**Definition:** Suppose that two random variables, either continuous or discrete, $X$ and $Y$ have joint density

$$f(x, y), \quad (x, y) \in \Lambda_{X,Y}$$

and marginal densities

$$f_X(x), \quad x \in \Lambda_X,$$
$$f_Y(y), \quad y \in \Lambda_Y.$$

The conditional probability density that $Y$ takes on the value $y$ given that $X$ is equal to $x$ is denoted by $f_{Y|x}(y)$ and is given by

$$f_{Y|x}(y) = \frac{f(x, y)}{f_X(x)} \quad (x, y) \in \Lambda_{X,Y}, x \in \Lambda_X$$

and the conditional probability density that $X$ takes on the value $x$ given that $Y$ is equal to $y$ is denoted by $f_{X|y}(x)$ and is given by

$$f_{X|y}(x) = \frac{f(x, y)}{f_Y(y)} \quad (x, y) \in \Lambda_{X,Y}, y \in \Lambda_Y.$$

**Example:** In a box, there are 4 white balls, 3 black balls, 2 red balls. We randomly select 2 balls without replacement. Define

$$X = \{\text{number of white balls}\}$$
$$Y = \{\text{number of black balls}\}$$

(1) Find density function $f(x, y)$ and marginal densities $f_X(x)$ and $f_Y(y)$.

(2) Find the conditional probability

$$\mathbb{P}(Y \leq 1|X = 0).$$

(3) Find the conditional probability

$$\mathbb{P}(X = 1|Y = 2).$$
Solution: (1) We can find

\[ P(X = 0, Y = 0) = \frac{C_2^2}{C_9^2} = \frac{1}{36}, \quad P(X = 0, Y = 1) = \frac{C_1^3 C_1^2}{C_9^2} = \frac{6}{36} \]

\[ P(X = 0, Y = 2) = \frac{C_2^3}{C_9^2} = \frac{3}{36}, \quad P(X = 1, Y = 0) = \frac{C_1^4 C_1^3}{C_9^2} = \frac{8}{36} \]

\[ P(X = 1, Y = 1) = \frac{C_1^4 C_1^3}{C_9^2} = \frac{12}{36}, \quad P(X = 2, Y = 0) = \frac{C_2^4}{C_9^2} = \frac{6}{36} \]

Dist. and marginal dists table

<table>
<thead>
<tr>
<th>Y \ X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>( f_Y(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{1}{36} )</td>
<td>( \frac{8}{36} )</td>
<td>( \frac{6}{36} )</td>
<td>( \frac{15}{36} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{6}{36} )</td>
<td>( \frac{12}{36} )</td>
<td>0</td>
<td>( \frac{18}{36} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{3}{36} )</td>
<td>0</td>
<td>0</td>
<td>( \frac{3}{36} )</td>
</tr>
<tr>
<td>( f_X(x) )</td>
<td>( \frac{10}{36} )</td>
<td>( \frac{20}{36} )</td>
<td>( \frac{6}{36} )</td>
<td></td>
</tr>
</tbody>
</table>

(2)

\[ f_{Y|X}(1) = \frac{f(0, 1)}{f_X(0)} = \frac{6/36}{10/36} = \frac{3}{5} \]

\[ f_{Y|X}(0) = \frac{f(0, 0)}{f_X(0)} = \frac{1/36}{10/36} = \frac{1}{10} \]

Thus,

\[ P(Y \leq 1|X = 0) = f_{Y|X}(0) + f_{Y|X}(1) = \frac{3}{5} + \frac{1}{10} = \frac{7}{10}. \]

\[ P(X = 1|Y = 2) = f_{X|Y}(1) = \frac{f(1, 2)}{f_Y(2)} = \frac{0}{3/36} = 0 \]
**Example:** Given the joint pdf

\[ f(x, y) = 2e^{-(x+y)} \quad 0 < x < y, \]

