(\frac{1}{C}v^{-1}(\langle \|W^{1/p'}\cdot\|\rangle_{I,p'})\right)^{p'/p},$$

which gives us (2.5) with $C_n = n^{2n}$. $\square$

The intuitive meaning of Lemma 2.3 is that $p \mapsto (v^{-1}(\langle \|W^{1/p}\cdot\|\rangle_{I,p}))^p$ is a decreasing function of $0 < p < \infty$. This would be the case if we knew that the constant $C_n = 1$ in (2.5). It is an open question whether inverse volume inequality holds with the constant $C_n = 1$.

Theorem 2.4. *If $W \in \mathcal{A}_{p,q}$, then for any $0 < \alpha < 1$, $W^\alpha \in \mathcal{A}_{p',q'}$, where $p' = \alpha(p-1)+1$ and $1/p' + 1/q' = 1$.*

Proof. Since $W \in \mathcal{A}_{p,0}$ reads

$$v^{-1}(\langle \|W^{1/p}\cdot\|\rangle_{I,p}) \leq C\exp\left(\frac{2}{p}\langle \log\det W\rangle_I\right),$$

we have

$$v^{-1}(\langle \|W^{1/p}\cdot\|\rangle_{I,p})^{p\alpha/p'} \leq C^\alpha\exp\left(\frac{2\alpha}{p'}\langle \log\det W\rangle_I\right).$$

By Jensen's inequality we have

$$\begin{aligned} v^{-1}(\langle \|W^{\alpha/p'}\cdot\|\rangle_{I,p'}) &\leq v^{-1}(\langle \|W^{\alpha/p'}\cdot\|\rangle_{I,p'/\alpha}) \\ &\leq C_n v^{-1}(\langle \|W^{1/p}\cdot\|\rangle_{I,p})^{p\alpha/p'}, \end{aligned}$$

where the last inequality is a consequence of Lemma 2.3 applied for exponents $p < p'/\alpha = p - 1 + 1/\alpha$. Combining the last inequalities, we have

$$v^{-1}(\langle \|W^{\alpha/p'}\cdot\|\rangle_{I,p'}) \leq C_n C^\alpha\exp\left(\frac{2\alpha}{p'}\langle \log\det W\rangle_I\right),$$

which means precisely that $W^\alpha \in \mathcal{A}_{p',0}$.

Similarly, since $W^{-q/p} \in \mathcal{A}_{q,0}$ reads

$$v^{-1}(\langle \|W^{-1/p}\cdot\|\rangle_{I,q}) \leq C\exp\left(-\frac{2}{p}\langle \log\det W\rangle_I\right),$$

we have

$$v^{-1}(\langle \|W^{-1/p}\cdot\|\rangle_{I,q})^{q/q'} \leq C^{q/q'}\exp\left(-\frac{2\alpha}{p'}\langle \log\det W\rangle_I\right),$$

(because $q/(pq') = \alpha/p'$). By Lemma 2.3 applied for exponents $q < q'$

$$\begin{aligned} C_n v^{-1}(\langle \|W^{-1/p} \cdot \|\rangle_{I,q})^{q/q'} &\geq v^{-1}(\langle \|W^{-q/(pq')} \cdot \|\rangle_{I,q'}) \\ &= v^{-1}(\langle \|W^{-\alpha/p'} \cdot \|\rangle_{I,q'}). \end{aligned}$$

The last two inequalities yield

$$v^{-1}(\langle \|W^{-\alpha/p'} \cdot \|\rangle_{I,q'}) \leq C_n C^{q/q'} \exp\left(-\frac{2\alpha}{p'}\langle \log\det W\rangle_I\right),$$

which precisely means that $W^{-\alpha q'/p'} \in \mathcal{A}_{q',0}$.

This finishes the proof because $W^\alpha \in \mathcal{A}_{p',0}$, and $W^{-\alpha q'/p'} \in \mathcal{A}_{q',0}$ iff $W^\alpha \in \mathcal{A}_{p',q'}$. ◻

Theorem 2.5. *Suppose* $1 < p < p' < \infty$. *If* $W \in \mathcal{A}_{p,q}$, *then* $W \in \mathcal{A}_{p',q'}$.

Proof. As before, $W \in \mathcal{A}_{p,0}$ reads

$$v^{-1}(\langle \|W^{1/p} \cdot \|\rangle_{I,p}) \leq C \exp\left(\frac{2}{p}\langle \log\det W\rangle_I\right).$$

Using Lemma 2.3, we have

$$v^{-1}(\langle \|W^{1/p'} \cdot \|\rangle_{I,p'}) \leq C_n v^{-1}(\langle \|W^{1/p} \cdot \|\rangle_{I,p})^{p/p'},$$

and by combining it with the previous inequality, we obtain

$$\begin{aligned} v^{-1}(\langle \|W^{1/p'} \cdot \|\rangle_{I,p'}) &\leq C_n \left(C \exp\left(\frac{2}{p}\langle \log\det W\rangle_I\right)\right)^{p/p'} \\ &= C_n C^{p/p'} \exp\left(\frac{2}{p'}\langle \log\det W\rangle_I\right), \end{aligned}$$

which says that $W \in \mathcal{A}_{p',0}$.

Similarly, $W^{-q/p} \in \mathcal{A}_{q,0}$ reads

$$v^{-1}(\langle \|W^{-1/p} \cdot \|\rangle_{I,q}) \leq C \exp\left(-\frac{2}{p}\langle \log\det W\rangle_I\right).$$

Using Lemma 2.3 for the matrix weight $W^{-q/p}$ and exponents $1 < q < qp'/p < \infty$, we have

$$\begin{aligned} C_n v^{-1}(\langle \|W^{-1/p} \cdot \|\rangle_{I,q})^{p/p'} &\geq v^{-1}(\langle \|W^{-1/p'} \cdot \|\rangle_{I,qp'/p}) \\ &\geq v^{-1}(\langle \|W^{-1/p'} \cdot \|\rangle_{I,q'}), \end{aligned}$$

where in the last step we could use Jensen's inequality, because $qp'/p > q'$ (since $p'/q' = p' - 1 > p - 1 = p/q$). By combining the two previous inequalities, we obtain

$$v^{-1}(\langle \|W^{-1/p'} \cdot \|\rangle_{I,q'} \le C_n C^{p/p'} \exp\left(-\frac{2}{p'}\langle \log\det W\rangle_I\right),$$

which says $W^{-q'/p'} \in \mathcal{A}_{q',0}$. □

Corollary 2.6. *Suppose* $W \in \mathcal{A}_{p,q}$*; then* $W^\alpha \in \mathcal{A}_{p,q}$ *for any* $0 < \alpha < 1$.

Proof. Immediate from Theorems 2.4 and 2.5. □

Remark 2.7. The proofs of Theorems 2.4 and 2.5 give the following quantitative statements. Suppose $0 < \alpha < 1$ and $W \in \mathcal{A}_{p,0}$ with constant C; then $W^\alpha \in \mathcal{A}_{\alpha(p-1)+1,0}$ with constant at most $C_n C$. Also, if $W \in \mathcal{A}_{p,0}$ (or $W^{-q/p} \in \mathcal{A}_{q,0}$) with constant C, then $W \in \mathcal{A}_{p',0}$ (or $W^{-q'/p'} \in \mathcal{A}_{q',0}$) with constant at most $C_n C$ for any $p' > p$, $1/p' + 1/q' = 1$. We will use these observations later.

