Example 12.1:
In order to guarantee a lower failure rate of one important supervisory duty, a military control center wants to install several same kind of computers for the supervisory. Suppose that each computer will fail independently with the same probability $p = 0.01$. The supervisory is successful if there is at least one computer is in working order. Find a minimum $n$ of computers which are required to be installed such that the failure rate is equal or less than 0.0001.

Solution: Suppose that there are $n$ computers installed. We need to find this number $n$. Let $X$ be the number of computers in working order. According to the question, we need to find $n$ such that the following inequality

$$P(X = 0) \leq 0.0001$$

satisfied. Since $P(X = 0) = C_0^n(1-p)^0 p^n = p^n$, it just finds $n$ such that $p^n \leq 0.0001$. Since

$$(0.01)^n \leq 0.0001 \iff n \geq 2$$

Therefore, $n = 2$.

Definition: Let $X$ be a rv defined on a sample space $\Omega$. For each real number $x \in \mathbb{R}$, we define

$$F(x) = P(X \leq x)$$

Thus in this way we have defined a function

$$F(x) : \mathbb{R} \rightarrow [0, 1]$$

This function is called the cumulative distribution function or simply distribution function of rv $X$. 
Example: A company has five applicants for three positions: three women and two men. Suppose that the five applicants are equally qualified and that no preference is given for choosing either gender. Let \( X \) equal the number of women chosen to fill the three positions. Find the probability distribution table and cumulative distribution table of \( X \), respectively.

Solution:

The Prob. Dist. table of rv \( X \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>( \frac{C_3^3C_2^2}{C_3^5} = \frac{3}{10} )</td>
<td>( \frac{C_3^3C_1^2}{C_3^5} = \frac{6}{10} )</td>
<td>( \frac{C_3^3C_0^2}{C_3^5} = \frac{1}{10} )</td>
</tr>
</tbody>
</table>

The Cumulative Dist. table of rv \( X \)

| \( x \) | \( x < 1 \) | \( 1 \leq x < 2 \) | \( 2 \leq x < 3 \) | \( 3 \leq x \) |
|---|---|---|---|
| \( F(x) \) | 0 | \( \frac{3}{10} \) | \( \frac{9}{10} \) | 1 |

Example: Let rv \( X \) have a density function

\[
f(x) = \begin{cases} 
6x(1-x) & \text{if } x \in [0,1] \\
0 & \text{if } x \notin [0,1]
\end{cases}
\]

Find the cumulative dist. function \( F(x) \) of \( X \) and its graph.

Solution: Since for \( 0 \leq y \leq 1 \)

\[
F(y) = \int_0^y 6x(1-x)dx = 3y^2 - 2y^3
\]

we have

\[
F'(y) = 6y(1-y) > 0
\]

Theorem: If \( F(x) \) is the cumulative distribution of rv \( X \), then

(a) \( \mathbb{P}(X > x) = 1 - F(x) \) for any \( x \in \mathbb{R} \);
(b) \( P(a < X \leq b) = F(b) - F(a) \) for any \( a, b \in \mathbb{R} \) and \( a < b \);
(c) If \( X \) is a continuous rv, then \( P(X = x) = 0 \) for any \( x \in \mathbb{R} \);
(d) If \( X \) is a continuous rv, then 
\[
F'(x) = f(x)
\]
where \( f(x) \) is the density function of \( X \).

We have introduced single rv and its density and distribution. In many situations, one rv can’t handle the problems. We need to introduce random vectors.

**Definition:** Suppose that \( X : \Omega \to \mathbb{R} \) and \( Y : \Omega \to \mathbb{R} \) are two discrete rv’s if there is a two variable, non-negative function \( f(x, y) \geq 0 \) such that

1. \( f(x, y) > 0 \) for any \( (x, y) \in \Lambda \subset \mathbb{R}^2 \);
2. \( \sum_{(x,y) \in \Lambda} f(x, y) = 1 \);
3. \( f(x, y) = P\{X = x, Y = y\} \) for any \( (x, y) \in \mathbb{R}^2 \).

Then, \((X, Y)\) is called a discrete random vector, \( f(x, y) \) is called the probability density function of \((X, Y)\), and \( \Lambda \) is called the range space of \((X, Y)\).