Find

(a) \( P(Y < 1|X = 1/2) \)
(b) \( P(Y < 1|X < 1) \)

**Solution:** (a) Since

\[ f_X(x) = \int_x^\infty 2e^{-x} e^{-y} dy = 2e^{-2x} \quad x > 0 \]

Thus,

\[ f_{Y|\frac{1}{2}}(y) = \frac{f(\frac{1}{2}, y)}{f_X(\frac{1}{2})} = \frac{2e^{-(y+\frac{1}{2})}}{2e^{-2(\frac{1}{2})}} = e^{-y+\frac{1}{2}}, \quad \frac{1}{2} < y \]

\[ P(Y < 1|X = 1/2) = \int_{\frac{1}{2}}^1 e^{\frac{1}{2}-y} dy = -\sqrt{e}e^{-y}\big|_{\frac{1}{2}}^1 = 1 - \frac{1}{\sqrt{e}} \]

(b) Since

\[ P(X < 1) = \int_0^1 2e^{-2x} dx = 1 - e^{-2} = 1 - 0.135 = 0.865 \]

\[ P(X < 1, Y < 1) = \int_0^1 dy[\int_0^y 2e^{-(x+y)} dx] \]

\[ = \int_0^1 dy\{2e^{-y}\left[-e^{-x}\big|_0^y\right]\} \]

\[ = \int_0^1 \{2e^{-y} - 2e^{-2y}\} dy \]

\[ = 1 - \frac{2}{e} + e^{-2} = 0.39957 \]

\[ P(Y < 1|X < 1) = \frac{0.39957}{0.865} = 0.46193 \]

Intuitively, a **mean** or an **expected value** of a rv \( X \) is \( X \)'s central tendency or average.

**Definition:**
Given a discrete random variable $X$ with range space $\Lambda = \{x_1, \ldots, x_n\}$ and the distribution table.

The distribution table of r.v $X$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\cdots$</th>
<th>$x_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>$p(x_1)$</td>
<td>$p(x_2)$</td>
<td>$\cdots$</td>
<td>$p(x_n)$</td>
</tr>
</tbody>
</table>

The mean or expected value of $X$ is defined by

$$\mu = \mathbb{E}(X) = \sum_{i=1}^{n} x_i p(x_i)$$

Example: In a class A, there are total 20 students. The students’ scores of the final exam are divided into five categories. Let $X$ be a rv which associates each student’s score of the final exam with a specified category. $X$ has following distribution table:

<table>
<thead>
<tr>
<th>$X$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
</table>

Find the average category of the students’ scores of the final exam.

Solution:

$$\mathbb{E}(X) = 1 \times 3/20 + 2 \times 6/20 + 3 \times 3/20 + 4 \times 3/20 + 5 \times 5/20 = 3.05$$

Example: A company has five applicants for two positions: two women and three men. Suppose that the five applicants are equally qualified and that no preference is given for choosing either gender. Let $X$ equal the number of women chosen to fill the two positions.
(a) Find the probability distribution of $X$;
(b) Find the average number of women who will be hired.

Solution: (a). The sample space
\[
\Omega = \{\omega_1 = \{M_1M_2\}, \omega_2 = \{M_1M_3\}, \omega_3 = \{M_2M_3\}, \]
\[
\omega_4 = \{W_1M_1\}, \omega_5 = \{W_1M_2\}, \omega_6 = \{W_1M_3\}, \omega_7 = \{W_2M_1\}, \]
\[
\omega_8 = \{W_2M_2\}, \omega_9 = \{W_2M_3\}, \omega_{10} = \{W_1W_2\}\]

and the range space
\[
\Lambda := \{x_1 = 0, x_2 = 1, x_3 = 2\}.
\]

\[
p(x_1) = p(0) = \mathbb{P}(X = 0)
= \mathbb{P}(\{\omega_1, \omega_2, \omega_3\})
= \mathbb{P}(\{\{M_1M_2\}, \{M_1M_3\}, \{M_2M_3\}\})
= \frac{3}{10},
\]