Corollary 2.8. *Suppose* $r > 1$ *and* $W^r \in \mathcal{A}_{p,q}$*; then* $W \in \mathcal{A}_{p',q'}$*, where* $p' = (p-1)/r + 1$ *and* $1/p' + 1/q' = 1$.

Proof. Immediate from Theorem 2.4. □

3. REVERSE INVERSE VOLUME INEQUALITY

Definition 3.1. We say that a matrix weight W satisfies the *reverse inverse volume inequality* for exponent $1 < p < \infty$, if there is some $1 < p' < p$ and a constant $C > 0$ such that

$$Cv^{-1}(\langle \|W^{1/p} \cdot \|\rangle_{I,p}) \ge v^{-1}(\langle \|W^{1/p'} \cdot \|\rangle_{I,p'})^{p'/p} \quad \text{for all } I \subset \mathbb{R}. \tag{3.1}$$

Note. Lemma 2.3 says that we always have

$$v^{-1}(\langle \|W^{1/p} \cdot \|\rangle_{I,p}) \le C_n v^{-1}(\langle \|W^{1/p'} \cdot \|\rangle_{I,p'})^{p'/p}$$

for $1 < p' < p < \infty$. If W is a scalar weight, then we simply have equality (without the constant C_n). More generally, if a matrix weight W is simultaneously diagonalizable, i.e., there exists a universal orthonormal basis $\{v_1, \dots, v_n\}$ of eigenvectors of $W(t)$ for all $t \in \mathbb{R}$, then both sides are comparable.

We start with a lemma about the equivalence of $\mathcal{A}_{p,0}$ and strong $\mathcal{A}_{p,0}$ conditions, which has appeared in a similar form both in [NT] and [V]. We say that a matrix weight W satisfies a strong $\mathcal{A}_{p,0}$ condition if there is a constant C so that

$$\langle \|W^{-1/p} \cdot \|\rangle_{I,0} \le C\langle \|W^{1/p} \cdot \|\rangle^{\star}_{I,p} \quad \text{for all } I \subset \mathbb{R}.$$

The author has decided to include a proof of this lemma to emphasize the interdependence of constants which will be used later.

Lemma 3.2. *Let* $1 < p < \infty$*. Suppose* V *is a matrix weight and for some interval* $I \subset \mathbb{R}$

$$v^{-1}(\langle \|V \cdot \|\rangle_{I,p}) \leq C_1 \exp(2\langle \log\det V\rangle_I). \tag{3.2}$$

Then

$$\langle \|V^{-1} \cdot \|\rangle_{I,0} \leq C_2 \langle \|V \cdot \|\rangle^{\star}_{I,p}, \tag{3.3}$$

with $C_2 = \sqrt{C_1} n^{3n/2}$*. Conversely,* (3.3) *implies* (3.2) *with constant* $C_1 = n^n (C_2)^{2n}$*.*

Proof. By John's Theorem, we can select a positive matrix V_I which is a reducing norm for $\langle \|V \cdot \|\rangle_{I,p}$, i.e.,

$$\frac{1}{\sqrt{n}} \|V_I \cdot \| \leq \langle \|V \cdot \|\rangle_{I,p} \leq \|V_I \cdot \|. \tag{3.4}$$

Then (3.4) implies

$$\frac{1}{n^n} (\det V_I)^2 \leq C_1 \exp(2\langle \log\det V\rangle_I). \tag{3.5}$$

Define a matrix valued function C by $C(t) = V(t)^{-1} V_I$. Note that

$$\frac{1}{\sqrt{n}} \|x\| \leq \langle \|C^{-1}(t)^{\star} x\|\rangle_{I,p} = \langle \|V(t) V_I^{-1} x\|\rangle_{I,p} \leq \|x\|. \tag{3.6}$$

Fix $x \in \mathbb{C}^n$ with $\|x\| = 1$. For any t in I let $y(t) \in \mathbb{C}^n$, $\|y(t)\| = 1$, be an eigenvector of a positive symmetric matrix $C(t)^{\star} C(t)$ corresponding to the least eigenvalue, that is $y(t)$ minimizes $\|C(t) y(t)\|$. Then

$$\begin{aligned} \det C(t) &= (\det C(t)^{\star} C(t))^{1/2} = \Big(\prod_{i=1}^{n} \langle C(t)^{\star} C(t) v_i, v_i\rangle \Big)^{1/2} \\ &\geq (\langle C(t)^{\star} C(t) x, x\rangle)^{1/2} (\langle C(t)^{\star} C(t) y(t), y(t)\rangle)^{(n-1)/2} \\ &= \|C(t) x\| \|C(t) y(t)\|^{n-1}, \end{aligned} \tag{3.7}$$

where $v_1, \ldots, v_n$ are eigenvectors of $C(t)^{\star} C(t)$. An integration of the logarithm of both sides of (3.7) yields

$$\langle \log \|C(t) x\|\rangle_I \leq \log\det V_I - \langle \log\det V(t)\rangle_I - (n-1)\langle \log \|C(t) y(t)\|\rangle_I.$$

Since

$$\begin{aligned} 1 &= \langle C^{-1}(t) C(t) y(t), y(t)\rangle = \langle C(t) y(t), C^{-1}(t)^{\star} y(t)\rangle \\ &\leq \|C(t) y(t)\| \|C^{-1}(t)^{\star} y(t)\|, \end{aligned}$$

we can substitute $\|C^{-1}(t)^\star y(t)\|^{-1}$ in place of $\|C(t)y(t)\|$ to obtain (after exponentiation of both sides)

$$\begin{aligned}\langle\|C(t)x\|\rangle_{I,0} &\le \det V_I \exp(-\langle\log\det V(t)\rangle_I)(\langle\|C^{-1}(t)^\star y(t)\|\rangle_{I,0})^{n-1}\\ &\le \sqrt{C_1}n^{n/2}(\langle\|C^{-1}(t)^\star y(t)\|\rangle_{I,p})^{n-1} \le \sqrt{C_1}n^{n/2}n^{n-1}\\ &\le \sqrt{C_1}n^{3n/2}\|x\|.\end{aligned}$$

By (3.5) and a crude estimate from (3.6)

$$\langle\|C^{-1}(t)^\star y(t)\|\rangle_{I,p} \le \Big\langle \sum_{i=1}^n \|C^{-1}(t)^\star e_i\|\Big\rangle_{I,p} \le \sum_{i=1}^n \langle\|C^{-1}(t)^\star e_i\|\rangle_{I,p} \le n.$$

Therefore

$$\langle\|V^{-1}(t)x\|\rangle_{I,0} \le \sqrt{C_1}n^{3n/2}\|V_I^{-1}x\| \le \sqrt{C_1}n^{3n/2}\langle\|V(t)x\|\rangle^\star_{I,p},$$

where the last inequality follows by taking duals of (3.4).

