10.7b The first part of this proof is exactly the first part of the proof in the book. The line \( L_x \) divides \( A \) into two pieces of equal area and divides \( B \) into pieces of area \( b_1(x) \) and \( b_2(x) \), where \( b_1 \) is the area of the piece furthest from \( x \) and \( b_2 \) is the area of the piece closest to \( x \). Consider the continuous function \( f(x) = b_1(x) - b_2(x) \). Notice that \( f(-x) = -f(x) \) because the piece furthest from \( x \) is the piece closest to \( -x \) and the piece closest to \( x \) is the piece furthest from \( -x \). So as we move around the circle from \( x \) to \( -x \) the function \( f \) changes sign, and thus it must be zero at an intermediate point. At this point, the line which divides \( A \) into two equal pieces also divides \( B \) into two equal pieces.

11.2 a Let \( \mathcal{W} \subseteq M \) be open and suppose \( M \) is an \( n \)-manifold. Let \( p \in \mathcal{W} \). Since \( M \) is a manifold, there is an open set \( p \in U \subseteq M \) and an open \( V \subseteq \mathbb{R}^n \) and a homeomorphism \( \varphi : U \to V \). Then \( \varphi|_{U \cap \mathcal{W}} : U \cap \mathcal{W} \to V \cap \mathcal{W} \) is a homeomorphism, and thus each point in \( \mathcal{W} \) has an open neighborhood in \( \mathcal{W} \) homeomorphic to an open set in \( \mathbb{R}^n \).

11.2 b Suppose \((z_1, z_2, \ldots, z_{n+1})\) represents a point in \( CP^n \). Then
\[
|z_1|^2 + |z_2|^2 + \ldots + |z_{n+1}|^2 = 1
\]
and therefore some \( z_i \neq 0 \). Suppose, for example, that \( z_{n+1} \neq 0 \). In that case, \( z_{n+1} \) can be written in “polar coordinates” as \( e^{i\theta}r \) where \( r \) is real and positive. Multiplying all the coordinates by \( e^{-i\theta} \) gives another representative of the same point; the new representative has the form \((w_1, w_2, \ldots, w_n, r)\) for complex numbers \( w_1, w_2, \ldots, w_n \). This new representative still lives in \( S^{2n+1} \), so
\[
|w_1|^2 + |w_2|^2 + \ldots + |w_n|^2 + r^2 = 1
\]
and therefore
\[
r = \sqrt{1 - |w_1|^2 - |w_2|^2 - \ldots - |w_n|^2}
\]
The conclusion is that points in \( CP^n \) with \( z_{n+1} \neq 0 \) have unique representatives of the form
\[
(w_1, w_2, \ldots, w_n, \sqrt{1 - |w_1|^2 - |w_2|^2 - \ldots - |w_n|^2})
\]
where \( |w_1|^2 + |w_2|^2 + \ldots + |w_n|^2 < 1 \) and thus
\[
(w_1, w_2, \ldots, w_n) \in D^{2n}
\]
where \( D^{2n} \) is the open disk of radius one.

Now I’d like to show formally that these provide local coordinates, i.e., that each point in \( CP^n \) has an open neighborhood homeomorphic to an open set in \( R^{2n} \). I’ll assume that my point has \( z_{n+1} \neq 0 \); analogous arguments work if some other \( z_i \neq 0 \).
Let $U$ be the set of all points in $CP^n$ whose representatives satisfy $z_{n+1} \neq 0$ and let $V$ be the open unit disk $D^{2n}$. Map $V \to U$ by

$$(w_1, w_2, \ldots, w_n) \to \left(w_1, w_2, \ldots, w_n, \sqrt{1 - |w_1|^2 - |w_2|^2 - \ldots - |w_n|^2}\right)$$

Map $U \to V$ by

$$(z_1, z_2, \ldots, z_n, z_{n+1}) \to \left(\frac{z_1}{z_{n+1}}|z_{n+1}|, \frac{z_2}{z_{n+1}}|z_{n+1}|, \ldots, \frac{z_n}{z_{n+1}}|z_{n+1}|\right)$$

These maps are inverse to each other, so each map is one-to-one and onto. We are done if $U$ is open and if both of these maps are continuous. But $U$ is open because its inverse image in $S^{2n+1}$ is open, being $\{(z_1, z_2, \ldots, z_{n+1}) \in S^{2n+1} \mid z_{n+1} \neq 0\}$. The first map is a map $D^{2n} \to CP^n$ induced by a continuous map $D^{2n} \to S^{2n+1}$ and so continuous. The bottom map is a map $U \subseteq CP^n \to D^{2n}$ induced from a continuous map from a subset of $S^{2n+1}$ to $D^{2n}$ and thus continuous.

