Additive Rule of Probability Given two events $A$ and $B$,
\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
If $A$ and $B$ are mutually exclusive, then
\[ P(A \cup B) = P(A) + P(B) \]

Multiplicative Rule of Probability Given two events $A$ and $B$, then
\[ P(A \cap B) = P(A)P(B|A) = P(B)P(A|B). \]
If $A$ and $B$ are independent, then
\[ P(A \cap B) = P(A)P(B). \]
Similarly, if $A$, $B$, and $C$ are independent (?), then
\[ P(A \cap B \cap C) = P(A)P(B)P(C). \]

Remark: Events $A$, $B$, and $C$ are independent if and only if
\[ P(A \cap B) = P(A)P(B), \]
\[ P(A \cap C) = P(A)P(C), \]
\[ P(B \cap C) = P(B)P(C), \]
\[ P(A \cap B \cap C) = P(A)P(B)P(C). \]
Example 5.9 Suppose that in a box there are 2 distinct, white balls and 3 distinct, black balls. We successively, randomly draw two balls with replacement. Define the typical classical probability on the sample space and define \( A = \{ \text{first drawing is a white ball} \} \) and \( B = \{ \text{Second drawing is a black ball} \} \). Find \( P(A \cup B) \) and \( P(A \cap B) \). Does \( P(A \cap B) = P(A)P(B) \) hold? How about changing the experiment into without replacement?

Solution: The sample space is

\[
\Omega = \{ (W_1, W_1), (W_1, W_2), (W_1, B_1), \ldots, (B_3, B_3) \}
\]

There are \( 5 \times 5 = 25 \) simple events. Since

\[
P(A) = \frac{C_2^1C_5^1}{25},
\]

\[
P(B) = \frac{C_5^1C_3^1}{25},
\]

and

\[
P(A \cap B) = \frac{C_2^1C_3^1}{25},
\]

we have

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

\[
= \frac{C_2^1C_5^1}{25} + \frac{C_5^1C_3^1}{25} - \frac{C_2^1C_3^1}{25} = \frac{19}{25}.
\]
With replacement,

\[ \mathbb{P}(A \cap B) = \frac{6}{25} = \frac{10}{25} \times \frac{15}{25} \]

\[ = \mathbb{P}(A)\mathbb{P}(B). \]

If without replacement,

\[ \Omega = \{(W_1, W_2), \ldots, \} \]

There are \(5 \times 4 = 20\) simple events. Since

\[ \mathbb{P}(A) = \frac{C^2_1 C^4_1}{20}, \]

\[ \mathbb{P}(B) = \frac{C^2_1 C^3_1 + C^3_1 C^2_1}{20}, \]

and

\[ \mathbb{P}(A \cap B) = \frac{C^2_1 C^3_1}{20}, \]

therefore,

\[ \mathbb{P}(A \cap B) = \frac{6}{20} \neq \frac{6}{25} = \frac{8}{20} \times \frac{12}{20} \]

\[ = \mathbb{P}(A)\mathbb{P}(B). \]

**Law of Total Probability**

Given a sequence of mutually exclusive events \(S_1, S_2, \ldots, S_n\). If event \(A \subset \bigcup_{i=1}^n S_i\) and \(\mathbb{P}(S_i) > 0\), then

\[ \mathbb{P}(A) = \mathbb{P}(S_1)\mathbb{P}(A|S_1) + \cdots + \mathbb{P}(S_n)\mathbb{P}(A|S_n) \]
**Proof:** Since \((A \cap S_i) \cap (A \cap S_j) = \phi\) for any \(i \neq j\), this means that the sequence \(\{(A \cap S_i), i = 1, \ldots, n\}\) are mutually exclusive, according to the second condition of the definition of probability, we have

\[ P(A) = P(A \cap S_1) + \cdots + P(A \cap S_n). \]

Since

\[ P(A \cap S_i) = P(S_i)P(A|S_i) \text{ for } i = 1, \ldots, n, \]

the law of total probability is proved.

**Example 7.1** A fair coin is flipped. If a head turns up, a fair die is tossed; if a tail turns up, two fair dice are tossed. What is the probability of the event \(B\) that the sum of the appearing number(s) is equal to 6?