Conversely, assume (3.3) holds. By applying v^{-1} to both (3.4) and the dual of (3.4) and multiplying the two resulting inequalities, we obtain

$$n^{-n} \le v^{-1}(\langle\|V\cdot\|\rangle_{I,p})v^{-1}(\langle\|V\cdot\|\rangle^\star_{I,p}) \le n^n. \tag{3.8}$$

Condition (3.3) implies $v^{-1}\langle\|V\cdot\|\rangle_{I,0}) \le (C_2)^{2n}v^{-1}(\langle\|V\cdot\|\rangle^\star_{I,p})$, hence

$$\begin{aligned}[v^{-1}(\langle\|V\cdot\|\rangle_{I,p})\exp(-\langle\log\det V\rangle_I)][v^{-1}(\langle\|V^{-1}\cdot\|\rangle_{I,0})\exp(-\langle\log\det V^{-1}\rangle_I)]\\ \le n^n(C_2)^{2n},\end{aligned}$$

since the exponential terms give 1 when multiplied. By Lemma 1.3 the second factor in the brackets (and the first as well) is at least 1, therefore we have (3.2) with constant $C_1 = n^n(C_2)^{2n}$. □

Lemma 3.3. *Suppose a matrix weight V satisfies* (3.3) *for some interval $I \subset \mathbb{R}$. Then for any $0 \ne x \in \mathbb{C}^n$,*

$$\langle\|Vx\|\rangle_{I,p} \le C_2\exp(\langle\log\|Vx\|\rangle_I). \tag{3.9}$$

Proof. Fix $x \ne 0$ and choose $0 \ne y' \in \mathbb{C}^n$ such that

$$\langle\|Vx\|\rangle_{I,p} = \sup_{y\ne 0}\frac{|\langle x,y\rangle|}{\langle\|V\cdot\|\rangle^\star_{I,p}(y)} = \frac{\langle x,y'\rangle}{\langle\|V\cdot\|\rangle^\star_{I,p}(y')}.$$

Using the above and (3.3) we obtain

$$\begin{aligned}\langle \|Vx\|\rangle_{I,p}\exp(-\langle\log\|Vx\|\rangle_I) &\le C_2\langle x,y'\rangle\exp(-\langle\log\|V^{-1}y'\|\rangle_I-\langle\log\|Vx\|\rangle_I)\\ &= C_2\langle x,y'\rangle\exp(-\langle\log(\|V^{-1}y'\|\cdot\|Vx\|)\rangle_I)\\ &\le C_2\langle x,y'\rangle\exp(-\langle\log\langle x,y'\rangle\rangle_I)=C_2.\end{aligned}$$

This proves inequality (3.9). ◻

The next lemma states that matrix weights satisfy a kind of self-improvement, which is an analog of the reverse Hölder inequality for scalar weights.

Lemma 3.4. *Suppose $1<p_0<\infty$ and $C>0$. Then there exist $r>1$ and $C'>0$, depending only on p_0 and C, having the property that if a matrix weight V satisfies for some $1<p\le p_0$*

$$v^{-1}(\langle\|V\cdot\|\rangle_{I,p})\le C\exp(2\langle\log\det V\rangle_I)\quad\text{for all } I\subset\mathbb{R}, \tag{3.10}$$

then

$$v^{-1}(\langle\|V\cdot\|\rangle_{I,pr})\le C'\exp(2\langle\log\det V\rangle_I)\quad\text{for all } I\subset\mathbb{R}. \tag{3.11}$$

Proof. Assume (3.10) holds; then by Lemmas 3.2 and 3.3 we have for any $x\ne 0$

$$\langle\|Vx\|\rangle_{I,p}\le\sqrt{C}n^{3n/2}\exp(\langle\log\|Vx\|\rangle_I)\quad\text{for all } I\subset\mathbb{R},$$

hence

$$\langle\|Vx\|^p\rangle_I\le C^{p/2}n^{3pn/2}\exp(\langle\log\|Vx\|^p\rangle_I)\quad\text{for all } I\subset\mathbb{R},$$

which means that a scalar weight $w(t)=\|V(t)x\|^p$ satisfies

$$\langle w\rangle_I\exp(-\langle\log w\rangle_I)\le\tilde{C}\quad\text{for all } I\subset\mathbb{R},$$

where $\tilde{C}=C^{p_0/2}n^{3p_0n/2}$. By [H, Theorem 1] (see also [GR, Theorem 2.15, p. 405]) $w\in A_\infty$, i.e., for all intervals $I\subset\mathbb{R}$ and all measurable subsets $E\subset Q$,

$$|E|\le\frac{1}{2}|I|\;\Rightarrow\;\int_E w(t)\,dt\le\delta\int_I w(t)\,dt,$$

with $\delta=\max(\frac{1}{2},4\tilde{C}^2/(1+4\tilde{C}^2))$. Hence by [S, Proposition 4, p. 202], weight w satisfies the reverse Hölder inequality, i.e., there exist $r>1$ and $c>0$ so that

$$\langle\|Vx\|\rangle_{I,rp}\le c\langle\|Vx\|\rangle_{I,p}\quad\text{for all } I\subset\mathbb{R}.$$

Hence

$$v^{-1}(\langle\|V\cdot\|\rangle_{I,rp}) \le c^{2n}v^{-1}(\langle\|V\cdot\|\rangle_{I,p}) \le C'\exp(2\langle\log\det V\rangle_I),$$

with $C' = c^{2n}C$. ◻

Corollary 3.5. *Suppose $W \in \mathcal{A}_{p,q}$. Then there exists $r > 1$ such that $W^r \in \mathcal{A}_{pr,q'}$, where $1/(pr) + 1/q' = 1$.*

Proof. Lemma 3.4 says that $W \in \mathcal{A}_{p,0}$ implies $W^r \in \mathcal{A}_{pr,0}$. Since $q' < q$ we have

$$v^{-1}(\langle\|W^{-1/p}\cdot\|\rangle_{I,q'}) \le v^{-1}(\langle\|W^{-1/p}\cdot\|\rangle_{I,q}) \le C\exp\left(-\frac{2}{p}\langle\log\det W\rangle_I\right),$$

which shows that $W^{-q'/p} = (W^r)^{-q'/(pr)} \in \mathcal{A}_{q',0}$. ◻

Theorem 3.6. *Suppose $W \in \mathcal{A}_{p_0,q_0}$. Then the following are equivalent:*

(i) *$W \in \mathcal{A}_{p_1,q_1}$ for some $1 < p_1 < p_0$, $1/p_1 + 1/q_1 = 1$,*
(ii) *$W^r \in \mathcal{A}_{p_0,q_0}$ for some $r > 1$,*
(iii) *W and W^{-q_0/p_0} satisfy reverse inverse volume inequalities for exponents p_0 and q_0, respectively.*

Proof of (i) ⇒ (iii). $W \in \mathcal{A}_{p_1,0}$ says that

$$v^{-1}(\langle\|W^{1/p_1}\cdot\|\rangle_{I,p_1}) \le C\exp\left(\frac{2}{p_1}\langle\log\det W\rangle_I\right).$$

Raising this to the power p_1/p_0 and combining with Lemma 1.3 applied to $\|W^{1/p_0}\cdot\|$ with exponent p_0, i.e.,

$$v^{-1}(\langle\|W^{1/p_0}\cdot\|\rangle_{I,p_0}) \ge \exp\left(\frac{2}{p_0}\langle\log\det W\rangle_I\right)$$

gives

$$C^{p_1/p_0}v^{-1}(\langle\|W^{1/p_0}\cdot\|\rangle_{I,p_0}) \ge (v^{-1}(\langle\|W^{1/p_1}\cdot\|\rangle_{I,p_1}))^{p_1/p_0}.$$

Hence W satisfies the reverse inverse volume inequality with exponent p_0.

