Forum. Math. Galley Proof, 41 pages
DOI 10.1515/FORM.2011.100

Forum Mathematicum
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Interacting superprocesses with discontinuous spatial motion

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Communicated by Ichiro Shigekawa

Abstract. A class of interacting superprocesses arising from branching particle systems with continuous spatial motions, called superprocesses with dependent spatial motion (SDSMs), has been introduced and studied by Wang and by Dawson, Li and Wang. In this paper, we extend the model to allow discontinuous spatial motions. Under Lipschitz condition for coefficients, we show that under a proper rescaling, branching particle systems with jump-diffusion underlying motions in a random medium converge to a measure-valued process, called stable SDSM. We further characterize this stable SDSM as a unique solution of a well-posed martingale problem. To prove the uniqueness of the martingale problem, we establish the $C^{2+\gamma}$-regularity for the transition semigroup of a class of jump-diffusion processes, which may be of independent interest.

Keywords. Superprocess, Brownian sheet, interaction, symmetric stable process, fractional Laplacian, $C^{2+\gamma}$-regularity of transition semigroup, branching mechanism, scaling limit, duality method, martingale problem.

2010 Mathematics Subject Classification. Primary 60J68, 60K37; secondary 60G52, 60J80.

1 Introduction

Dawson–Watanabe superprocesses arise naturally as the scaling limit of branching processes where each particle moves independently of each other. In this paper, we are concerned with the scaling limit of an interacting branching particle system where the motion of each particle is governed by the following equation: for each $i \in \mathbb{N}$,

$$z_t^i - z_0^i = \int_0^t c(z_{s-}^i)dB_s^i + \int_0^t\int_{\mathbb{R}} h(y - z_{s-}^i)W(dy, ds) + \int_0^t b(z_{s-}^i)\, dS_s^i, \tag{1.1}$$

The research is supported in part by NSF Grants DMS-0600206 and DMS-0906743 (first author), by a research Grant of University of Oregon (second author) and by NSF Grant DMS-0906907 (third author).

where $\{B^i; i \geq 1\}$ are independent standard Brownian motions on $\mathbb{R}$, W is a Brownian sheet on $\mathbb{R}^2$ (see definition below) and $\{S^i; i \geq 1\}$ are independent 1-dimensional symmetric α-stable processes with $\alpha \in (0,2)$. The processes W, $\{B^i; i \geq 1\}$ and $\{S^i; i \geq 1\}$ are assumed to be independent of each other. Equation (1.1) says that each particle is subject to a common random medium force, which is described by the Brownian sheet W, and each particle has its own jump-diffusion dynamics, which are described by an individual Brownian motion and an individual symmetric stable process. We will show in this paper that under suitable conditions and scaling, the branching particle system converges weakly to a measure-valued process, which we call an interacting superprocess. We will further show that this measure-valued process is the unique solution to a related martingale problem. When $b = 0$ (that is, there is no stable-noise motion S^i in the particle system), such a model has been studied in Wang [32,33], which has been further investigated in subsequent papers by Dawson et al. [10] and Li et al. [24,25]. However, many physical, economic and biological systems are better modeled by using discontinuous stochastic processes because the jumps exhibited by discontinuous processes reveal sudden big changes when viewed in a long time scale. Thus it is natural and meaningful to study the scaling limit of interacting branching processes with discontinuous spatial motion such as those described by (1.1). The presence of jumps introduces many new challenges. For example, to establish the uniqueness of the martingale problem of the interacting superprocess, one needs to establish the C^2-regularity for the semigroup of the n-particle system of (1.1). However unlike the diffusion case, due to the presence of (discontinuous) stable Lévy motion, the infinitesimal of such a semigroup is a non-local operator. Establishing regularity results for pseudo-differential operators is typically a challenging problem and is, in fact, of current research interest (see, e.g., [2], especially item 6 of its Section 7.7). One of the main results of this paper is to show that under certain conditions there is some $\gamma \in (0,1)$ so that for every $n \geq 1$, the semigroup of the n-particle system of (1.1) maps $C_b^{2+\gamma}(\mathbb{R}^n)$ into itself. The following is a more detailed description on the content of this paper.

Let $L^2(\mathbb{R})$ be the Hilbert space of square-integrable functions on $\mathbb{R}$ and let $\mathcal{B}_b(\mathbb{R}^m)$ be the space of bounded Lebesgue measurable functions on $\mathbb{R}^m$. Denote by $C(\mathbb{R}^m)$ and $C^j(\mathbb{R}^m)$ the space of continuous functions on $\mathbb{R}^m$ and the space of continuous functions on $\mathbb{R}^m$ with continuous derivatives up to and including order j, respectively. Denote by $C_b(\mathbb{R}^m)$ and $C_b^k(\mathbb{R}^m)$ the space of bounded continuous functions on $\mathbb{R}^m$ and the space of bounded continuous functions that have bounded derivatives up to and including order k, respectively. We use $\mathrm{Lip}(\mathbb{R})$ to denote the space of Lipschitz functions on $\mathbb{R}$; that is, $f \in \mathrm{Lip}(\mathbb{R})$ if there is a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x-y|$ for every $x, y \in \mathbb{R}$. The class of bounded Lipschitz functions on $\mathbb{R}$ will be denoted by $\mathrm{Lip}_b(\mathbb{R})$.

Under the condition that $b, c \in \mathrm{Lip}_b(\mathbb{R})$ and $h \in L^2(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, we will show that the interacting stochastic differential equation (SDE) (1.1) has a unique strong solution. From the strong solution of (1.1), we can construct a family of interacting branching particle systems. Let $\lambda > 0$ and $\theta > 1$ be fixed constants. For $n \geq 1$, suppose that initially there are $m_0^{(n)}$ number of particles located at z_0^i, $1 \leq i \leq m_0^{(n)}$, and each has mass θ^{-n}. These particles evolve according to (1.1) and branch independently at rate $\lambda\theta^n$. The branching mechanism for each particle is assumed to be state independent, independent to each other and identically distributed. After branching, the offsprings of each particle evolves independently according to (1.1) and then branches again. The common n^{th}-stage branching mechanism $q^{(n)} := \{q_k^{(n)}; k = 0, 1, \dots\}$ is assumed to be critical (that is, the average number of offspring is 1), and it cannot produce 1 or more than n number of children. Under the assumption that the initial state $\theta^{-n} \sum_{k=1}^{m^{(n)}} \delta_{z_0^k}$ of the particles converges to a measure μ_0 and that the branching function $q^{(n)}$ converges uniformly to a limiting branching function having finite second moment and some additional conditions, we show that the empirical process $\theta^{-n} \sum_{k\geq 1} \delta_{z_t^k}$ converges to a measure-valued process. By Itô's formula and the independence of motions and branching, we will show that the limiting measure-valued process has the following formal generators (usually called pregenerators):

$$\mathcal{L}F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}F(\mu), \tag{1.2}$$

$$\mathcal{B}F(\mu) := \frac{1}{2}\lambda\sigma^2 \int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta\mu(x)^2} \mu(dx) \tag{1.3}$$

and

$$\begin{aligned} \mathcal{A}F(\mu) := \int_{\mathbb{R}} \left(\tfrac{1}{2} a(x) \frac{d^2}{dx^2} + |b(x)|^\alpha \Delta^{\alpha/2} \right) \frac{\delta F(\mu)}{\delta\mu(x)} \mu(dx) \\ + \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(x-y) \frac{\partial^2}{\partial x \partial y} \left(\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)} \right) \mu(dx)\mu(dy) \end{aligned} \tag{1.4}$$

for $F(\mu) \in \mathcal{D}(\mathcal{L}) \subset C(M_F(\mathbb{R}))$. Here $\lambda > 0$ is related to the branching rate of the particle system and $\sigma^2 > 0$ is the variance of the limiting offspring distribution; $M_F(\mathbb{R})$ denotes the Polish space of all finite measures on $\mathbb{R}$ with weak topology, $C(M_F(\mathbb{R}))$ is the space of all continuous functions on $M_F(\mathbb{R})$. The variational derivative is defined by