To complete the argument, we need only show that $CP^n$ is Hausdorff. Since $S^{2n+1}$ is compact Hausdorff, we can use theorem 8.11, so it suffices to show that $\pi : S^{2n+1} \to CP^n$ is a closed map. Thus we want to show that if $A \subseteq S^{2n+1}$ is closed, then $\pi^{-1}(A)$ is closed.

This set is the set of all points in $S^{2n+1}$ which are equivalent to points in $A$. Equivalently, it is the image of $S^1 \times A$ under the map

$$S^1 \times A \to S^1 \times S^{2n+1} \to S^{2n+1}$$

where this last map is

$$\lambda \times (z_1, z_2, \ldots, z_{n+1}) \to (\lambda z_1, \lambda z_2, \ldots, \lambda z_{n+1})$$

But $A \subseteq S^{2n+1}$ is closed, so compact. Thus $S^1 \times A$ is compact, and so its image in $S^{2n+1}$ is compact, and so closed.

11.2 Each point in $M$ has an open neighborhood $U$ homeomorphic to an open $V \subseteq R^n$. Shrinking $V$ if necessary, we can suppose that $V$ is a disk. Magnifying, we can suppose that $V$ is the open disk of radius 1, and thus that $U$ is homeomorphic to such a disk.

$M$ is covered by the union of these $U$ and so by finitely many of them, $U_1, \ldots, U_k$. By 8.14j, $X/(X - U_i)$ is homeomorphic to $U_i^\infty$, and thus homeomorphic to $(\partial D)^\infty$, which is homeomorphic to $S^n$.

The map $M \to M/(M - U_i)$ is continuous, so $M \to M/(M - U_i) \to S^n$ is continuous. Putting these maps together for all $i$ gives a map

$$M \to M/(M - U_1) \times \ldots \times M/(M - U_k) \to S^n \times \ldots \times S^n \subseteq R^{n+1} \times \ldots \times R^{n+1} = R^{k(n+1)}.$$
We now claim this map is one-to-one. If so, we are done, because $M$ is compact Hausdorff, so a continuous one-to-one map onto its image is automatically a homeomorphism.

Suppose $x$ and $y \in M$ and $x \neq y$. If $x \in U_i$ and $y \in U_i$, then $x \not\in M - U_i$ and $y \not\in M - U_i$, so $x$ and $y$ represent different elements in $M/(M - U_i)$ and thus map to different points in $\mathbb{R}^{k(n+1)}$. If $x \in U_i$ and $y \not\in U_i$, then $x \not\in X - U_i$ and $y \in M - U_i$ and so $x$ and $y$ represent different points in $M/(M - U_i)$. But $x$ is certainly in some $U_i$.

**Extra Problem 1**

Notice that the two top $b$ arrows are glued together, so their ends become the same point. But these ends are the two ends of $a$. So the ends of $b$ and both ends of $a$ are the same point.

On the right side, we see that the end of $a$ is the start of $b$, so both ends of $b$ and both ends of $a$ all glue to the same point. The same reasoning applied to the bottom of the diagram shows that both ends of $d$ and both ends of $c$ glue to the same point.

But the $a$ on the right goes from one end of $d$ to an end of $b$, so all ends of $a$ and $b$ are glued to all ends of $c$ and $d$. 
Extra Problem 2

cut shaded region from here
and glue it here
Extra Problem 3

Reading from bottom to top counterclockwise, the end of $a$ is glued to the start of $b$, which is glued to the end of $b$, which is glued to the start of $c$. Moreover, from the top left, the end of $c$ is glued to the end of $a$. So the start and end of $c$ are glued to the start and end of $b$ which are glued to the end of $a$. But the start of $c$ is glued to the start of $a$. So all vertices are glued together.
Extra Problem 4

now relabel and change some arrows!