Similarly, $W^{-q_1/p_1} \in \mathcal{A}_{q_1,0}$ says that

$$v^{-1}(\langle\|W^{-1/p_1}\cdot\|\rangle_{I,q_1}) \le C\exp\left(-\frac{2}{p_1}\langle\log\det W\rangle_I\right).$$

Taking this to the power p_1/p_0, and combining with Lemma 1.3 applied to $\|W^{-1/p_0}\cdot\|$ with exponent q_0

$$v^{-1}(\langle\|W^{-1/p_0}\cdot\|\rangle_{I,q_0}) \geq \exp\left(-\frac{2}{p_0}\langle\log\det W\rangle_I\right)$$

gives us

$$C^{p_1/p_0}v^{-1}(\langle\|W^{-1/p_0}\cdot\|\rangle_{I,q_0}) \geq (v^{-1}(\langle\|W^{-1/p_1}\cdot\|\rangle_{I,q_1}))^{p_1/p_0}.$$

Since $\langle\|W^{-1/p_1}\cdot\|\rangle_{I,q_1} \geq \langle\|W^{-1/p_1}\cdot\|\rangle_{I,q_0p_1/p_0}$ (by $q_1 > q_0p_1/p_0$) we have

$$C^{p_1/p_0}v^{-1}(\langle\|W^{-1/p_0}\cdot\|\rangle_{I,q_0}) \geq (v^{-1}(\langle\|W^{-1/p_1}\cdot\|\rangle_{I,q_0p_1/p_0}))^{p_1/p_0},$$

hence W^{-q_0/p_0} satisfies the reverse inverse volume inequality with exponent q_0. ◻

Proof of (iii) ⇒ (ii). By our hypothesis, there exists $1 < p_1 < p_0$ such that

$$Cv^{-1}(\langle\|W^{1/p_0}\cdot\|\rangle_{I,p_0}) \geq (v^{-1}(\langle\|W^{1/p_1}\cdot\|\rangle_{I,p_1}))^{p_1/p_0}.$$

This can be rewritten as

$$Cv^{-1}(\langle\|W^{1/p_0}\cdot\|\rangle_{I,p_0}) \geq (v^{-1}(\langle\|W^{r_0/p_0}\cdot\|\rangle_{I,p_0/r_0}))^{1/r_0},$$

where $r_0 = p_0/p_1 > 1$. Combining it with $W \in \mathcal{A}_{p_0,0}$ yields

$$(v^{-1}(\langle\|W^{r_0/p_0}\cdot\|\rangle_{I,p_0/r_0}))^{1/r_0} \leq C\exp\left(\frac{2}{p_0}\langle\log\det W\rangle_I\right),$$

hence

$$v^{-1}(\langle\|W^{r_0/p_0}\cdot\|\rangle_{I,p_0/r_0}) \leq C^{r_0}\exp\left(\frac{2r_0}{p_0}\langle\log\det W\rangle_I\right).$$

By the inverse volume inequality (Lemma 2.3)

$$\begin{aligned} v^{-1}(\langle\|W^{s/p_0}\cdot\|\rangle_{I,p_0/s}) &\leq C_n(v^{-1}(\langle\|W^{r_0/p_0}\cdot\|\rangle_{I,p_0/r_0}))^{s/r_0} \\ &\leq C^{r_0}C_n\exp\left(\frac{2s}{p_0}\langle\log\det W\rangle_I\right), \end{aligned}$$

for all $1 \leq s \leq r_0$.

By Lemma 3.4 applied for exponent p_0, constant $C^{r_0}C_n$, and weights $V = W^{s/p_0}$ there exists $r_1 > 1$ and $C' > 0$ so that

$$v^{-1}(\langle \|W^{s/p_0}\cdot\|\rangle_{I,r_1p_0/s}) \le C' \exp\left(\frac{2s}{p_0}\langle \log\det W\rangle_I\right), \tag{3.12}$$

for all $1 \le s \le r_0$. We can also assume that $r_1 \le r_0$ (otherwise use Hölder inequality to decrease r_1). Take $s = r_1$ in (3.12) to obtain $W^{r_1} \in \mathcal{A}_{p_0,0}$.

Thus we have shown that $W \in \mathcal{A}_{p_0,0}$ and W satisfying the reverse inverse volume inequality with exponent p_0 implies that there exists $r > 1$ so that $W^r \in \mathcal{A}_{p_0,0}$. Applying the above for the weight $W^{-q_0/p_0} \in \mathcal{A}_{q_0,0}$ satisfying the reverse inverse volume inequality with exponent q_0, we have $r_2 > 1$ so that $W^{-r_2q_0/p_0} \in \mathcal{A}_{q_0,0}$. Finally take $r = \min(r_1, r_2) > 1$ and use Corollary 2.6 to conclude that both $W^r \in \mathcal{A}_{p_0,0}$ and $W^{-rq_0/p_0} \in \mathcal{A}_{q_0,0}$, so $W^r \in \mathcal{A}_{p_0,q_0}$. □

Proof of (ii) $\Rightarrow$ (i). Follows from Corollary 2.8. □

4. AN EXAMPLE OF A MATRIX WEIGHT IN $\mathcal{A}_{2,2}$

In this section we exhibit a matrix weight in $\mathcal{A}_{2,2}$ that violates the open ended property enjoyed by scalar weights. This is the first example of such a weight, and it suggested to the author a wider class of matrix weights breaking the open ended property for any $1 < p < \infty$. Although Proposition 5.3 gives an alternative proof to the one below, the author decided to present this example to display the argument used in the simpler case $p = 2$.

Lemma 4.1. *Suppose* $-\infty < a < b < \infty$ *and* $0 < c < 1$*; then*

$$\frac{1}{b-a}\int_a^b |t|^{-c}\,dt \le \frac{2}{1-c}\max(|a|,|b|)^{-c},$$
$$\frac{1}{b-a}\left|\int_a^b \operatorname{sign}(t)|t|^{-c}\,dt\right| \le \frac{1}{1-c}\max(|a|,|b|)^{-c}.$$

Proof. It is enough to show

$$\frac{1}{b-a}\int_a^b |t|^{-c}\,dt \le \frac{1}{1-c}b^{-c}$$

for any $0 \le a < b < \infty$, which in turn follows easily by differentiation of left hand side as a function of a for fixed b. □

Theorem 4.2. *There is a* 2×2 *matrix weight* $W \in \mathcal{A}_{2,2}$ *such that for any* $r > 1$, $W^r \notin \mathcal{A}_{2,2}$.

Proof. Let

$$W(t) = U(t)^\star \begin{pmatrix} 1 & 0 \\ 0 & b(t) \end{pmatrix} U(t), \quad U(t) = \begin{pmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{pmatrix},$$

where

$$b(t) = \begin{cases} |t|^{1/2} & -1 \le t \le 1, \\ 1 & \text{otherwise,} \end{cases}$$

$$\alpha(t) = \begin{cases} \operatorname{sign}(t)|t|^{1/4} & -1 \le t \le 1, \\ \operatorname{sign} t & \text{otherwise.} \end{cases}$$

By direct computations we have

$$W = \begin{pmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix} + |t|^{1/2} \begin{pmatrix} \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha \end{pmatrix},$$

and

$$W^{-1} = \begin{pmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix} + |t|^{-1/2} \begin{pmatrix} \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha \end{pmatrix}.$$

Let $\Delta = \{(a,b) : a \le b,\ a,\ b \in \mathbb{R}\} \setminus \{(0,0)\} \subset \mathbb{R}^2$. Define a function $N : \Delta \to \mathbb{R}$ by

$$N(a,b) = \|\langle W^{-1}\rangle_I^{1/2}\langle W\rangle_I^{1/2}\|, \quad \text{where } I = [a,b],$$

and $\langle W\rangle_I$ for $I = [a,a]$ means $W(a)$. We claim that N is continuous on Δ and, moreover, $N(a,b)$ tends to 1 as $a \to \infty$ or as $b \to \infty$.