\[
p(x_2) = p(1) = \mathbb{P}(X = 1) = \mathbb{P}(\{\omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9\})
= \mathbb{P}(\{\{W_1M_1\}, \{W_1M_2\}, \{W_1M_3\}, \{W_2M_1\}, \{W_2M_2\}, \{W_2M_3\}\})
= \frac{6}{10},
\]

\[
p(x_3) = p(2) = \mathbb{P}(X = 2) = \mathbb{P}(\{\omega_{10} = \{W_1W_2\}\}) = \frac{1}{10}.
\]
Therefore,

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>3/10</td>
<td>6/10</td>
<td>1/10</td>
</tr>
</tbody>
</table>

(b)

$\mu = \mathbb{E}(X) = \sum_{x_i \in \Lambda} x_i p(x_i)$

$= 0(3/10) + 1(6/10) + 2(1/10) = 8/10.$

**Definition:**
Given a continuous random variable $X$ with range space $\Lambda$ and the density function: $f(x), x \in \Lambda$. The **mean** or **expected value** of $X$ is defined by

$$\mu = \mathbb{E}(X) = \int_{\Lambda} x f(x) dx$$

**Example:** Let $X$ be a rv which represents the lifetime of a bulb. $X$ has pdf $f(x) = \lambda e^{-\lambda x}, x > 0$, where $\lambda > 0$ is a constant. Find the average lifetime of a bulb.

**Solution:** Using integration by part, we can get

$$\mu = \mathbb{E}(X) = \int_0^\infty \lambda xe^{-\lambda x} dx$$

$$= 1/\lambda$$

**Theorem:** Let $X_1, \cdots, X_n$ be rv’s for which $\mathbb{E}(X_i) < \infty, i = 1, 2, \cdots, n$ and $a_1, \cdots, a_n$ be constants. Then,

$$\mathbb{E}(a_1X_1 + \cdots + a_nX_n) = a_1\mathbb{E}(X_1) + \cdots + a_n\mathbb{E}(X_n)$$

**Theorem:** (1) Let $X$ be a discrete rv with range space $\Lambda = \{x_1, \cdots, x_n\}$ and the distribution table.
The distribution table of r.v $X$

<table>
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</tr>
</tbody>
</table>

Let $g$ be a function such that

$$\sum_{x_i \in \Lambda} |g(x_i)|p(x_i) < \infty$$

The mean of $g(X)$ is given by

$$\mathbb{E}g(X) = \sum_{i=1}^{n} g(x_i)p(x_i)$$

(2) Let $Y$ be a continuous rv with range space $\Lambda$ and the density function: $h(y), y \in \Lambda$. Let $g$ be a function such that

$$\int_{y \in \Lambda} |g(y)|h(y)dy < \infty$$

The mean of $g(Y)$ is given by

$$\mathbb{E}g(Y) = \int_{\Lambda} g(y)h(y)dy$$
**Theorem:** If $X$ and $Y$ are two independent rv’s and $\mathbb{E}(X) < \infty, \mathbb{E}(Y) < \infty$, then, we have

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y).$$

**Example:** Let $X$, $Y$ be two independent rv’s with density functions:

$$f_X(x) = 2x, \quad 0 < x < 1$$

and

$$f_Y(y) = 1/2 \quad 0 < y < 2$$

Find $\mathbb{E}(2X - 3Y)^2 = \mathbb{E}[(2X - 3Y)^2]$.  