**Solution:** Since \(P(H) = \frac{1}{2}\) and \(P(T) = \frac{1}{2}\), and

\[ P(B|H) = \frac{1}{6} \]

\[ P(B|T) = \frac{5}{36} \]

\{(1,5), (2,4), (3,3), (4,2), (5,1)\}. Therefore,

\[ P(B) = P(H)P(B|H) + P(T)P(B|T) \]

\[ = \frac{1}{2} \times \frac{1}{6} + \frac{1}{2} \times \frac{5}{36} = 0.15. \]

**Bayes’ Rule** Given a sequence of mutually exclusive
events $S_1, S_2, \cdots, S_n$ and $\mathbb{P}(S_i) > 0$. If event $A \subset \bigcup_{i=1}^n S_i$, then
\[
\mathbb{P}(S_i|A) = \frac{\mathbb{P}(S_i)\mathbb{P}(A|S_i)}{\sum_{k=1}^n \mathbb{P}(S_k)\mathbb{P}(A|S_k)}
\]

**Proof:** According to the definition of conditional probability, we have
\[
\mathbb{P}(S_i|A) = \frac{\mathbb{P}(S_i \cap A)}{\mathbb{P}(A)}.
\]

Since
\[
\mathbb{P}(S_i \cap A) = \mathbb{P}(S_i)\mathbb{P}(A|S_i),
\]
and according to the law of total probability,
\[
\mathbb{P}(A) = \mathbb{P}(S_1)\mathbb{P}(A|S_1) + \cdots + \mathbb{P}(S_n)\mathbb{P}(A|S_n),
\]
this proves the Bayes’ rule.

**Example 7.2** A man takes either a bus or the subway to work with probabilities 0.3 and 0.7, respectively. When he takes the bus, he is late 30% of the days. When he takes the subway, he is late 20% of the days. If the man is late for work on a particular day, what is the probability that he took the bus?

**Solution:** Define $B = \{ \text{The man takes a bus} \}$, $S = \{ \text{The man takes the subway} \}$, and $L = \{ \text{The man is late on the day} \}$. Then,
\[
\mathbb{P}(B) = 0.3, \mathbb{P}(S) = 0.7, \mathbb{P}(L|B) = 0.3, \mathbb{P}(L|S) = 0.2.
\]
By Bayes’ rule, we have

\[ P(B|L) = \frac{P(B)P(L|B)}{P(S)P(L|S) + P(B)P(L|B)} = \frac{0.3 \times 0.3}{0.7 \times 0.2 + 0.3 \times 0.3} = \frac{0.09}{0.23} = 0.3913. \]

Example 7.3 To evaluate the effectiveness of a screening procedure, we will evaluate the probability of a false negative or a false positive using the following notation:

- \( T^+ \): The test is positive and indicate that the person has the disease.
- \( T^- \): The test is negative and indicate that the person does not have the disease.
- \( D_c \): The person really does not have the disease.
- \( D \): The person really has the disease.

According to the test results, we found that the sensitivity of the test has following conditional probabilities:

\[ P(T^+|D) = 0.98, \]

and

\[ P(T^-|D_c) = 0.99. \]

If the proportion of the general population infected with this disease is 2 per million, what is (a) the probability of a false positive,

\[ P(D_c|T^+) \]
(b) the probability of a false negative,

\[ P(D|T^-) \]

Solution: From the given information, we know the following:

\[
P(D) = 0.000002, \quad P(D^c) = 0.999998
\]
\[
P(T^+|D) = 0.98 \quad P(T^-|D) = 0.02
\]
\[
P(T^+|D^c) = 0.01 \quad P(T^-|D^c) = 0.99
\]

(a) From Bayes’ Rule,

\[
P(D^c|T^+) = \frac{P(D^c \cap T^+)}{P(T^+)} = \frac{P(D^c)P(T^+|D^c)}{P(D^c)P(T^+|D^c) + P(D)P(T^+|D)}
\]

Therefore,

\[
P(D^c|T^+) = \frac{0.00999998}{0.01000194} = 0.999804038
\]

(b) Using a similar calculation,

\[
P(D|T^-) = \frac{P(D \cap T^-)}{P(T^-)} = \frac{P(D)P(T^-|D)}{P(D^c)P(T^-|D^c) + P(D)P(T^-|D)}
\]
Therefore,

\[ P(D|T^-) = \frac{0.00000004}{0.98999806} = 0.00000004 \]

Hence, the probability of a false positive is near 1 and very likely, while the probability of a false negative is quite small and very unlikely.

**Example 7.4** If men constitute 47% of the population and tell the truth 78% of the time, while women tell the truth 63% of the time, what is the probability that a person selected at random will answer a question truthfully?

**Solution:** Define

\[ B = \{ \text{The person interviewd answers truthfully} \} \]

\[ A = \{ \text{The person interviewed is a man} \} \]

According to the law of total probability, we have

\[ P(B) = P(A)P(B|A) + P(A^c)P(B|A^c) \]

\[ = (0.47)(0.78) + (0.53)(0.63) = 0.70 \]

**Example 7.4** A worker-operated machine produces a defective item with probability 0.01 if the worker follows the machine’s operating instructions exactly, and with probability 0.03 if he does not. If the worker follows the instructions 90% of time, what proportion of all items produced by the machine will be defective?
Given that a defective item is produced, what is the conditional probability of the event that the worker exactly follows the machine operating instructions?