$$\frac{\delta F(\mu)}{\delta\mu(x)} := \lim_{\varepsilon \downarrow 0} \frac{F(\mu + \varepsilon\delta_x) - F(\mu)}{\varepsilon}; \tag{1.5}$$

the fractional Laplacian $\Delta^{\alpha/2}$ on $C_b^2(\mathbb{R})$ is defined for $\phi \in C_b^2(\mathbb{R})$,

$$\Delta^{\alpha/2}\phi(x) := \int_{\mathbb{R}\setminus\{0\}} \left(\phi(x+\xi) - \phi(x) - \phi'(x)\xi \mathbb{1}_{\{|\xi|\le 1\}}\right) \frac{c_\alpha}{|\xi|^{1+\alpha}} d\xi \tag{1.6}$$

and

$$\rho(x) := \int_{\mathbb{R}} h(y-x)h(y)dy, \quad a(x) := c^2(x) + \rho(0), \tag{1.7}$$

and $\mathcal{D}(\mathcal{L})$, the domain of the pregenerator $\mathcal{L}$, consists of functions of the form

$$F(\mu) = f(\langle \phi_1, \mu\rangle, \ldots, \langle \phi_k, \mu\rangle),$$

with $\phi_i \in C_c^2(\mathbb{R})$, $f \in C_b^2(\mathbb{R}^k)$, $k \in \mathbb{N}$. In the usual models (for example, (α, d, β)-superprocesses, see Dawson [8], Dynkin [13] or Dynkin et al. [14]) in which the motions of particles are independent and the motions are independent of branching, the particle systems have the following multiplicative property. If two branching Markov processes evolve independently, with initial state m_1 and m_2 respectively, then their sum has the same distribution as the branching process with initial state $m_1 + m_2$. It is well known that the log-Laplace functional (or evolution equation) technique can be applied to these models in order to construct the limiting measure-valued process. However, for our model and pregenerator, it is obvious that the motions of particles are not independent and this destroys the multiplicative property. Thus, just as in Wang [33] and Dawson et al. [11], the usual log-Laplace functional method is not applicable to our new model. In order to construct the branching particle system, we show that under Lipschitz condition on functions c, h and b, SDE (1.1) has a strong solution and the solution is pathwise unique. Since the symmetric α-stable process S^i is not square-integrable and is a martingale only when $\alpha \in (1, 2)$, the Picard's successive approximation method is not directly applicable to (1.1) and a truncation procedure on the jumps of S^i is needed.

To prove the uniqueness of the martingale problem for $\mathcal{L}$ for the measure-valued interacting process, we use a duality method due to Dawson and Kurtz [9]. To apply Dawson and Kurtz's duality method in our context, we need to find, for every integer $m \ge 1$, a linear subspace $\mathcal{D}$ of $C_b^2(\mathbb{R}^m)$ so that $P_t^m \mathcal{D} \subset \mathcal{D}$ for every $t > 0$. Here $\{P_t^m; t \ge 0\}$ is the transition semigroup of the underlying motion of m-particles given by (1.1). We show in this paper that we can take $C^{2+\gamma}(\mathbb{R}^m)$ for such $\mathcal{D}$ for some $\gamma \in (0, 1)$. That is, we show that there is some $\gamma \in (0, 1)$ so that $P_t^m f \in C_b^{2+\gamma}(\mathbb{R}^m)$ for each fixed $t > 0$ and $f \in C_b^{2+\gamma}(\mathbb{R}^m)$. This result has its own independent interest and is one of the main results of this paper. For earlier regularity results for non-local operators, we refer the reader to [2, 6, 28]

and the references therein. Note that the infinitesimal generator of m-particles $(z^1, \dots, z^m)$ of (1.1) is given by (see the proof of (3.4) below)

$$\begin{aligned}
&\mathcal{L}^{(m)} f(x_1, \dots, x_m) \\
&\quad := \frac{1}{2} \sum_{i,j=1}^{m} \Gamma_{ij}(x_1, \dots, x_m) \frac{\partial^2}{\partial x_i \partial x_j} f(x_1, \dots, x_m) \\
&\qquad + \sum_{j=1}^{m} |b(x_j)|^\alpha \Delta_{x_j}^{\alpha/2} f(x_1, \dots, x_m) \\
&\quad = \frac{1}{2} \sum_{i,j=1}^{m} \Gamma_{ij}(x_1, \dots, x_m) \frac{\partial^2}{\partial x_i \partial x_j} f(x_1, \dots, x_m) \\
&\qquad + \sum_{j=1}^{m} \int_{\mathbb{R} \setminus \{0\}} \Big(f(x_1, \dots, x_j + w, \dots, x_m) - f(x_1, \dots, x_m) \\
&\qquad\qquad - \frac{\partial}{\partial x_j} f(x_1, \dots, x_m) w \mathbb{1}_{\{|w| \le 1\}} \Big) \frac{c_\alpha |b(x_j)|^\alpha}{|w|^{1+\alpha}} dw,
\end{aligned} \tag{1.8}$$

where

$$\Gamma_{ij}(x_1, \dots, x_m) := \begin{cases} a(x_i) & \text{if } i = j, \\ \rho(x_i - x_j) & \text{if } i \neq j. \end{cases} \tag{1.9}$$

Here $\Delta_{x_j}^{\alpha/2} f$ is the fractional Laplacian defined in (1.6) applied to the function

$$x_j \mapsto f(x_1, \dots, x_j, \dots, x_m).$$

The $C^{2+\gamma}$-regularity of P_t^m will be proved by a perturbation method, treating the non-local operator part $\sum_{j=1}^m |b(x_j)|^\alpha \Delta_{x_j}^{\alpha/2}$ of $\mathcal{L}^{(m)}$ as a perturbation of the elliptic differential operator $\frac{1}{2} \sum_{i,j=1}^m \Gamma_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$.

In this paper, to better illustrate our approach to the study of interacting superprocesses whose underlying spatial motions are discontinuous, we restrict ourselves to a specific type of jump-diffusions for the spatial motions given by (1.1). However the method of this paper works for a more general class of jump-diffusions. For example, there can be a drift term $\int_0^t r(z_s^i) ds$ in (1.1) and the symmetric stable process S^i can be replaced by some other Lévy process of pure jump type such as mixed stable processes. Moreover, $\{S^i; i \ge 1\}$ can even be replaced by a certain family of independent random measures (see [20]). Furthermore, the spatial motions z^i in (1.1) can also be multidimensional.

While the component of α-stable noise in the spatial motion is captured by the fractional Laplacian $\Delta^{\alpha/2}$ in the expression (1.4) of $\mathcal{A}$ for the pregenerator $\mathcal{L}$ of the interacting superprocess, the superprocess itself has continuous trajectories. This is because in this paper, we have critical branching at every stage and the branching function $q^{(n)}$ converges to a limiting branching function in which the offspring distribution has a finite second moment. These assumptions imply that the limiting interacting measure-valued process has null Lévy (jumping) measure for its branching mechanism and therefore the superprocess is continuous. This is in analog to the case of Dawson–Watanabe process that it has continuous paths if and only if the Lévy (jumping) measure for the branching mechanism is zero (see Fitzsimmons [18] and El Karoui–Roelly [15]).

If the offspring distribution determined by the limiting branching function of $q^{(n)}$ has infinite second moment, for example, a stable branching mechanism or a general branching mechanism, then the limiting measure-valued process is discontinuous. For the latter case, we believe that the conditional log-Laplace functional would be a good tool to construct the limiting measure-valued process. See [8] and [25] for related work in this direction on interacting superprocesses with continuous spatial motion.