**Remark:** Here each component, say \( X \), is a single rv.

**Example:** The experiment consists of flipping a fair coin and rolling a fair die. The sample space is
\[
\Omega = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6),
(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}
\]
Define a rv \( X : \omega \in \Omega \to X(\omega) \in \mathbb{R} \) by
\[
X(\omega) = \begin{cases} 
0 & \text{if } \omega = (H, i), i = 1, \ldots 6 \\
1 & \text{if } \omega = (T, i), i = 1, \ldots 6 
\end{cases}
\]
and define a rv \( Y : \omega \in \Omega \rightarrow Y(\omega) \in \mathbb{R} \) by

\[ Y(\omega) = i \quad \text{if} \ \omega = (x, i), x = H \text{ or } T \]

Define

\[ \Lambda = \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), \]
\[ (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\} \]

and

\[ f(x, y) = \begin{cases} \frac{1}{12} & \text{if } (x, y) \in \Lambda \\ 0 & \text{if } (x, y) \notin \Lambda \end{cases} \]

Then, \((X, Y)\) is a discrete random vector with range space \( \Lambda \) and density function \( f(x, y) \). (Check the definition!) Find the probability \( \mathbb{P}\{X \leq 0; 2 \leq Y < 5\} \).

\[
\mathbb{P}\{X \leq 0; 2 \leq Y < 5\} = \mathbb{P}((H, 2), (H, 3), (H, 4)) = 3/12
\]

Similarly we can define a continuous random vector as follows.

**Definition:** Suppose that \( X : \Omega \rightarrow \mathbb{R} \) and \( Y : \Omega \rightarrow \mathbb{R} \) are two continuous rv’s if there is a two variable, non-negative function \( f(x, y) \geq 0 \) such that

1. \( f(x, y) > 0 \) for any \((x, y) \in \Lambda \subset \mathbb{R}^2\);
2. \( \int \int_{(x,y) \in \Lambda} f(x, y) dx dy = 1 \);
3. \( \mathbb{P}(X, Y) \in [a, b] \times [c, d]) = \int_a^b \int_c^d f(x, y) dx dy \) for any interval \([a, b] \times [c, d] \in \mathbb{R}^2\).

Then, \((X, Y)\) is called a continuous random vector, \( f(x, y) \) is called the probability density function of \((X, Y)\), and \( \Lambda \) is called the range space of \((X, Y)\).

**Remark:** For a continuous rv or rv’s, usually a question only
describes the density function and range space. The sample space is not given.

**Example:** A continuous rv’s \((X, Y)\) has range space \(\Lambda = \{0 < x < 1, 0 < y < 1\}\) and density function as follows:

\[
f(x, y) = \begin{cases} 
4xy & \text{if } (x, y) \in \Lambda \\
0 & \text{if } (x, y) \notin \Lambda 
\end{cases}
\]

Find the probability \(\mathbb{P}(X \leq 1/2, Y < 1/4)\).

\[
\mathbb{P}(X \leq 1/2, Y < 1/4) = \int_0^{1/2} \int_0^{1/4} 4xy \, dx \, dy = \frac{1}{64}
\]
Example: A continuous rv’s \((X, Y)\) has range space \(\Lambda = \{0 < x, 0 < y\}\) and density function as follows:

\[
f(x, y) = \begin{cases} 
  ye^{-xy-y} & \text{if } (x, y) \in \Lambda \\
  0 & \text{if } (x, y) \notin \Lambda
\end{cases}
\]

Find the probability \(P(X \leq 1, Y < 2)\).

Solution:

\[
P(X \leq 1, Y < 2) = \int_0^1 \int_0^2 ye^{-xy-y}dxdy
\]

\[
= \int_0^2 \left\{ \int_0^1 e^{-xy}dx \right\} ye^{-y}dy = \int_0^2 \left\{ \frac{e^{-xy}}{-y} \right\}_0^1 ye^{-y}dy
\]

\[
= \int_0^2 \left\{ \frac{-e^{-y}}{y} + \frac{1}{y} \right\} ye^{-y}dy = \int_0^2 \left\{ -e^{-2y} + e^{-y} \right\}dy
\]

\[
= \frac{e^{-2y}}{2} \bigg|_0^2 - e^{-y} \bigg|_0^2 = 0.5 + 0.5e^{-4} - e^{-2}
\]

Similar to the case of single rv, we can define the cumulative distribution function of rv’s.

Definition: For a given rv’s \((X, Y)\), define

\[
F(x, y) = P(X \leq x, Y \leq y)
\]

then, \(F(x, y)\) is called the joint cumulative distribution function of \(X\) and \(Y\). According to \((X, Y)\) is discrete or continuous, we have

\[
F(x, y) = \sum_{u \leq x} \sum_{v \leq y} f(u, v)
\]

or

\[
F(x, y) = \int_{-\infty}^x \int_{-\infty}^y f(u, v)dudv.
\]
The density function

\[ f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) \]

Example: Suppose that rv’s \(X\) and \(Y\) vary in accordance with the joint pdf

\[ f(x, y) = c(x + y) \quad 0 < x < y < 1 \]

Find \(c\).

Solution: Since

\[
1 = \mathbb{P}(\Omega) = \int_0^1 \left[ \int_0^y c(x + y) \, dx \right] dy = \int_0^1 \left[ c\left(\frac{x^2}{2} + xy\right)\right]_0^y dy = \int_0^1 c\left(\frac{y^2}{2} + y^2\right) dy = \left[ c\frac{y^3}{6} \right]_0^1 = \frac{c}{2},
\]

we get \(c = 2\).