To show $W \in \mathcal{A}_{2,2}$, we need $N(a,b)$ be bounded as $\Delta \ni (a,b) \to (0,0)$. First notice that

$$W(t) = \begin{pmatrix} 1 + O(|t|^{1/2}) & -\operatorname{sign}(t)|t|^{1/4} + O(|t|^{3/4}) \\ -\operatorname{sign}(t)|t|^{1/4} + O(|t|^{3/4}) & 2|t|^{1/2} + O(|t|) \end{pmatrix},$$

and

$$W^{-1}(t) = \begin{pmatrix} 2 + O(|t|^{1/2}) & -\operatorname{sign}(t)|t|^{-1/4} + O(|t|^{1/4}) \\ -\operatorname{sign}(t)|t|^{-1/4} + O(|t|^{1/4}) & |t|^{-1/2} + O(1) \end{pmatrix},$$

for t close to 0.

Suppose that $I = [a, b]$ for $b > 0$, $b \geq |a|$ and take any $x = (x_1, x_2) \in \mathbb{C}^2$.

$$\begin{aligned}\langle\langle W\rangle_I x, x\rangle &\leq (1 + O(b^{1/2}))|x_1|^2 + 2(b^{1/4} + O(b^{3/4}))|x_1|\,|x_2| \\ &\quad + (2b^{1/2} + O(b))|x_2|^2 \\ &\leq 2(|x_1|^2 + 2b^{1/4}|x_1|\,|x_2| + 2b^{1/2}|x_2|^2) \\ &\leq 2(2|x_1|^2 + 3b^{1/2}|x_2|^2).\end{aligned}$$

Hence

$$\langle W\rangle_I \leq 2\begin{pmatrix} 2 & 0 \\ 0 & 3b^{1/2}\end{pmatrix}.$$

Analogously, using Lemma 4.1 we have

$$\begin{aligned}\langle\langle W^{-1}\rangle_I x, x\rangle &\leq (2 + O(b^{1/2}))|x_1|^2 + 2\left(\frac{4}{3}b^{-1/4} + O(b^{3/4})\right)|x_1|\,|x_2| \\ &\quad + (4b^{-1/2} + O(1))|x_2|^2 \\ &\leq 2(2|x_1|^2 + 2b^{-1/4}|x_1|\,|x_2| + 3b^{-1/2}|x_2|^2) \\ &\leq 2(3|x_1|^2 + 4b^{-1/2}|x_2|^2).\end{aligned}$$

Hence

$$\langle W^{-1}\rangle_I \leq 2\begin{pmatrix} 3 & 0 \\ 0 & 4b^{-1/2}\end{pmatrix}.$$

Analogously, if $I = [a, b]$ for $a < 0$, $|a| \geq |b|$ we obtain the estimates

$$\langle W\rangle_I \leq 2\begin{pmatrix} 2 & 0 \\ 0 & 3|a|^{1/2}\end{pmatrix},$$
$$\langle W^{-1}\rangle_I \leq 2\begin{pmatrix} 3 & 0 \\ 0 & 4|a|^{-1/2}\end{pmatrix},$$

which show that N is bounded if a, b are close to 0. Therefore N must be bounded on all of Δ, which means $W \in \mathcal{A}_{2,2}$.

On the other hand, by computing averages over symmetric intervals $I = [-a, a]$, $a > 0$, we see that W^r for $r > 1$ can not belong to $\mathcal{A}_{2,2}$. Indeed, since

$$W^{\pm r} = \begin{pmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin^2\alpha\end{pmatrix} + |t|^{\pm r/2}\begin{pmatrix} \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha\end{pmatrix},$$

we have for small $a > 0$

$$\langle W^r \rangle_I = \begin{pmatrix} 1 + O(a^{1/2}) & 0 \\ 0 & \frac{2}{3}a^{1/2} + O(a^{r/2}) \end{pmatrix},$$

$$\langle W^{-r} \rangle_I = \begin{pmatrix} \frac{2}{(3-r)}a^{(1-r)/2} + O(1) & 0 \\ 0 & \frac{2}{(2-r)}a^{-r/2} + O(a^{(1-r)/2)}) \end{pmatrix},$$

because the off-diagonal entries vanish.

Now it is clear that $N(-a,a) = \|\langle W^{-r}\rangle_I^{1/2}\langle W^r\rangle_I^{1/2}\| \to \infty$ as $a \to 0$. ◻

Corollary 4.3 (counterexample to the open ended property)**.** *There exists a* 2×2 *matrix weight* $W \in \mathcal{A}_{2,2}$ *which is not in class* $\mathcal{A}_{p,q}$ *for any* $p < 2$, $1/p + 1/q = 1$.

Proof. Immediate from Theorems 4.2 and 3.6. ◻

5. Examples of weights in $\mathcal{A}_{p,q}$

Lemma 5.1. *There is a universal constant* $C > 0$, *depending on dimension* n, *such that for any* $1 \le p \le \infty$ *and any unitary matrix* U

$$\frac{1}{C}\|x\| \le \|Ux\|_p \le C\|x\| \quad \text{for } x \in \mathbb{C}^n.$$

Here for $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ *we denote*

$$\|x\|_p = \begin{cases} \Big(\sum\limits_{i=1}^{n} |x_i|^p\Big)^{1/p} & 1 \le p < \infty, \\ \sup |x_i| & p = \infty. \end{cases}$$

Proof. Since

$$\|x\|_\infty \le \|x\|_p \le \|x\|_1 \quad \text{for any } 1 \le p \le \infty,$$

and $\|U^{-1}x\| = \|x\|$, it suffices to show that there exists $C > 0$ such that

$$\frac{1}{C}\|x\| \le \|x\|_\infty \quad \text{and} \quad \|x\|_1 \le C\|x\|.$$

It is easy to see that $C = \sqrt{n}$ works in both inequalities. ◻

Lemma 5.2. *Let* $1 \le p \le \infty$ *and* A *be any non singular* $n \times n$ *matrix. If* $\rho(x) = \|Ax\|_p$, *then* $\rho^\star(x) = \|(A^{-1})^\star x\|_q$, *where* $1/p + 1/q = 1$.

Proof. Let $B_p = \{x \in \mathbb{C}^n : \|x\|_p \le 1\}$ denote the unit ball of $\|\cdot\|_p$. Then the unit ball B_ρ of the norm ρ is $A^{-1}(B_p)$.

$$\rho^\star(y) = \sup_{x\in B_\rho} \langle y, x\rangle = \sup_{x\in B_p} \langle y, A^{-1}x\rangle = \sup_{x\in B_p} \langle (A^{-1})^\star y, x\rangle = \|(A^{-1})^\star y\|_q.$$

We used here that $\|\cdot\|_p^\star \cong \|\cdot\|_q$, where the duality is realized by the standard inner product $\langle\cdot,\cdot\rangle$. ◻

Proposition 5.3. *Suppose a 2×2 matrix weight W is given by*

$$W(t) = U(t)^\star \begin{pmatrix} 1 & 0 \\ 0 & b(t) \end{pmatrix} U(t), \quad \text{where } U(t) = \begin{pmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{pmatrix},$$
$$\alpha(t) = \begin{cases} \operatorname{sign}(t)|t|^\delta & \text{for } |t| \le 1, \\ \operatorname{sign}(t) & \text{otherwise,} \end{cases} \qquad b(t) = \begin{cases} |t|^\varepsilon & \text{for } |t| \le 1, \\ 1 & \text{otherwise,} \end{cases}$$

and $-1 < \varepsilon < p/q$, $\delta > 0$, and $1/p + 1/q = 1$. Then $W \in \mathcal{A}_{p,q}$ if and only if $-p\delta \le \varepsilon \le p\delta$.