The remainder of this paper is organized as follows. Section 2 is devoted to establish the strong existence and pathwise uniqueness for solutions of (1.1). This is the basis for the construction of the branching particle systems. In Section 3, we construct branching particle system. The tightness of the corresponding empirical measure-valued processes $\{\mu_t^n; t \geq 0\}$ is established in Section 4. We prove the L^2-convergence of each term in the the decomposition of $\langle \phi, \mu_t^n \rangle$ and then derive the corresponding decomposition of $\langle \phi, \mu_t^0 \rangle$ for any weak sequential limit $\{\mu_t^0; t \geq 0\}$ of $\{\mu_t^n; t \geq 0\}_{n\geq 1}$. The latter implies that the measure-valued process $\{\mu_t^0; t \geq 0\}$ solves the martingale problem for the generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ of (1.2). Moreover, we show that $\{\mu_t^0; t \geq 0\}$ is a continuous process taking values in $M_F(\mathbb{R})$. We then use Dawson–Kurtz's duality method to show that the martingale problem for $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is well-posed. This in particular implies that $\{\mu_t^n; t \geq 0\}$ converges weakly in the Skorokhod space $\mathbb{D}([0, \infty), M_F(\mathbb{R}))$ to a continuous measure-value process that solves the martingale problem for $\mathcal{L}$. The proof of the $C^{2+\gamma}$-regularity property of the semigroup $P_t^{(m)}$, which is used in a crucial way in the proof of well-posedness of the aforementioned martingale problem, is given in Section 5.

For the reader's convenience, let us recall the definition of the Brownian sheet W on $\mathbb{R}$. For a $D = \mathbb{R}^m$ or $D = \mathbb{R} \times \mathbb{R}_+$, let $\mathcal{B}(D)$ be the Borel σ-field on D. By abusing the notation, the Lebesgue measure on $\mathbb{R}$ and on $\mathbb{R}^2$ will all be denoted by $\mathfrak{m}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space with a right

continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A random set function W on $\mathcal{B}(\mathbb{R}\times\mathbb{R}_+)$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is called a *Brownian sheet* or a *space-time white noise* on $\mathbb{R}$ if

(i) For $A \in \mathcal{B}(\mathbb{R}\times\mathbb{R}_+)$ having finite Lebesgue measure, $W(A)$ is a Gaussian random variable with mean zero and variance $\mathfrak{m}(A)$.

(ii) If $A_i \in \mathcal{B}(\mathbb{R}\times\mathbb{R}_+)$, $i = 1, 2$, have finite Lebesgue measure and satisfy $A_1 \cap A_2 = \emptyset$, then $W(A_1)$ and $W(A_2)$ are independent and

$$W(A_1 \cup A_2) = W(A_1) + W(A_2) \quad \mathbb{P}\text{-a.s.}$$

(iii) For every $A \in \mathcal{B}(\mathbb{R})$ having finite Lebesgue measures,

$$M(A)_t := W(A \times [0, t]),$$

as a process in $t \geq 0$, is a square-integrable $\{\mathcal{F}_t\}$-martingales.

In fact, it follows from (i) and (ii) that for every $A \in \mathcal{B}(\mathbb{R})$ having finite Lebesgue measures, $t \mapsto M(A)_t$ is a centered Gaussian process with independent stationary increments with $\mathbb{E}[M(A)_t^2] = \mathfrak{m}(A)t$. Thus it has a continuous modification and is a Brownian motion in $\mathbb{R}$. Condition (iii) above puts in an additional requirement that $M(A)$ be an $\{\mathcal{F}_t\}$-martingale. It is a consequence of above (ii) and (iii) that for every $A, B \in \mathcal{B}(\mathbb{R})$ with finite Lebesgue measures, the covariance process for martingales $M(A)$ and $M(B)$ satisfies

$$\langle M(A), M(B)\rangle_t = \mathfrak{m}(A \cap B)\, t, \quad t \geq 0.$$

For more detailed information on Brownian sheet, the reader is referred to Walsh [31, Chapter 2] and Dawson [8, Section 7.1].

2 Strong existence and pathwise uniqueness of underlying motions

Recall that a one-dimensional symmetric stable process of index $\alpha \in (0, 2]$ is a Lévy process $S := \{S_t; t \geq 0\}$ such that

$$\mathbb{E}\Big[e^{i\xi(S_t - S_0)}\Big] = e^{-t|\xi|^\alpha} \quad \text{for every } \xi \in \mathbb{R}.$$

Note that when $\alpha = 2$, S is just a Brownian motion with speed running twice fast as the standard Brownian motion. More generally, given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a process $S := \{S_t; t \geq 0\}$ is said to be a $\{\mathcal{F}_t\}$-sym-

metric α-stable process if S_t is $\mathcal{F}_t$-measurable for every $t \geq 0$ and for every $t > r > 0$, $S_t - S_r$ is independent of $\mathcal{F}_r$ and

$$\mathbb{E}\Big[e^{i\xi(S_t - S_r)}\Big] = e^{-(t-r)|\xi|^\alpha} \quad \text{for every } \xi \in \mathbb{R}.$$

The Lévy measure for such a process is given by $\frac{c_\alpha}{|w|^{1+\alpha}}dw$, where c_α is a constant depending only on α. Its role can be understood in the following Lévy system formula for a one-dimensional $\{\mathcal{F}_t\}$-symmetric α-stable process S. For any $\{\mathcal{F}_t\}$-predictable process $H := \{H_t; t \geq 0\}$ and any Borel measurable function F on $\mathbb{R}^2$ satisfying $\int_0^t \int_{\mathbb{R}} |H_u F(S_u, S_u + w)| \frac{c_\alpha}{|w|^{1+\alpha}} dw\, du < \infty$, we have for every $t > 0$,

$$t \mapsto \sum_{u \leq t} H_u F(S_{u-}, S_u) - \int_0^t \int_{\mathbb{R}} H_u F(S_u, S_u + w) \frac{c_\alpha}{|w|^{1+\alpha}} dw\, du \tag{2.1}$$

is a local martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. It is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if in addition H is a bounded process and

$$\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} |H_u F(S_u, S_u + w)| \frac{c_\alpha}{|w|^{1+\alpha}} dw du\right] < \infty$$

for every $t > 0$. Here for any process with jumps we use $X_{s-} := \lim_{t\uparrow s} X_t$ to denote the left hand limit and $\triangle X_s := X_s - X_{s-}$ to denote the jump at time s. See, for example, [4], [26], [20] and [30] for more information on the stochastic analysis of processes with jumps. The L^2-infinitesimal generator $\mathcal{K}$ of a symmetric α-stable process is the fractional Laplacian $-(-\Delta)^{\alpha/2}$. It can be defined in terms of Fourier transform as follows:

$$\mathrm{Dom}(\mathcal{K}) := \left\{ f \in L^2(\mathbb{R}) : |\xi|^\alpha \widehat{f}(\xi) \in L^2(\mathbb{R}) \right\},$$

$$\widehat{\mathcal{K} f}(\xi) := |\xi|^\alpha \widehat{f}(\xi) \quad \text{for } f \in \mathrm{Dom}(\mathcal{K}).$$

Here for $f \in L^2(\mathbb{R})$, $\widehat{f}(\xi) := \int_{\mathbb{R}} e^{i\xi x} f(x) dx$. It can be shown (see, e.g., [3]) that $C_c^2(\mathbb{R}) \subset \mathrm{Dom}(\mathcal{K})$ and $\mathcal{K} f = \Delta^{\alpha/2} f$ for $f \in C_c^2(\mathbb{R})$, where the latter is defined by (1.6).