**Solution:** Since $(2X - 3Y)^2 = 4X^2 - 12XY + 9Y^2$, we have

$$\mathbb{E}(2X - 3Y)^2 = 4\mathbb{E}(X^2) - 12\mathbb{E}(X)\mathbb{E}(Y) + 9\mathbb{E}(Y^2).$$

Since

$$\mathbb{E}(X) = \int_0^1 x2xdx = 2/3$$

$$\mathbb{E}(Y) = \int_0^2 y(1/2)dy = 1$$

$$\mathbb{E}(X^2) = \int_0^1 x^22xdx = 1/2$$

$$\mathbb{E}(Y^2) \int_0^2 y^2(1/2)dy = 4/3$$

Thus,

$$\mathbb{E}(2X - 3Y)^2 = 4(1/2) - 12(2/3)1 + 9(4/3) = 6$$

**Definition:**

Given a discrete random variable $X$ with range space $\Lambda = \{x_1, \cdots, x_n\}$ and distribution table as follows:
The distribution table of r.v $X$

<table>
<thead>
<tr>
<th>$X$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\cdots$</th>
<th>$x_n$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$p(x_1)$</td>
<td>$p(x_2)$</td>
<td>$\cdots$</td>
<td>$p(x_n)$</td>
</tr>
</tbody>
</table>

Let $\mu = \mathbb{E}(X) = \sum_{i=1}^{n} x_i p(x_i)$ be its mean. Then, the variance of $X$ is denoted by $\sigma^2$ or $\text{Var}(X)$, is defined by

$$\text{Var}(X) = \sigma^2 = \mathbb{E}(X - \mu)^2 = \sum_{x_i \in \Lambda} (x_i - \mu)^2 p(x_i)$$

The standard deviation of $X$ is defined as $\sigma = \sqrt{\sigma^2}$.

Remark: The variance of a rv $X$ is a measurement to indicate the dispersion of the distribution of $X$.

Example: A company has five applicants for two positions: two women and three men. Suppose that the five applicants are equally qualified and that no preference is given for choosing either gender. Let $X$ equal the number of women chosen to fill the two positions.

Find the mean and standard deviation of $X$.

Solution: The sample space

$$\Omega = \{ \omega_1 = \{M_1 M_2\}, \omega_2 = \{M_1 M_3\}, \omega_3 = \{M_2 M_3\}, \omega_4 = \{W_1 M_1\}, \omega_5 = \{W_1 M_2\}, \omega_6 = \{W_1 M_3\}, \omega_7 = \{W_2 M_1\}, \omega_8 = \{W_2 M_2\}, \omega_9 = \{W_2 M_3\}, \omega_{10} = \{W_1 W_2\} \}$$

and the range space

$$\Lambda := \{ x_1 = 0, x_2 = 1, x_3 = 2 \}.$$

\[ p(x_1) = p(0) = \mathbb{P}(X = 0) \\
= \mathbb{P}(\{\omega_1, \omega_2, \omega_3\}) \\
= \mathbb{P}(\{\{M_1M_2\}, \{M_1M_3\}, \{M_2M_3\}\}) \\
= \frac{3}{10}, \]

\[ p(x_2) = p(1) = \mathbb{P}(X = 1) = \mathbb{P}(\{\omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9\}) \\
= \mathbb{P}(\{\{W_1M_1\}, \{W_1M_2\}, \{W_1M_3\}, \{W_2M_1\}, \{W_2M_2\}, \{W_2M_3\}\}) \\
= \frac{6}{10}, \]

\[ p(x_3) = p(2) = \mathbb{P}(X = 2) = \mathbb{P}(\{\omega_{10} = \{W_1W_2\}\}) = \frac{1}{10}. \]

Therefore,

**The distribution table of r.v \(X\)**

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p(x))</td>
<td>3/10</td>
<td>6/10</td>
<td>1/10</td>
</tr>
</tbody>
</table>

The mean is

\[
\mu = \mathbb{E}(X) = \sum_{x_i \in \Lambda} x_i p(x_i) \\
= 0(3/10) + 1(6/10) + 2(1/10) = 8/10.
\]