Proof. Assume first that $0 < \varepsilon < p/q$. Let $\rho_t(x) = \|W^{1/p}(t)x\|$, where $x = (x_1, x_2) \in \mathbb{C}^2$. Since

$$U(t)W^{1/p}(t) = \begin{pmatrix} \cos(\alpha(t)) & -\sin(\alpha(t)) \\ |t|^{\varepsilon/p}\sin(\alpha(t)) & |t|^{\varepsilon/p}\cos(\alpha(t)) \end{pmatrix},$$

Lemma 5.1 implies that

$$\begin{aligned} \frac{1}{C}\rho_t(x) &\le \left(|x_1\cos(\alpha(t)) - x_2\sin(\alpha(t))|^p + |t|^\varepsilon|x_1\sin(\alpha(t)) + x_2\cos(\alpha(t))|^p\right)^{1/p} \\ &\le C\rho_t(x), \end{aligned}$$

and similarly

$$\begin{aligned} \frac{1}{C}\rho_t^\star(x) &\le \big(|x_1\cos(\alpha(t)) - x_2\sin(\alpha(t))|^q + |t|^{-\varepsilon q/p}|x_1\sin(\alpha(t)) \\ &\qquad + x_2\cos(\alpha(t))|^q\big)^{1/q} \\ &\le C\rho_t^\star(x). \end{aligned}$$

Suppose that $\varepsilon \le p\delta$. Then for t close to 0

$$\begin{aligned}
\frac{1}{C^p}(\rho_t(x))^p &\le |x_1(1+O(|t|^{2\delta})) - x_2\,\mathrm{sign}(t)(|t|^\delta + O(|t|^{3\delta}))|^p \\
&\quad + |t|^\varepsilon |x_1\,\mathrm{sign}(t)(|t|^\delta + O(|t|^{3\delta})) + x_2(1+O(|t|^{2\delta}))|^p \\
&\le 2^p|x_1|^p[(1+O(|t|^{2\delta}))^p + |t|^\varepsilon(|t|^\delta + O(|t|^{3\delta}))^p] \\
&\quad + 2^p|x_2|^p[(|t|^\delta + O(|t|^{3\delta}))^p + |t|^\varepsilon(1+O(|t|^{2\delta}))^p] \\
&\le 2^{p+1}[|x_1|^p + |t|^\varepsilon|x_2|^p].
\end{aligned}$$

Therefore, if $I = [a,b]$ is some interval with a, b close to 0 and $d = \max(|a|,|b|)$, then

$$\langle\rho\rangle_{I,p} \le 4C(|x_1|^p + d^\varepsilon|x_2|^p)^{1/p} = 4C(|x_1|^p + |d^{\varepsilon/p}x_2|^p)^{1/p}.$$

After taking the dual norm, by Lemma 5.2 we have

$$\langle\rho\rangle_{I,p}^\star \ge \frac{1}{4}C(|x_1|^q + d^{-\varepsilon q/p}|x_2|^q)^{1/q}.$$

Analogously,

$$\begin{aligned}
\frac{1}{C^q}(\rho_t^\star(x))^q \le& 2^q|x_1|^q[(1+O(|t|^{2\delta}))^q + |t|^{-\varepsilon q/p}(|t|^\delta + O(|t|^{3\delta}))^q] \\
&+ 2^q|x_2|^q[(|t|^\delta + O(|t|^{3\delta}))^q + |t|^{-\varepsilon q/p}(1+O(|t|^{2\delta}))^q] \\
\le& 2^{q+1}[|x_1|^q + |t|^{-\varepsilon q/p}|x_2|^q],
\end{aligned}$$

therefore by $\varepsilon q/p < 1$ and Lemma 4.1

$$\langle\rho^\star\rangle_{I,q} \le 4C\left(|x_1|^q + \frac{2}{1-\varepsilon q/p}d^{-\varepsilon q/p}|x_2|^q\right)^{1/q}.$$

To finish the argument, define $\Delta = \{(a,b) : a \le b,\ a,\ b \in \mathbb{R}\}\setminus\{(0,0)\} \subset \mathbb{R}^2$ and a function $N : \Delta \to \mathbb{R}$ by

$$N(a,b) = \sup_{0\ne x\in\mathbb{C}^2} \frac{\langle\rho^\star\rangle_{I,q}(x)}{\langle\rho\rangle_{I,p}^\star(x)}, \quad \text{where } I = [a,b],$$

and $\langle\rho\rangle_{I,p}(x)$ for $I = [a,a]$ means simply $\rho_a(x)$. We claim that N is continuous on Δ and, moreover, $N(a,b)$ tends to 1 as $a \to \infty$ or as $b \to \infty$. By collecting the two previous inequalities together we obtain

$$\langle\rho^\star\rangle_{I,q} \le (4C)^2\left(\frac{2}{1-\varepsilon q/p}\right)^{1/q}\langle\rho\rangle_{I,p}^\star,$$

which means that N is bounded as a, b are close to 0. Therefore N must be bounded on all of Δ, which means $W \in \mathcal{A}_{p,q}$.

On the other hand, assume $\varepsilon > \delta p$; then for t close to 0 we have

$$\begin{aligned} C^p \rho_t(x)^p \geq {} & |x_1(1 + O(|t|^{2\delta})) - x_2 \operatorname{sign}(t)(|t|^\delta + O(|t|^{3\delta}))|^p \\ & + |t|^\varepsilon |x_1 \operatorname{sign}(t)(|t|^\delta + O(|t|^{3\delta})) + x_2(1 + O(|t|^{2\delta}))|^p. \end{aligned}$$

Consider a matrix V given by

$$V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then the matrix $U = 1/\sqrt{2}V$ is unitary, and by Lemma 5.1 $\|Uy\|_p \geq 1/C^2\|y\|_p$ for any $y \in \mathbb{C}^2$, hence $\|Vy\|_p \geq \sqrt{2}/C^2\|y\|_p$. Hence, if t is close to 0 we have

$$\begin{aligned} & C^p(\rho_t(x)^p + \rho_{-t}(x)^p) \\ & \quad \geq 2^{p/2} C^{-2p}[|x_1|^p(1 + O(|t|^{2\delta}))^p + |x_2|^p(|t|^\delta + O(|t|^{3\delta}))^p \\ & \qquad + |t|^\varepsilon(|x_1|^p(|t|^\delta + O(|t|^{3\delta}))^p + |x_2|^p(1 + O(|t|^{2\delta}))^p)] \\ & \quad \geq C^{-2p}[|x_1|^p + |x_2|^p|t|^{\delta p}]. \end{aligned}$$

An integration of the above inequality on the interval $I = [0, a]$, where $a > 0$ is close to 0, yields

$$\frac{1}{a}\int_0^a (\rho_t(x)^p + \rho_{-t}(x)^p)\,dt \geq C^{-3p}\left(|x_1|^p + \frac{1}{1+\delta p}a^{\delta p}|x_2|^p\right),$$

hence

$$\langle \rho \rangle_{I,p} \geq 2^{-1/p}C^{-3}\left(|x_1|^p + \frac{1}{1+\delta p}a^{\delta p}|x_2|^p\right)^{1/p},$$

where $I = [-a, a]$. After taking the dual norm, by Lemma 5.2 we have

$$\langle \rho \rangle_{I,p}^\star(x) \leq 2^{1/p}C^3(|x_1|^q + (1+\delta p)^{q/p}a^{-\delta q}|x_2|^q)^{1/q}.$$

Using the above technique we could also estimate the norm $\langle \rho^\star \rangle_{I,q}$. However, to obtain the desired conclusion it is enough to find values of these norms on the standard unit vector e_1.