Theorem 2.1. *Let $h \in L^2(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, $b, c \in \mathrm{Lip}_b(\mathbb{R})$. Let W be a Brownian sheet, $B := \{B_t; t \geq 0\}$ be a standard one-dimensional Brownian motion and $S := \{S_t; t \geq 0\}$ be a one-dimensional symmetric α-stable process with $\alpha \in (0, 2)$. Assume that W, B and S are defined on a common filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and are independent of each other. Then for every*

$\mathcal{F}_0$-measurable random starting point $Z_0 \in \mathbb{R}$ the stochastic integral equation

$$Z_t = Z_0 + \int_0^t c(Z_{s-})dB_s + \int_0^t b(Z_{s-})\,dS_s + \int_0^t\int_{\mathbb{R}} h(y - Z_{s-})W(dy, ds) \tag{2.2}$$

has a unique strong solution.

Proof. Note that the symmetric α-stable process S is not square-integrable and is a martingale only when $\alpha \in (1, 2)$. So the standard Picard's successive approximation method does not work here. However, it can be made to work with a truncation argument. Such a truncation method is known to experts. However, for the reader's convenience, we will spell out the details of the proof.

We make the convention $S_{0-} = S_0$. With the above notations, it is well known that

$$\widehat{S}_t := S_t - \sum_{s \le t} \triangle S_s \mathbb{1}_{\{|\triangle S_s| > 1\}}, \quad t \ge 0, \tag{2.3}$$

is a Lévy process with Lévy measure $\frac{c_\alpha}{|w|^{1+\alpha}}\mathbb{1}_{\{|w|\le 1\}}dw$ and that $\widehat{S}$ and $S - \widehat{S}$ are independent (cf. [4, pp. 13–15]). Note that the process $\widehat{S}$ is a pure jump, square-integrable martingale and

$$\widehat{S}_t = S_t \quad \text{for } t < T_1 := \inf\{r > 0 : |\triangle S_r| > 1\}. \tag{2.4}$$

We now use Picard's successive approximation method to construct a strong solution Y for (2.2) but with $\widehat{S}$ in place of S. Later on, we will use Y to construct the strong solution for (2.2).

For any given real-valued $\mathcal{F}_0$-measurable random variable Z_0, let $Y_t^{(1)} := Z_0$ for $t \ge 0$, and define for $k \ge 2$ and $t \ge 0$,

$$\begin{aligned} Y_t^{(k)} := Z_0 &+ \int_0^t c(Y_{s-}^{(k-1)})dB_s + \int_0^t b(Y_{s-}^{(k-1)})\,d\widehat{S}_s \\ &+ \int_0^t\int_{\mathbb{R}} h(y - Y_{s-}^{(k-1)})W(dy, ds). \end{aligned} \tag{2.5}$$

Thus for $k \ge 3$ and $t \ge 0$, we have

$$\begin{aligned} Y_t^{(k)} - Y_t^{(k-1)} = &\int_0^t \Big(c(Y_{s-}^{(k-1)}) - c(Y_{s-}^{(k-2)})\Big)\,dB_s \\ &+ \int_0^t \Big(b(Y_{s-}^{(k-1)}) - b(Y_{s-}^{(k-2)})\Big)\,d\widehat{S}_s \\ &+ \int_{\mathbb{R}} \Big(h(y - Y_{s-}^{(k-1)}) - h(y - Y_{s-}^{(k-2)})\Big)\,W(dy, ds). \end{aligned}$$

As the three terms on the right hand side are square-integrable martingales that are independent to each other, by Doob's maximal inequality we have

$$
\begin{aligned}
f_k(t) &\equiv \mathbb{E}\Big[\sup_{0\le s\le t}\Big|Y_s^{(k)}-Y_s^{(k-1)}\Big|^2\Big] \\
&\le 12\mathbb{E}\Big[\Big|Y_t^{(k)}-Y_t^{(k-1)}\Big|^2\Big] \\
&= 12\mathbb{E}\Big[\int_0^t\Big(c(Y_{s-}^{(k-1)})-c(Y_{s-}^{(k-2)})\Big)^2 ds\Big] \\
&\qquad + 12\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\Big(b(Y_{s-}^{(k-1)})-b(Y_{s-}^{(k-2)})\Big)^2\frac{c_\alpha|w|^2}{|w|^{1+\alpha}}\mathbb{1}_{\{|w|\le 1\}}dwds\Big] \\
&\qquad + 4\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}}\Big(h(y-Y_{s-}^{(k-1)})-h(y-Y_{s-}^{(k-2)}\Big)^2 dy\,ds\Big] \\
&\le c\left(\|c'\|_\infty^2+\|b'\|_\infty^2+\|h'\|_\infty^2\right)\mathbb{E}\Big[\int_0^t\Big(Y_s^{(k-1)})-Y_s^{(k-2)}\Big)^2 ds\Big] \\
&\le c_1\int_0^t f_{k-1}(s)ds, \qquad (2.6)
\end{aligned}
$$

where $\|\cdot\|_\infty$ is the supremum norm. A similar calculation shows that

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{0\le s\le t}\Big|Y_s^{(2)}-Y_s^{(1)}\Big|^2\Big] &\le 4\mathbb{E}\Big[\Big|Y_t^{(2)}-Y_t^{(1)}\Big|^2\Big] \\
&= 4\mathbb{E}\Big[\Big(\int_0^t c(Z_0)dB_s+\int_0^t b(Z_0)d\widehat{S}_s \\
&\qquad +\int_0^t\int_{\mathbb{R}} h(y-Z_0)W(dy,ds)\Big)^2\Big] \\
&= 4\mathbb{E}\Big[c(Z_0)^2\,t+b(Z_0)^2\int_{\mathbb{R}}\frac{c_\alpha|w|^2}{|w|^{1+\alpha}}\mathbb{1}_{\{|w|\le 1\}}dw \\
&\qquad +\int_0^t\int_{\mathbb{R}} h(y-Z_0)^2 dyds\Big] \\
&\le c_2\left(\|c\|_\infty^2+\|b\|_\infty^2+\rho(0)\right)t =: c_3\,t.
\end{aligned}
$$

It then follows from above that, for $n\ge 3$,

$$
\mathbb{E}\Big[\sup_{0\le s\le t}\Big|Y_s^{(n)}-Y_s^{(n-1)}\Big|^2\Big]\le c_4^{n-1}\frac{t^{n-1}}{(n-1)!},
$$

where $c_4 = c_1 \vee c_3$. This implies that for every $t > 0$,

$$\sqrt{\mathbb{E}\left[\left(\sum_{n=1}^{\infty} \sup_{0\le s\le t} \left|Y_s^{(n)} - Y_s^{(n-1)}\right|\right)^2\right]}$$
$$\le \sum_{n=1}^{\infty} \sqrt{\mathbb{E}\left[\sup_{0\le s\le t} \left|Y_s^{(n)} - Y_s^{(n-1)}\right|^2\right]} < \infty.$$

Therefore $\sum_{n=1}^{\infty} \sup_{0\le s\le t} \left|Y_s^{(n)} - Y_s^{(n-1)}\right|$ converges a.s. and consequently Y^n converges uniformly on each $[0, t]$ to a càdlàg process Y a.s. Clearly we also have from above that for every $t > 0$,

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\le s\le t} \left|Y_s^{(n)} - Y_s\right|^2\right] = 0.$$

Thus we deduce by the same argument as that for (2.6) that

$$\int_0^t c(Y_{s-}^{(k-1)})dB_s + \int_0^t b(Y_{s-}^{(k-1)})\, d\widehat{S}_s + \int_0^t\int_{\mathbb{R}} h(y - Y_{s-}^{(k-1)})W(dy, ds)$$

converges in L^2 as $k \to \infty$ to

$$\int_0^t c(Y_{s-})dB_s + \int_0^t b(Y_{s-})\, d\widehat{S}_s + \int_0^t\int_{\mathbb{R}} h(y - Y_{s-})W(dy, ds).$$

This implies by (2.5) that

$$Y_t = z + \int_0^t c(Y_{s-})dB_s + \int_0^t b(Y_{s-})\, d\widehat{S}_s + \int_0^t\int_{\mathbb{R}} h(y - Y_{s-})W(dy, ds). \quad (2.7)$$

Moreover, Y is the unique strong solution for (2.7). Indeed, suppose $\widehat{Y}$ is another strong solution of (2.7). Then for every $t \ge 0$,

$$Y_t - \widehat{Y}_t = \int_0^t \left(c(Y_{s-}) - c(\widehat{Y}_{s-})\right) dB_s + \int_0^t \left(b(Y_{s-}) - b(\widehat{Y}_{s-})\right) d\widehat{S}_t$$
$$+ \int_{\mathbb{R}} \left(h(y - Y_{s-}) - h(y - \widehat{Y}_{s-})\right) W(dy, ds).$$

By a similar argument as that for (2.6), we have

$$\mathbb{E}\left[\sup_{0\le s\le t} \left|Y_s - \widehat{Y}_s\right|^2\right] \le c_5 t\, \mathbb{E}\left[\sup_{0\le s\le t} \left|Y_s - \widehat{Y}_s\right|^2\right].$$

It follows that $\sup_{0\le s\le t}\left|Y_s-\widehat{Y}_s\right|=0$ a.s. for every $t\in[0,1/c_5)$, and therefore for every $t\in[k/c_5,\,(k+1)/c_5)$. This proves that $Y=\widehat{Y}$ a.s.

