1 The Cayley-Dickson Construction

This material is taken from the paper The Octonions by John C. Baez, published in the Bulletin of the AMS in 2002 and also available on the web at http://math.ucr.edu/home/baez/octonions/

Our goal is describe an 8-dimensional algebra satisfying the conditions of Hurwitz’s theorem. This algebra was discovered by a friend of Hamilton’s, John T. Graves, on December 26th of 1843. It was independently discovered by Cayley. The algebra is known as the octonions or Cayley numbers.

We'll describe a later treatment of this algebra by Dickson.

An admissible structure on $\mathbb{R}^n$ is a bilinear product with unit on $\mathbb{R}^n$ and a conjugation operation $v \to \overline{v}$ on $\mathbb{R}^n$, such that

- If $u$ is the unit, $\overline{u} = u$
- We have $\overline{a} = a$ for all $a$, and $\overline{ab} = \overline{b} \overline{a}$ for all $a$ and $b$
- For any $a$, $a + \overline{a}$ is a multiple of the unit
- For any $a$, $a\overline{a}$ is $||a||^2$ times the unit.

Suppose we have a bilinear product on $\mathbb{R}^n$. Suppose this product has a unit, and identify the real numbers with multiples of this unit. Suppose that we have a conjugation operation $v \to \overline{v}$ on $\mathbb{R}^n$, satisfying $\overline{a} = a$ and $\overline{ab} = \overline{b} \overline{a}$. Suppose that $a + \overline{a}$ is a real multiple of the unit, and $v\overline{v} = ||v||^2$ times the unit.

We identify the real numbers with the scalar multiples of the unit.

We'll describe a construction called the Cayley-Dickson construction, which produces a similar algebra on $\mathbb{R}^{2n}$. By definition, an element of this new algebra is a pair $(a, b)$ with
\(a, b \in \mathbb{R}^n\). Define

\[(a, b)(c, d) = (ac - db, \overline{ad} + cb)\]

\[\overline{(a, b)} = (\overline{a}, -b)\]

We now must show that this new algebra has all the properties required of the original algebra. If 1 is the unit of the original algebra, \((1, 0)\) is the unit of the new algebra because

\[(1, 0)(a, b) = (a, 1b) = (a, b) = (a, b)(1, 0) = (a, b)\]

This product and conjugation have all the required properties. Indeed

\[\overline{(1, 0)} = (\overline{1}, -0) = (1, 0)\]

and

\[\overline{(a, b)} = (\overline{a}, -b) = (\overline{a}, b) = (a, b)\]

Note also that

\[\overline{(a, b)(c, d)} = \overline{(ac - db, \overline{ad} + cb)} = (ac - db, -\overline{ad} - cb) = (\overline{\overline{a}} \overline{c} - b \overline{d}, -\overline{ad} - cb)\]

and

\[\overline{(c, d)} \overline{(a, b)} = (\overline{c}, -d)(\overline{a}, -b) = (\overline{c} \overline{a} - b \overline{d}, -\overline{cd} - \overline{ad})\]

We have \((a, b) + (\overline{a}, b) = (a, b) + (\overline{a}, -b) = (a + \overline{a}, 0)\), which is a multiple of \((1, 0)\).

Finally, notice that

\[(a, b)(\overline{a}, -b) = (a, b)(\overline{a}, -b) = (a \overline{a} + b \overline{b}, -\overline{ab} + \overline{a}b) = (a \overline{a} + b \overline{b}, 0) = ||a||^2 + ||b||^2 = ||(a, b)||^2\]

**Examples**

We can start the construction with the usual product and trivial conjugation on \(\mathbb{R}^2\). Then we get an algebra structure on \(\mathbb{R}^2\) satisfying

\[(a, b)(c, d) = (ac - db, \overline{ad} + cb) = (ac - bd, ad + bc)\]

\[\overline{(a, b)} = (\overline{a}, -b) = (a, -b)\]

Clearly this gives the complex numbers.

Next apply the Cayley-Dickson construction to the complex numbers. We claim that we get the quaternions. To see this, notice that \(q = a + bi + cj - dk = (a + bi) + j(c + di) = A + jB\) where \(A\) and \(B\) are complex. Also notice that for complex \(A\), \(jA = \overline{A}j\). So

\[(A + jB)(C + jD) = AC + jBC + AjD + jBjD = AC + jBC + j\overline{A}D + j^2\overline{B}D\]
Thus

\[(A + jB)(C + jD) = (AC - BD) + j(AD + BC)\]

and since complex numbers commute, this agrees with the formula

\[(a, b)(c, d) = (ac - db, an + cb)\]

Moreover

\[(A + jB) = \overline{A} + \overline{B}(-j) = \overline{A} - jB\]

agrees with the general formula

\[\overline{(a, b)} = (\overline{a}, -b)\]

Applying the construction once more gives an algebra structure on \(R^8\). This structure is not associative, so great care is required when working with it. However, we will show that \(||o_1 o_2||^2 = ||o_1||^2 ||o_2||^2\). It follows that it satisfies the conditions of Hurwitz’s theorem, that non-zero elements have multiplicative inverses, and that the algebra has no zero divisors.

All the remaining Cayley-Dickson algebras have zero divisors.

## 2 The Octonions

By definition, the octonions or Cayley numbers are the result of applying the Cayley-Dickson construction to the quaternions.

We want to prove that the octonions satisfy the Hurwitz condition. To see this, let \(a\) and \(b\) be octonions. We want to prove that \(||ab||^2 = ||a||^2 ||b||^2\). A naive proof would proceed as follows:

\[||ab||^2 = (ab)(\overline{ab}) = (ab)(\overline{b} \overline{a}) = a(b\overline{b})\overline{a}\]

Unfortunately, this last step uses associativity, which isn’t always true in the octonions. But ignoring this, we could note that \(b\overline{b} = ||b||^2\) is real and thus commutes with all octonions, so this is \(||b||^2a\overline{a} = ||b||^2 ||a||^2\).

Consequently, we try to prove this from first principles. Consider two octonions \((a, b)\) and \((c, d)\). We form

\[(a, b)(c, d) = (ac - db, \overline{a}d + cb)\]

Then

\[(a, b)(c, d)(a, b)(c, d) = (ac - db, \overline{a}d + cb)(\overline{c} \overline{a} - b\overline{d}, -\overline{a}d - cb)\]

The second component of this product is

\[(\overline{c} \overline{a} - b\overline{d})(-\overline{a}d - cb) + (\overline{c} \overline{a} - b\overline{d})(\overline{a}d + cb) = 0\]
The first component of the product is

$$(ac - db)(\vec{c} \vec{a} - b\vec{d}) - (-ad - cb)(\vec{d}a + b\vec{c})$$

This product has eight terms. Four are

$$||a||^2||c||^2 + ||b||^2||d||^2 + ||a||^2||d||^2 + ||b||^2||a||^2 = (||a||^2 + ||b||^2)(||c||^2 + ||d||^2)$$

This is just

$$||(a, b)||^2||(c, d)||^2$$

exactly the result we desire The final four terms are

$$-acb\vec{d} - d\vec{b} \times \vec{a} + ad\vec{b}\vec{c} + cb\vec{d}a$$

This can be rewritten

$$-2Re(acb\vec{d}) + 2Re(cb\vec{d}a)$$

and consequently equals a purely real quaternion. On the other hand, it is the real part of the difference

$$(cb\vec{d})a - a(cb\vec{d})$$

But the real part of a product of two quaternions $(r, v)(s, w)$ is $rs - v \cdot w$ and this does not depend on the order of the terms. So our real part is zero. QED.