For t close to 0 we have

$$C^q \rho_t^\star(e_1)^q \geq (1 + O(|t|^{2\delta}))^q + |t|^{-\varepsilon q/p}(|t|^\delta + O(|t|^{3\delta}))^q \geq \frac{1}{2}|t|^{(\delta p - \varepsilon)q/p}.$$

An integration of the above inequalities over an interval $I = [-a, a]$, where $a > 0$ is close to 0, yields

$$\langle \rho^\star \rangle_{I,q}(e_1) \geq \frac{1}{2C}\left(1 + \frac{(\delta p - \varepsilon)q}{p}\right)^{-1/q} a^{(\delta p - \varepsilon)/p}.$$

Now we see that $N(-a, a) \geq \langle \rho^\star \rangle_{I,q}(e_1)/\langle \rho \rangle^\star_{I,p}(e_1) \to \infty$ as $a \to 0$. Therefore, the assumption $\varepsilon > \delta p$ excludes W from being in class $\mathcal{A}_{p,q}$. This ends the proof of the proposition in the case $0 < \varepsilon < p/q$.

If $\varepsilon = 0$, then $W(t) = Id$ belongs to $\mathcal{A}_{p,q}$ class. Finally assume $-1 < \varepsilon < 0$. By the duality, i.e., $W \in \mathcal{A}_{p,q} \iff W^{-q/p} \in \mathcal{A}_{q,p}$ we deduce that $W \in \mathcal{A}_{p,q}$ (where W is defined as in Proposition 5.3) iff $-\varepsilon q/p \leq q\delta$, i.e., $\varepsilon \geq -p\delta$. This ends the proof of Proposition 5.3. ◻

Remark 5.4. For any $1 < p < \infty$, by taking $\varepsilon = \delta p$, Proposition 5.3 gives a weight W in $\mathcal{A}_{p,q}$, which is not in $\mathcal{A}_{p',q'}$ for any $p' < p$. It is not hard to see that W satisfies the reverse inverse volume inequality for exponent p, but by Theorem 3.6, $W^{-q/p}$ cannot satisfy this inequality for exponent q. By the duality (take $\varepsilon = -p\delta$ in Proposition 5.3) we can find a matrix weight W which does not satisfy the reverse inverse volume inequality for exponent p, but $W^{-q/p}$ does satisfy it.

6. Logarithm of a weight and BMO

The following theorem is an immediate consequence of the result by Steven Bloom in [B].

Theorem 6.1. *Suppose $W \in \mathcal{A}_{p,q}$ for some $1 < p < \infty$. Then the self-adjoint matrix valued function $\log W$ belongs to BMO, i.e., each entry of $\log W$ is in BMO.*

Proof. Suppose $W \in \mathcal{A}_{p,q}$. Then by Theorem 2.4, there exists $0 < \alpha \leq 1$ such that $W^\alpha \in \mathcal{A}_{2,2}$ (if $p \leq 2$, it suffices to take $\alpha = 1$, by Theorem 2.5). Therefore by [B, Theorem 2.5], $\log W^\alpha = \alpha \log W$ is in BMO, and so is $\log W$. ◻

In the scalar case it is known that the converse statement is true, that is, if $b \in$ BMO, then $\exp(\varepsilon b)$ is in A_p for ε sufficiently close to 0. However, in the matrix case there is a self-adjoint matrix valued function $B(t)$ in BMO, such that $\exp(\varepsilon B(t)) \notin \mathcal{A}_{2,2}$ for any $\varepsilon \neq 0$, see [B]. Therefore, by Theorem 2.4, $\exp(\varepsilon B(t)) \notin \mathcal{A}_{p,q}$ for any $\varepsilon \neq 0$ and $1 < p < \infty$. Also, in the scalar case, if $b \in$ VMO, then $\exp(b)$ is in A_p. This again turns out to be false in the matrix case. The construction of our example is based on the counterexample to Peller's conjecture, found in [TV3].

Theorem 6.2. *There is a self-adjoint matrix valued B belonging to VMO such that $\exp(\varepsilon B)$ is not in $\mathcal{A}_{p,q}$ $(1 < p < \infty)$ for any $\varepsilon \neq 0$.*

Proof. Let

$$B(t) = U(t)^\star \begin{pmatrix} 0 & 0 \\ 0 & b(t) \end{pmatrix} U(t), \quad U(t) = \begin{pmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{pmatrix},$$

where

$$b(t) = \begin{cases} (-\log|t|)^{1/2} & -\frac{1}{2} \le t \le \frac{1}{2}, \\ (-\log\frac{1}{2})^{1/2} & \text{otherwise,} \end{cases}$$

$$\alpha(t) = \begin{cases} \dfrac{\operatorname{sign} t}{\log|t|} & -\frac{1}{2} \le t \le \frac{1}{2}, \\ \dfrac{\operatorname{sign} t}{\log\frac{1}{2}} & \text{otherwise.} \end{cases}$$

By a simple calculation

$$B(t) = \begin{pmatrix} \sin^2\alpha(t)b(t) & \sin\alpha(t)\cos\alpha(t)b(t) \\ \sin\alpha(t)\cos\alpha(t)b(t) & \cos^2\alpha(t)b(t) \end{pmatrix}.$$

It is clear that all entries of the matrix B, except $(B)_{2,2}$, are continuous. But $(B)_{2,2} \sim b(t) = (-\log|t|)^{1/2}$ has vanishing mean oscillation for intervals I as $|I| \to 0$, because $(-\log|t|)^{1/2}$ is in VMO. Since $B(t)$ is constant for $|t| \ge \frac{1}{2}$, the mean oscillation will also vanish for I as $|I| \to \infty$. Therefore the matrix function B is in VMO.

Consider the matrix weight W given by

$$W(t) = \exp(\varepsilon B(t)) = U(t)^\star \begin{pmatrix} 1 & 0 \\ 0 & \exp(\varepsilon b(t)) \end{pmatrix} U(t).$$

By direct computations we have

$$W = \begin{pmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix} + \exp(\varepsilon b(t)) \begin{pmatrix} \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha \end{pmatrix},$$

and

$$W^{-1} = \begin{pmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin^2\alpha \end{pmatrix} + \exp(-\varepsilon b(t)) \begin{pmatrix} \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha \end{pmatrix}.$$