Recall the stopping time T_1 in (2.4). Define

$$Z_t:=\begin{cases}Y_t & \text{if } t<T_1,\\ Y_{T_1-}+b(Y_{T_1-})(S_{T_1}-S_{T_1-}) & \text{if } t=T_1.\end{cases}$$

Then, in view of (2.3), Z is the unique strong solution for SDE (2.2) over the random time interval $[0,T_1]$.

For $k\ge 1$, define

$$T_k:=T_{k-1}+T_1\circ\theta_{T_{k-1}},$$

where $\{\theta_t;t>0\}$ are shift operators on the canonical path space Ω. In other words, T_k is the kth time that the symmetric α stable process S has a jump of size larger than 1. Since $t\mapsto S_t$ is right continuous and has left hand limit, S can only have finite many jumps of size larger than 1 over any compact time intervals. So $\lim_{k\to\infty}T_k=\infty$ almost surely.

Note that $\{B_{T_1+t}-B_{T_1};t\ge 0\}$ is a Brownian motion, $\{S_{T_1+t}-S_{T_1};t\ge 0\}$ is symmetric α-stable process and $\{W((y_1,y_2],[T_1,T_1+t]);t\ge 0\}$ is a Brownian sheet that are independent to each other and are independent of $\mathcal{F}_{T_1}$. Suppose $\{Y^2_t;t\ge 0\}$ is the strong solution of (2.7) but with Z_{T_1} in place of Z_0, and $\{B_{T_1+t}-B_{T_1};t\ge 0\}$, $\{S_{T_1+t}-S_{T_1};t\ge 0\}$ and $W((y_1,y_2],[T_1,T_1+t])$ in place of B, S and $W((y_1,y_2],[0,t])$ there, respectively. Define

$$Z_{T_1+t}:=\begin{cases}Y^2_t & \text{if } t<T_1\circ\theta_{T_1},\\ Y^2_{T_1\circ\theta_{T_1}-}+b(Y^2_{T_1\circ\theta_{T_1}-})(S_{T_2}-S_{T_2-}) & \text{if } t=T_1\circ\theta_{T_1}.\end{cases}$$

It is easy to see that such defined Z solves SDE (2.2) for $t\le T_2$. Iterating this procedure, we can define Z on $[0,T_k]$ for every $k\ge 1$ and hence on $[0,\infty)$. Thus we have obtained a strong solution Z for (2.2) for $t\in[0,\infty)$. Such a strong solution is unique as it is unique on each time interval $[T_{k-1},T_k]$. Here we take $T_0=0$. □

3 Branching particle systems

In order to construct branching particle systems, first we need to introduce an index set to identify each particle in the branching tree structure. Let $\Re$ be the set of all multi-indices, i.e., strings of the form $\xi=n_1n_2\ldots n_k$, where the n_i's are non-negative integers. Let $|\xi|$ denote the length of ξ. We provide $\Re$ with the arboreal ordering: $m_1m_2\ldots m_p\prec n_1n_2\ldots n_q$ if and only if $p\le q$ and

$m_1 = n_1, \dots, m_p = n_p$. If $|\xi| = p$, then ξ has exactly $p-1$ predecessors, which we shall denote respectively by $\xi - 1, \xi - 2, \dots, \xi - |\xi| + 1$. For example, with $\xi = 6879$, we get $\xi - 1 = 687$, $\xi - 2 = 68$ and $\xi - 3 = 6$.

Let $\{B^\xi; \xi \in \Re\}$ be an independent family of standard Brownian motions in $\mathbb{R}$, $\{S^\xi; \xi \in \Re\}$ be an independent family of one-dimensional symmetric α-stable processes with $\alpha \in (0,2)$ and W be a Brownian sheet. Assume further that W, $\{B^\xi; \xi \in \Re\}$ and $\{S^\xi; \xi \in \Re\}$ are defined on a common filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and independent of each other. For every index $\xi \in \Re$ and initial data z_0^ξ, by Theorem 2.1 there is a unique strong solution z_t^ξ for

$$\begin{aligned} z_t^\xi = z_0^\xi &+ \int_0^t c(z_{s-}^\xi) dB_s^\xi + \int_0^t b(z_{s-}^\xi)\, dS_s^\xi \\ &+ \int_0^t \int_{\mathbb{R}} h(y - z_{s-}^\xi) W(dy, ds). \end{aligned} \tag{3.1}$$

Since the strong solution of (3.1) only depends on the initial state z_0^ξ, the Brownian motion $B^\xi := \{B_t^\xi; t \geq 0\}$, the symmetric α-stable process $S^\xi := \{S_t^\xi; t \geq 0\}$ and the common white noise W, we can write $z_t^\xi = \Phi(z_0^\xi, B^\xi, S^\xi, t)$ for some measurable real-valued map Φ (omitting W in the notation as it is selected and fixed once and for all).

Lemma 3.1. *There is a constant $c > 0$ such that for every $\phi \in C_b^2(\mathbb{R})$,*

$$\|\Delta^{\alpha/2}\phi\|_\infty \leq c\left(\|\phi\|_\infty + \|\phi''\|_\infty\right).$$

Proof. By the mean-value theorem, we have

$$\begin{aligned} \|\Delta^{\alpha/2}\phi(x)\|_\infty &= \sup_{x\in\mathbb{R}} \left| \int_{\mathbb{R}\setminus\{0\}} \left(\phi(x+w) - \phi(x) - \phi'(x) w \mathbb{1}_{\{|w|\leq 1\}}\right) \frac{c_\alpha}{|w|^{1+\alpha}} dw \right| \\ &\leq c\left(\|\phi''\|_\infty \int_{|w|\leq 1} |w|^{1-\alpha} dw + \|\phi\|_\infty \int_{|w|>1} |w|^{-1-\alpha} dw \right) \\ &< \infty. \end{aligned}$$

This proves the lemma. □

Define

$$\widehat{S}_t^\xi := S_t^\xi - \sum_{s\leq t} \Delta S_s^\xi \mathbb{1}_{\{|\Delta S_s^\xi|>1\}},$$

which is a Lévy process independent of $S^\xi - \widehat{S}^\xi$. Note that $\Delta z_t^\xi = b(z_{t-}^\xi)\Delta S_t^\xi$.

