Example: Suppose that 10 defective computers are included in a shipment of 1000 computers. If you test 20 computers in this 1000 computers. Let X be the number of defective computers you found in your test. Find out the distribution table of X.

Solution:

The distribution table of rv X

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(x)</td>
<td>( \frac{C^0_{10} C^{990}<em>{990}}{C^{1000}</em>{10}} )</td>
<td>( \frac{C^1_{10} C^{990}<em>{990}}{C^{1000}</em>{20}} )</td>
<td>( \frac{C^2_{10} C^{990}<em>{18}}{C^{1000}</em>{20}} )</td>
<td>( \frac{C^3_{10} C^{990}<em>{17}}{C^{1000}</em>{20}} )</td>
<td>( \frac{C^4_{10} C^{990}<em>{16}}{C^{1000}</em>{20}} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(x)</td>
<td>( \frac{C^5_{10} C^{990}<em>{15}}{C^{1000}</em>{20}} )</td>
<td>( \frac{C^6_{10} C^{990}<em>{14}}{C^{1000}</em>{20}} )</td>
<td>( \frac{C^7_{10} C^{990}<em>{13}}{C^{1000}</em>{20}} )</td>
<td>( \frac{C^8_{10} C^{990}<em>{12}}{C^{1000}</em>{20}} )</td>
<td>( \frac{C^9_{10} C^{990}<em>{11}}{C^{1000}</em>{20}} )</td>
<td>( \frac{C^{10}<em>{10} C^{990}</em>{10}}{C^{1000}_{20}} )</td>
</tr>
</tbody>
</table>

Theorem: Suppose that an urn contains \( p \) white chips and \( q \) black chips and \( p + q = N \). If \( n \) chips are drawn out at random, without replacement, let \( X \) be the number of white chips selected, then

\[
\mathbb{P}(X = k) = \frac{C^p_k C^q_{n-k}}{C^N_n} \quad 0 \leq k \leq p; k \leq n
\]

\( X \) is called a rv having hypergeometric distribution.

Example: A city has 4050 children under the age of 10, including 514 who have not been vaccinated for measles. 65 of the city’s children are enrolled in the ABC Day Care Center. Suppose the municipal health department sends a doctor and a nurse to ABC to immunize any child who has not already been vaccinated. Let \( X \) be the number of children at ABC who have not been vaccinated. Find the distribution of \( X \).

Solution:

\[
\mathbb{P}(X = k) = \frac{C^{514}_k C^{3536}_{65-k}}{C^{4050}_{65}} \quad 0 \leq k \leq 65
\]
Repeated Independent Trials and Binomial Distribution

A binomial experiment is one that has following three characteristics:

1. The experiment consists of \( n \) independent, identical trials.
2. Each trial has two possible outcomes: success \( A \), failure \( \bar{A} \).
3. \( 0 < P(A) = p < 1 \) and \( P(\bar{A}) = 1 - p = q \).

Remark: In binomial experiment, if \( p \neq q \), then simple events are not equally likely. Therefore, it is not a classical probability model.

Example 10.1 Consider a binomial experiment of flipping a biased coin three times. Let \( A \) be the event of observing a head and \( \bar{A} \) be the event of observing a tail.

Suppose that

\[
P(A) = \frac{1}{3} \quad P(\bar{A}) = \frac{2}{3}
\]

Then

\[
\Omega = \{AAA, A\bar{A}A, A\bar{A}\bar{A}, \bar{A}AA, \bar{A}\bar{A}\bar{A}, A\bar{A}\bar{A}, \bar{A}\bar{A}\bar{A}\}
\]

\[
P(AAA) = P(A)P(A)P(A) = \frac{1}{27}
\]

\[
P(A\bar{A}A) = P(A)P(\bar{A})P(A) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{27}
\]

Therefore, it is not a classical probability model.

Let \( X \) be number of heads observed. Find the probability \( P(X = 2) \).

\[
P(X = 2) = P(\{AA\bar{A}\}, \{A\bar{A}A\}, \{\bar{A}AA\})
\]

\[
P(\{AA\bar{A}\}) = P(A)P(A)P(\bar{A}) = ppq
\]

\[
P(\{\bar{A}AA\}) = P(A)P(A)P(\bar{A}) = ppq
\]

\[
P(\{A\bar{A}A\}) = P(A)P(\bar{A})P(\bar{A}) = ppq
\]

\[
P(X = 2) = C_2^3ppq = C_2^3p^2q^{3-2}
\]
Generally, consider a binomial experiment with \( n \) trials. Let \( X \) be the number of successes in the \( n \) trials. What is the probability of event \( \{X = k\} \). Then,

\[
P(X = k) = C_k^n p^k q^{n-k}
\]

(For \( C_k^n \), Consider there are \( n \) positions and how many ways to choose \( n - k \) positions to put bars on tops of these positions.)

**Definition 10.2** A binomial experiment consists of \( n \) independent, identical trials with probability of success \( p \) and probability of failure \( q = 1 - p \) on each trial. Let \( X \) be the number of successes in the \( n \) trials. Then,

The distribution table of r.v \( X \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\cdots</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>( C_0^n p^0 q^n )</td>
<td>( C_1^n p^1 q^{n-1} )</td>
<td>( C_2^n p^2 q^{n-2} )</td>
<td>\cdots | ( C_n^n p^n q^0 )</td>
<td></td>
</tr>
</tbody>
</table>

is called the binomial distribution.

**Example 10.2** Consider a binomial experiment of flipping a biased coin twenty times. Let \( A \) be the event of observing a head and \( \bar{A} \) be the event of observing a tail.

Suppose that

\[
P(A) = \frac{1}{5}, \quad P(\bar{A}) = \frac{4}{5}
\]

Let \( X \) be number of heads observed in the twenty flippings.

(a) Find the probability \( P(X = 8) \).
(b) Find the probability \( P(X > 2) \).

**solution:** (a) According to the binomial distribution formula, we have

\[
P(X = 8) = C_8^{20} \left( \frac{1}{5} \right)^8 \left( \frac{4}{5} \right)^{20-8} = 0.022.
\]
We have $\mathbb{P}(X \leq 8) = 0.990$ and $\mathbb{P}(X \leq 7) = 0.968$ according to the table 1 ($n = 20$, $k = 8$, $k = 7$, $p = 0.2$).

$$\mathbb{P}(X = 8) = \mathbb{P}(X \leq 8) - \mathbb{P}(X \leq 7) = 0.022$$

(b) $\mathbb{P}(X > 2) = 1 - \mathbb{P}(X \leq 2) = 1 - 0.206 = 0.794$

Example 10.3 Consider an experiment of randomly drawing a chip 8 times with replacement from a box containing two black chips and three white chips. The chips are identical except their colors. Let $X$ be the number of black chips observed in the 8 drawings.

(a) Find the probability $\mathbb{P}(X = 5)$.

(b) Find the probability $\mathbb{P}(X > 2)$.

solution: (a) According to the binomial distribution formula, we have

$$\mathbb{P}(X = 5) = \binom{8}{5}(\frac{2}{5})^5(\frac{3}{5})^{8-5}.$$

We have $\mathbb{P}(X \leq 5) = 0.950$ and $\mathbb{P}(X \leq 4) = 0.826$ according to the table 1 ($n = 8$, $k = 5$, $k = 4$, $p = 0.4$).

$$\mathbb{P}(X = 5) = \mathbb{P}(X \leq 5) - \mathbb{P}(X \leq 4) = 0.124$$

(b) $\mathbb{P}(X > 2) = 1 - \mathbb{P}(X \leq 2) = 1 - 0.315 = 0.685$