The variance of \(X\) is

\[
\sigma^2 = \mathbb{E}[(X - \mu)^2] = \sum_{x_i \in \Lambda} (x_i - \mu)^2 p(x_i) \\
= (0 - 8/10)^2(3/10) + (1 - 8/10)^2(6/10) + (2 - 8/10)^2(1/10) = 0.36
\]

and

\[
\sigma = \sqrt{\sigma^2} = 0.6
\]
Definition: 
Given a continuous random variable $X$ with density function 

$$f(x), \quad x \in \Lambda.$$ 

Let $\mu = \mathbb{E}(X) = \int_{\Lambda} xf(x)dx$ be its mean. Then, the variance of $X$ is denoted by $\sigma^2$ or $\text{Var}(X)$, is defined by 

$$\text{Var}(X) = \sigma^2 = \mathbb{E}(X - \mu)^2 = \int_{x \in \Lambda} (x - \mu)^2 f(x)dx$$

The standard deviation of $X$ is defined as $\sigma = \sqrt{\sigma^2}$.

Theorem: Let $X$ be a rv, discrete or continuous, have mean $\mu$ and finite $\mathbb{E}(X^2)$. Then,

$$\text{Var}(X) = \sigma^2 = \mathbb{E}(X - \mu)^2 = \mathbb{E}(X^2) - \mu^2$$

Theorem: Let $X$ be a rv, discrete or continuous, and $a$, $b$ be two constants. Then,

$$\text{Var}(aX + b) = a^2 \text{Var}(X)$$

Theorem: Let $X_1, \ldots, X_n$ be independent rv’s, discrete or continuous. Then,

$$\text{Var}(X_1 + \cdots + X_n) = \text{Var}(X_1) + \cdots + \text{Var}(X_n)$$

Example: Let $X, Y$ be two independent rv’s with following pdf’s:

$$f_X(x) = 2x, \quad 0 < x < 1$$

$$f_Y(y) = 6y^5, \quad 0 < y < 1$$

Find $\text{Var}(7X + 8Y)$.

Solution:

$$\mathbb{E}(X) = \int_0^1 x 2xdx = 2/3 x^3 |^1_0 = 2/3$$
\[ \mathbb{E}(Y) = \int_{0}^{1} y6y^5 \, dy = 6/7x^7|_{0}^{1} = 6/7 \]

\[ \text{Var}(7X + 8Y) = \text{Var}(7X) + \text{Var}(8Y) \]
\[ = 7^2 \text{Var}(X) + 8^2 \text{Var}(Y) \]
\[ = 49(\mathbb{E}X^2 - 4/9) + 64(\mathbb{E}Y^2 - 36/49) \]

Since
\[ \mathbb{E}X^2 = \int_{0}^{1} x^2 2x \, dx = 1/2 \]
\[ \mathbb{E}Y^2 = \int_{0}^{1} x^2 6x^5 \, dx = 3/4 \]

Thus,
\[ \text{Var}(7X + 8Y) = 49(\mathbb{E}X^2 - 4/9) + 64(\mathbb{E}Y^2 - 36/49) \]
\[ = 49(1/2 - 4/9) + 64(3/4 - 36/49) = 3265/882 \]

**Theorem:** Let \( X \) be a rv with binomial distribution \( b(n, p) \). Then, \( \mu = \mathbb{E}(X) = np \) and \( \text{Var}(X) = np(1 - p) \).

**Example:** Using
\[ \text{Var}(X_1 + \cdots + X_n) = \text{Var}(X_1) + \cdots + \text{Var}(X_n) \]
if \( X_1, \cdots, X_n \) are independent, prove above theorem.

**Example:** Let \( X_1, \cdots, X_{100} \) be independent rv’s and each with binomial distribution \( b(10, 1/3) \). Find
\[ \mathbb{E}\left\{ \frac{1}{100} \sum_{i=1}^{100} X_i \right\} \]
and
\[ \text{Var}\left( \frac{1}{100} \sum_{i=1}^{100} X_i \right) \]