So by the property of Lévy measure (see (2.1)), we have for every $\phi \in C_b^2(\mathbb{R})$,

$$\begin{aligned} M_t^{\phi,\xi} := \sum_{s\le t} \Big[\phi(z_s^\xi) - \phi(z_{s-}^\xi) - \big(\phi' b\big)(z_{s-}^\xi)\triangle S_s^\xi \mathbb{1}_{\{|\triangle S_s^\xi|\le 1\}}\Big] \\ - \int_0^t \Big(\int_{\mathbb{R}\setminus\{0\}} \Big(\phi\left(z_s^\xi + b(z_s^\xi)w\right) - \phi(z_s^\xi) \\ - \big(\phi' b\big)(z_s^\xi)\, w\mathbb{1}_{\{|w|\le 1\}}\Big)\frac{c_\alpha}{|w|^{1+\alpha}}dw\Big)ds \end{aligned}$$

is a square-integrable martingale with

$$\begin{aligned} \mathbb{E}\Big[\Big(M_t^{\phi,\xi}\Big)^2\Big] &= \mathbb{E}\Big[\int_0^t\int_{\mathbb{R}} \Big(\phi(z_s^\xi + b(z_s^\xi)w) - \phi(z_s^\xi) \\ &\qquad - (\phi' b)(z_s^\xi)w\mathbb{1}_{\{|w|\le 1\}}\Big)^2 \frac{c_\alpha}{|w|^{1+\alpha}}dwds\Big] \\ &\le \mathbb{E}\Big[\int_0^t \mathbb{1}_{\{z_s^\xi\neq\partial\}}\Big(\int_{\mathbb{R}}\Big(\frac{1}{2}\|\phi'' b^2\|_\infty |w|^2\mathbb{1}_{\{|w|\le 1\}} \\ &\qquad + \|2\phi\|_\infty^2\mathbb{1}_{\{|w|>1\}}\Big)\frac{c_\alpha}{|w|^{1+\alpha}}dw\Big)ds\Big] \\ &\le c\left(\|b^2\phi''\|_\infty + \|\phi\|_\infty\right)t. \end{aligned} \tag{3.2}$$

Observe that due to the symmetry of the kernel $c_\alpha/|w|^{1+\alpha}$,

$$\Delta^{\alpha/2}\phi(z) = \int_{\mathbb{R}\setminus\{0\}} \left(\phi(z+w) - \phi(z) - \phi'(z)w\mathbb{1}_{\{|w|\le h\}}\right)\frac{c_\alpha}{|w|^{1+\alpha}}\,dw$$

for every $h > 0$. So by a change of variable, we have

$$\begin{aligned} &\int_{\mathbb{R}\setminus\{0\}} \left(\phi(x+b(x)w) - \phi(x) - (\phi' b)(x)w\mathbb{1}_{\{|w|\le 1\}}\right)\frac{c_\alpha}{|w|^{1+\alpha}}dw \\ &\qquad = |b(x)|^\alpha \Delta^{\alpha/2}\phi(x). \end{aligned}$$

It follows that

$$\begin{aligned} M_t^{\phi,\xi} = \sum_{s\le t}\Big(\phi(z_s^\xi) - \phi(z_{s-}^\xi) - \big(\phi' b\big)(z_{s-}^\xi)\triangle S_s^\xi\mathbb{1}_{\{|\triangle S_s^\xi|\le 1\}}\Big) \\ - \int_0^t \Big(|b|^\alpha\Delta^{\alpha/2}\phi\Big)(z_s^\xi)ds. \end{aligned} \tag{3.3}$$

With $\phi \in C_b^2(\mathbb{R})$, we have by Itô's formula that for every $t > 0$,

$$\begin{aligned}
\phi(z_t^\xi) - \phi(z_0^\xi) &= \int_0^t \left(\phi' c\right)(z_{s-}^\xi) dB_s^\xi + \int_0^t \int_{\mathbb{R}} \phi'(z_{s-}^\xi) h(y - z_{s-}^\xi) W(dy, ds) \\
&\quad + \int_0^t \left(\phi' b\right)(z_{s-}^\xi)\, dS_s^\xi + \frac{1}{2} \int_0^t (\rho(0) + c^2(z_s^\xi))\phi''(z_s^\xi) ds \\
&\quad + \sum_{s \le t} \left[\phi(z_s^\xi) - \phi(z_{s-}^\xi) - \left(\phi' b\right)(z_{s-}^\xi) \triangle S_s^\xi\right] \\
&= \int_0^t \left(\phi' c\right)(z_s^\xi) dB_s^\xi + \int_0^t \int_{\mathbb{R}} \phi'(z_s^\xi) h(y - z_s^\xi) W(dy, ds) \\
&\quad + \int_0^t \left(\phi' b\right)(z_{s-}^\xi)\, d\widehat{S}_s^\xi + \frac{1}{2} \int_0^t \left(a\phi''\right)(z_s^\xi) ds \\
&\quad + \sum_{s \le t} \left[\phi(z_s^\xi) - \phi(z_{s-}^\xi) - \left(\phi' b\right)(z_{s-}^\xi) \triangle S_s^\xi \mathbb{1}_{\{|\triangle S_s^\xi| \le 1\}}\right] \\
&= \int_0^t \left(\phi' c\right)(z_s^\xi) dB_s^\xi + \int_0^t \int_{\mathbb{R}} \phi'(z_s^\xi) h(y - z_s^\xi) W(dy, ds) \\
&\quad + \int_0^t \left(\phi' b\right)(z_{s-}^\xi)\, d\widehat{S}_s^\xi + M_t^{\phi,\xi} \\
&\quad + \int_0^t \left(\tfrac{1}{2} a\phi'' + |b|^\alpha \Delta^{\alpha/2}\phi\right)(z_s^\xi) ds. \qquad (3.4)
\end{aligned}$$

We now consider the branching particle systems in which each particle's spatial motion is modeled by the SDE (3.1). For every positive integer $n \ge 1$, there is an initial system of m_0^n particles. Each particle has mass θ^{-n} and branches independently at rate $\lambda\theta^n$. The branching mechanism is assumed to be state independent. Let q_k^n denote the probability of having k offsprings. We assume that

$$q_k^n = 0 \quad \text{if } k = 1 \text{ or } k \ge n + 1,$$

and

$$\sum_{k=0}^{n} k q_k^n = 1 \quad \text{and} \quad \lim_{n\to\infty} \sup_{k \ge 0} |q_k^n - p_k| = 0,$$

where $\{p_k, k = 0, 1, 2, \dots\}$ is the limiting offspring distribution which is assumed to satisfy following conditions:

$$p_1 = 0, \quad \sum_{k=0}^{\infty} k p_k = 1 \quad \text{and} \quad m_2 := \sum_{k=0}^{\infty} k^2 p_k < \infty.$$

Let $m_c^n := \sum_{k=0}^n (k-1)^4 q_k^n$. Here $\{m_c^n; n \geq 1\}$ may be unbounded, but we assume that

$$\lim_{n\to\infty} \frac{m_c^n}{\theta^{2n}} = 0 \quad \text{for any } \theta \geq 2.$$

We will see that the limiting offspring distribution is the offspring distribution of the Stable SDSM, the limiting measure-valued process we will construct. We assume that the initial number of particles $m_0^n \leq \hbar\, \theta^n$, where $\hbar > 0$ and $\theta \geq 2$ are fixed constants. Define $m_2^n := \sum_{k=0}^n k^2 q_k^n$, $\sigma_n^2 := m_2^n - 1$ and $\sigma^2 := m_2 - 1$. Note that σ_n^2 and σ^2 are the variance of the n-stage and the limiting offspring distribution, respectively. We have $\sigma_n^2 < \infty$ and $\lim_{n\to\infty} \sigma_n^2 = \sigma^2$.

Let $\{C^\xi; \xi \in \Re\}$ be a family of i.i.d. real-valued exponential random variables with parameter $\lambda\theta^n$, which serve as lifetimes of the particles, and $\{D^\xi; \xi \in \Re\}$ be a family of i.i.d. random variables with $\mathbb{P}(D^\xi = k) = q_k^n$ for $k = 0, 1, 2, \ldots$, where $\{q_k^n; k = 0, 1, \ldots\}$ is the offspring distribution satisfying above conditions. We assume W, $\{B^\xi; \xi \in \Re\}$, $\{S^\xi; \xi \in \Re\}$, $\{C^\xi; \xi \in \Re\}$ and $\{D^\xi; \xi \in \Re\}$ are all independent.