If we consider only symmetric intervals $I = [-a, a]$, $a > 0$, then the off-diagonal entries of the averaged matrices $\langle W\rangle_I$ and $\langle W^{-1}\rangle_I$ vanish because the functions $\sin\alpha(t)$ and $\cos\alpha(t)$ are, respectively, odd and even. It's not hard to see that, for small $a > 0$, we can find a constant $C > 0$ independent of a such that

$$\sin\alpha\left(\frac{a}{2}\right) > C\sin\alpha(a),$$

and consequently we have

$$\frac{1}{|I|}\int_I \sin^2\alpha(t)\,dt \geq \frac{C^2}{2}\sin^2\alpha(a).$$

Therefore we can deduce that

$$\langle W\rangle_I \geq \exp(\varepsilon b(a))\begin{pmatrix} \frac{C^2}{2}\sin^2\alpha(a) & 0 \\ 0 & \cos^2\alpha(a)\end{pmatrix},$$

$$\langle W^{-1}\rangle_I \geq \begin{pmatrix} \cos^2\alpha(a) & 0 \\ 0 & \frac{C^2}{2}\sin^2\alpha(a)\end{pmatrix},$$

and

$$\langle W\rangle_I^{-1} \leq \exp(-\varepsilon b(a))\begin{pmatrix} 2C^{-2}\sin^{-2}\alpha(a) & 0 \\ 0 & \cos^{-2}\alpha(a)\end{pmatrix}.$$

The $\mathcal{A}_{2,2}$ condition requires that there exists a constant $A > 0$ such that for all intervals $I \subset \mathbb{R}$

$$\langle W^{-1}\rangle_I \leq A\langle W\rangle_I^{-1},$$

which in our case would imply that $\exp(\varepsilon b(a))\sin^2\alpha(a)\cos^2\alpha(a)$ must be bounded for small $a > 0$. But

$$\exp(\varepsilon(-\log|a|)^{1/2})\sin^2\alpha(a)\cos^2\alpha(a) \sim \frac{\exp(\varepsilon(-\log|a|)^{1/2})}{\log^2|a|} \to \infty \quad \text{as } a \to 0,$$

by Lemma 6.3. Therefore the matrix weight $W = \exp(\varepsilon B)$ does not belong to $\mathcal{A}_{2,2}$ for any $\varepsilon \neq 0$, and consequently $\exp(\varepsilon B) \notin \mathcal{A}_{p,q}$ for any $1 < p < \infty$. □

Lemma 6.3. *For any ε, δ, $N > 0$*

$$\lim_{x\to\infty}\frac{\exp(\varepsilon(\log|x|)^{\delta})}{\log^N|x|} = \infty.$$

Proof. L'Hôpital rule. □

7. APPENDIX: MINKOWSKI'S INEQUALITY

Suppose (S,μ), (T,ν) are two measure spaces (μ, ν are positive and σ-finite). Suppose that a function $K : S \times T \to [0,\infty)$ is measurable. The *Minkowski inequality* for $1 \le q < \infty$ asserts that

$$\left(\int_T \left(\int_S K(s,t)\, d\mu(s) \right)^q d\nu(t) \right)^{1/q} \le \int_S \left(\int_T K(s,t)^q\, d\nu(t) \right)^{1/q} d\mu(s). \tag{7.1}$$

By a simple change of exponents, $p = 1/q$, we obtain Minkowski's inequality for $0 < p \le 1$,

$$\left(\int_T \left(\int_S K(s,t)\, d\mu(s) \right)^p d\nu(t) \right)^{1/p} \ge \int_S \left(\int_T K(s,t)^p\, d\nu(t) \right)^{1/p} d\mu(s). \tag{7.2}$$

Assume additionally that $K(s,t) > 0$ for a.e. $(s,t) \in S \times T$, and $\nu(T) = 1$. The limiting case of Minkowski's inequality as p approaches 0 asserts that

$$\begin{aligned} \exp\left(\int_T \log\left(\int_S K(s,t)\, d\mu(s) \right) d\nu(t) \right) \\ \ge \int_S \exp\left(\int_T \log K(s,t)\, d\nu(t) \right) d\mu(s). \end{aligned} \tag{7.3}$$

We refer to (7.3) as the *exp-log Minkowski inequality*. This inequality can be applied only if the function

$$t \mapsto \log_+ \left(\int_S K(s,t)\, d\mu(s) \right) \quad \text{is integrable on } T, \tag{7.4}$$

where $\log_+ x = \max(0, \log x)$. Then for a.e. $s \in S$ the function $t \mapsto \log_+ K(s,t)$ is integrable, and under the convention $\exp(-\infty) = 0$ both sides of (7.3) are always meaningful. To show (7.3) we use the following well known fact, see [Bo, Ch. IV Section 6, Ex. 7c] or [DS, Ex. 32, p. 535].

Fact 7.1. Suppose (T,ν) is a measure space with $\nu(T) = 1$. If $f : T \mapsto [0,\infty)$ is a measurable function such that $\int_T f(t)^p\, d\nu(t) < \infty$ for some $p > 0$, then we have

$$\left(\int_T f(t)^p\, d\nu(t) \right)^{1/p} \to \exp\left(\int_T \log f(t)\, d\nu(t) \right) \quad \text{as } p \to 0^+. \tag{7.5}$$

Proof of (7.3). We claim that it suffices to show (7.3) under the hypothesis $\mu(S) < \infty$. Indeed, if $\mu(S) = \infty$, we express $S = \bigcup_{m=1}^\infty S_m$, where $S_m \subset S_{m+1}$, and $\mu(S_m) < \infty$ for all m. Since (7.4) holds with S replaced by S_m, we have (7.3) with S replaced by S_m. We obtain (7.3) by letting $m \to \infty$ and using the Monotone Convergence Theorem.

Suppose now $\mu(S) < \infty$. First we show (7.3) under the assumption that K is bounded. By Fatou's Lemma, (7.2), and (7.5)

$$\begin{aligned}
&\int_S \exp\left(\int_T \log K(s,t)\,d\nu(t)\right) d\mu(s) \\
&\quad = \int_S \lim_{p\to 0^+} \left(\int_T K(s,t)^p\,d\nu(t)\right)^{1/p} \\
&\quad \le \liminf_{p\to 0^+} \int_S \left(\int_T K(s,t)^p\,d\nu(t)\right)^{1/p} d\mu(s) \\
&\quad \le \liminf_{p\to 0^+} \left(\int_T \left(\int_S K(s,t)\,d\mu(s)\right)^p d\nu(t)\right)^{1/p} \\
&\quad = \exp\left(\int_T \log\left(\int_S K(s,t)\,d\mu(s)\right) d\nu(t)\right).
\end{aligned}$$

This shows (7.3) for K bounded. Suppose now that K is an arbitrary non-negative function on $S \times T$ such that (7.4) holds. For every $\varepsilon > 0$ define K_ε by $K_\varepsilon(s,t) = \max(\varepsilon, K(s,t))$. By (7.3) and the Monotone Convergence Theorem applied to the sequence of functions $(\min(m, K_\varepsilon))_{m=1}^\infty$ we have

$$\begin{aligned}
\exp\left(\int_T \log\left(\int_S K_\varepsilon(s,t)\,d\mu(s)\right) d\nu(t)\right)& \\
\ge \int_S \exp\left(\int_T \log K_\varepsilon(s,t)\,d\nu(t)\right) d\mu(s).&
\end{aligned} \tag{7.6}$$

By (7.4) and

$$\varepsilon\mu(S) \le \int_S K_\varepsilon(s,t)\,d\mu(s) \le \varepsilon\mu(S) + \int_S K(s,t)\,d\mu(s), \tag{7.7}$$

the left side of (7.6) is finite. In particular, by (7.6) with $\varepsilon = 1$ we have $t \mapsto \log K_1(s,t) = \log_+ K(s,t)$ is integrable for a.e. $s \in S$. By (7.4), (7.7), and the Monotone Convergence Theorem the left side of (7.6) converges to the left side of (7.3) as $\varepsilon \to 0$. Since the right side of (7.6) is greater than the right side of (7.3), this shows (7.3). ◻

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