The birth time $\beta(\xi)$ of the particle x^ξ is given by

$$\beta(\xi) := \begin{cases} \sum_{j=1}^{|\xi|-1} C^{\xi-j} & \text{if } D^{\xi-j} \geq 2 \text{ for every } j = 1, \ldots, |\xi|-1, \\ \infty, & \text{otherwise.} \end{cases}$$

The death time of x^ξ is given by $\zeta(\xi) = \beta(\xi) + C^\xi$ and the indicator function of the lifespan of x^ξ is denoted by $h_t^\xi := \mathbb{1}_{[\beta(\xi),\zeta(\xi))}(t)$.

Recall that ∂ denotes the cemetery point. Define $x_t^\xi = \partial$ if either $t < \beta(\xi)$ or $t \geq \zeta(\xi)$. We make a convention that any function f defined on $\mathbb{R}$ is automatically extended to $\mathbb{R} \cup \{\partial\}$ by setting $f(\partial) = 0$. This convention allows us to keep track of only those particles that is alive at any given time.

Given $\mu_0 \in M_F(\mathbb{R})$, let $\mu_0^n := \theta^{-n} \sum_{\xi=1}^{m_0^n} \delta_{x_0^\xi}$ be such that $\mu_0^n \Rightarrow \mu_0$ as $n \to \infty$ (see [33]). We are thus provided with collection of initial starting points $\{x_0^\xi\}$ for each $n \geq 1$.

Let $\mathcal{N}_1^n := \{1, 2, \ldots, m_0^n\}$ be the set of indices for the first generation of particles. For any $\xi \in \mathcal{N}_1^n \cap \Re$, define

$$x_t^\xi := \begin{cases} \Phi(x_0^\xi, B^\xi, S^\xi, t) & \text{if } 0 \leq t < C^\xi, \\ \partial & \text{if } t \geq C^\xi, \end{cases} \tag{3.5}$$

and

$$x_t^\xi \equiv \partial \quad \text{for any } \xi \in (\mathbb{N} \setminus \mathcal{N}_1^n) \cap \Re \text{ and } t \geq 0.$$

If $\xi_0 \in \mathcal{N}_1^n$ and $D^{\xi_0}(\omega) = k \geq 2$, define for every $\xi \in \{\xi_0 i; i = 1, 2, \ldots, k\}$,

$$x_t^\xi := \begin{cases} \Phi(x^{\xi_0}_{\zeta(\xi_0)-}, B^\xi, S^\xi, t - \beta(\xi)) & \text{if } t \in [\beta(\xi), \zeta(\xi)), \\ \partial & \text{otherwise.} \end{cases} \tag{3.6}$$

If $D^{\xi_0}(\omega) = 0$, define $x_t^\xi \equiv \partial$ for $0 \leq t < \infty$ and for every $\xi \in \{\xi_0 i; i \geq 1\}$.

More generally for an integer $m \geq 1$, let $\mathcal{N}_m^n \subset \mathfrak{R}$ be the set of all indices for the living particles in m^{th}-generation. If $\xi_0 \in \mathcal{N}_m^n$ and $D^{\xi_0}(\omega) = k \geq 2$, define for $\xi \in \{\xi_0 i; i = 1, 2, \ldots, k\}$

$$x_t^\xi := \begin{cases} \Phi(x^{\xi_0}_{\zeta(\xi_0)-}, B^\xi, S^\xi, t - \beta(\xi)) & \text{if } t \in [\beta(\xi), \zeta(\xi)), \\ \partial & \text{otherwise.} \end{cases} \tag{3.7}$$

If $D^{\xi_0}(\omega) = 0$, define

$$x_t^\xi \equiv \partial \quad \text{for } 0 \leq t < \infty \text{ and for every } \xi \in \{\xi_0 i; i \geq 1\}.$$

Continuing in this way, we get a branching tree of particles for any given ω with random initial state taking values in $\{x_0^1, x_0^2, \ldots, x_0^{m_0^n}\}$. This gives us our branching particle systems on the real line where particles undergo a finite-variance branching at independent exponential times and have interacting spatial motion powered by jump-diffusions and a common white noise.

4 Tightness and the limiting martingale problem

Recall the branching particle system constructed in the last section. Define its associated empirical process by

$$\mu_t^n(A) := \frac{1}{\theta^n} \sum_{\xi \in \mathfrak{R}} \delta_{x_t^\xi}(A) \quad \text{for } A \in \mathcal{B}(\mathbb{R}). \tag{4.1}$$

In the following, we will show that $\{\mu_t^n; t \geq 0\}$ converges in $\mathbb{D}([0, \infty), M_F(\mathbb{R}))$ weakly as $n \to \infty$ and its weak limit is the stable SDSM.

For any $t > 0$, define

$$M^n(A \times (0, t]) := \sum_{\xi \in \mathfrak{R}} \frac{(D^\xi - 1)}{\theta^n} \mathbb{1}_{\{x^\xi_{\zeta(\xi)-} \in A,\, \zeta(\xi) \leq t\}}, \tag{4.2}$$

which gives the total mass along each survived trajectory in set A up to time t.

Note that (3.4) implies that for every $\phi \in C_b^2(\mathbb{R})$,

$$\begin{aligned}\langle \phi, \mu_t^n\rangle - \langle \phi, \mu_0^n\rangle = \frac{1}{\sqrt{\theta^n}} U_t^n(\phi) + \frac{1}{\sqrt{\theta^n}} H_t^n(\phi) + X_t^n(\phi) + Y_t^n(\phi) \\ + \frac{1}{\sqrt{\theta^n}} J_t^n(\phi) + M_t^n(\phi),\end{aligned} \tag{4.3}$$

where, recalling $h_s^\xi = \mathbb{1}_{[\beta(\xi),\, \zeta(\xi))}(s)$,

$$\begin{aligned}
U_t^n(\phi) &:= \frac{1}{\sqrt{\theta^n}} \sum_{\xi\in\mathfrak{R}} \int_0^t h_s^\xi\, \phi'(x_s^\xi) c(x_s^\xi) dB_s^\xi\,,\\
X_t^n(\phi) &:= \int_0^t \int_{\mathbb{R}} \langle h(y-\cdot)\phi'(\cdot), \mu_s^n\rangle W(dy, ds)\,,\\
Y_t^n(\phi) &:= \int_0^t \langle \tfrac{1}{2} a\phi'' + |b|^\alpha \Delta^{\alpha/2}\phi, \mu_s^n\rangle ds,\\
H_t^n(\phi) &:= \frac{1}{\sqrt{\theta^n}} \sum_{\xi\in\mathfrak{R}} \int_0^t h_s^\xi\, \phi'(x_{s-}^\xi) b(x_{s-}^\xi) d\widehat{S}_s^\xi\,,\\
J_t^n(\phi) &:= \frac{1}{\sqrt{\theta^n}} \sum_{\xi\in\mathfrak{R}} M_t^{\phi,\xi}\\
M_t^n(\phi) &:= \int_0^t \int_{\mathbb{R}} \phi(x) M^n(dx, ds)\\
&= \sum_{\xi\in\mathfrak{R}} \frac{(D^\xi - 1)}{\theta^n} \phi(x_{\zeta(\xi)-}^\xi)\, \mathbb{1}_{\{\zeta(\xi)\le t\}}.
\end{aligned}$$

The six terms in (4.3) represent the respective components of the overall motion of the finite particle systems $\langle \phi, \mu_t^n\rangle$ contributed by the individual Brownian motions ($U_t^n(\phi)$), the random medium ($X_t^n(\phi)$), the mean effect of jump-diffusive motions ($Y_t^n(\phi)$), the branching mechanism ($M_t^n(\phi)$), the individual stable motions ($H_t^n(\phi)$) with jump size bounded by one and the individual compensated jump sums ($J_t^n(\phi)$). Using a result of Dynkin [12, Theorem 10.25 on p. 325], we get at once the following theorem.

Theorem 4.1. *For any $n \in \mathbb{N}$, μ_t^n defined by* (4.1) *is a right continuous strong Markov process which is the unique strong solution of* (4.3) *in the sense that it is a unique solution of* (4.3) *for a given probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and given W, $\{B^\xi\}$, $\{S^\xi\}$, $\{C^\xi\}$, $\{D^\xi\}$ defined on* $(\Omega, \mathcal{F}, \mathbb{P})$. *Furthermore, $\{\mu_t^n; t \ge 0\}$ are all defined on the common probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

For each $t \geq 0$, let $\mathcal{F}_t^n$ denote the σ-algebra generated by the collection of processes

$$\Big\{\mu_t^n(\phi), U_t^n(\phi), X_t^n(\phi), Y_t^n(\phi), M_t^n(\phi), H_t^n(\phi), J_t^n(\phi),\ \phi \in C_b^2(\mathbb{R}),\ t \geq 0\Big\}.$$

Note that according to our assumption, the fourth moment of D^ξ,

$$m_c^n := \mathbb{E}\{(D^\xi - 1)^4\},$$

is finite and $\lim_{n\to\infty} m_c^n \theta^{-2n} = 0$ for any $\theta \geq 2$.

Lemma 4.2. *Based on the above notations, we have the following:*

(i) *For every $\phi \in C_b^2(\mathbb{R})$, $M^n(\phi) := \{M_t^n(\phi); t \geq 0\}$ is a purely discontinuous square-integrable martingale with*

$$\langle M^n(\phi)\rangle_t = \lambda\sigma_n^2 \int_0^t \langle \phi^2, \mu_u^n\rangle du \quad \textit{for every } t \geq 0.$$

(ii) *For any $t \geq 0$ and $n \geq 1$, we have*

$$\mathbb{E}\Big[\sup_{0\leq s\leq t} \langle 1, \mu_s^n\rangle^2\Big] \leq 2\langle 1, \mu_0^n\rangle^2 + \ 8\lambda\sigma_n^2 t\langle 1, \mu_0^n\rangle.$$

There is a constant $\kappa > 0$ such that for every $t \geq 0$,

$$\mathbb{E}\Big[\sup_{0\leq s\leq t} \langle 1, \mu_s^n\rangle^4\Big] \leq \kappa\Big(\lambda^3\sigma_n^6 t^3\langle 1, \mu_0^n\rangle + \lambda^2\sigma_n^4 t^2\langle 1, \mu_0^n\rangle^2 + \frac{\lambda m_c^n}{\theta^{2n}} t\langle 1, \mu_0^n\rangle + \langle 1, \mu_0^n\rangle^4\Big).$$

(iii) *Let $\{\mu_t^n; t \geq 0\}$ be defined by* (4.1). *Then $\{\mu_t^n; t \geq 0\}$ is tight as a family of processes with sample paths in $\mathbb{D}([0,\infty), M_F(\mathbb{R}))$.*

Proof. (i) Recall that $\{C^\xi; \xi \in \Re\}$ are i.i.d. exponential random variables with parameter $\lambda\theta^n$, $\{D^\xi - 1; \xi \in \Re\}$ are i.i.d. random variables with zero mean and these two families are independent. Thus $\mathbb{E}\{M_t^n(\phi)\} = 0$ for every $t > 0$ and $\phi \in \mathcal{S}(\mathbb{R})$. Since this holds for any initial state μ_0^n, by the Markov property of $\{\mu_t^n; t \geq 0\}$, we have for every $t, s > 0$,

$$\mathbb{E}\left[M_{t+s}^n(\phi) - M_t^n(\phi) \mid \mathcal{F}_t^n\right] = \mathbb{E}_{\mu_t^n}\left[M_s^n(\phi) - M_0^n(\phi)\right] = 0.$$

This shows that $M^n(\phi)$ is a martingale. Clearly, it is purely discontinuous.

Note 1 add some wording or replace period by colon?

$$
\begin{aligned}
&\mathbb{E}\Big[\phi^2(x^{\xi}_{\zeta(\xi)-});\ \zeta(\xi)\le t\Big]\\
&\quad=\mathbb{E}\Big[\mathbb{1}_{[0,t]}(\beta(\xi)+C^{\xi})\phi^2(x^{\xi}_{(\beta(\xi)+C^{\xi})-})\Big]\\
&\quad=\mathbb{E}\Big[\int_0^\infty \mathbb{1}_{[0,t]}(\beta(\xi)+u)\phi^2(x^{\xi}_{(\beta(\xi)+u)-})\lambda\theta^n e^{-\lambda\theta^n u}du\Big]\\
&\quad=\mathbb{E}\Big[\int_0^\infty \mathbb{1}_{[0,t]}(\beta(\xi)+u)\phi^2(x^{\xi}_{(\beta(\xi)+u)-})\lambda\theta^n \mathbb{1}_{\{C^{\xi}>u\}}du\Big] \quad \text{by independence}\\
&\quad=\mathbb{E}\Big[\int_0^\infty \mathbb{1}_{[0,t]}(\beta(\xi)+u)\phi^2(x^{\xi}_{\beta(\xi)+u})\lambda\theta^n du\Big] \qquad \text{by the definition of } x^{\xi}\\
&\quad=\mathbb{E}\Big[\int_{\beta(\xi)}^\infty \mathbb{1}_{[0,t]}(v)\phi^2(x^{\xi}_{v})\lambda\theta^n dv\Big]\\
&\quad=\lambda\theta^n\,\mathbb{E}\Big[\int_0^t \mathbb{1}_{[\beta(\xi),\zeta(\xi))}(v)\phi^2(x^{\xi}_{v})dv\Big].
\end{aligned}
$$

As $\{C^{\xi};\xi\in\Re\}$ and $\{D^{\xi};\xi\in\Re\}$ are all independent and $\mathbb{E}D^{\xi}=1$, we conclude that

$$
\begin{aligned}
\mathbb{E}\left[M^n_t(\phi)^2\right] &= \sum_{\xi\in\Re}\theta^{-2n}\mathbb{E}\left[(D^{\xi}-1)^2\right]\mathbb{E}\Big[\phi^2(x^{\xi}_{\zeta(\xi)-})\mathbb{1}_{(\zeta(\xi)\le t)}\Big]\\
&= \theta^{-2n}\sigma_n^2\sum_{\xi\in\Re}\lambda\theta^n\mathbb{E}\Big[\int_0^t \mathbb{1}_{[\beta(\xi),\zeta(\xi))}(v)\phi^2(x^{\xi}_{v})dv\Big] \qquad (4.4)\\
&= \lambda\sigma_n^2\mathbb{E}\Big[\int_0^t \langle\phi^2,\ \mu^n_v\rangle\, dv\Big].
\end{aligned}
$$

Note that identity (4.4) holds for any initial distribution μ^n_0. Again by the Markov property of $\{\mu^n_t;t\ge 0\}$, we have for every $t,s>0$,

$$
\begin{aligned}
&\mathbb{E}\Big[M^n_{t+s}(\phi)^2-M^n_t(\phi)^2-\lambda\sigma_n^2\int_t^{t+s}\langle\phi^2,\ \mu^n_v\rangle\, dv \,\Big|\, \mathcal{F}^n_t\Big]\\
&\qquad=\mathbb{E}_{\mu^n_t}\Big[M^n_s(\phi)^2-\lambda\sigma_n^2\int_0^s\langle\phi^2,\ \mu^n_v\rangle\, dv\Big]=0.
\end{aligned}
$$

This shows that $M^n_t(\phi)^2-\lambda\sigma_n^2\int_0^t\langle\phi^2,\ \mu^n_v\rangle\, dv$ is a martingale. Hence we conclude that $M^n(\phi)$ is a purely discontinuous square-integrable martingale with

$$
\langle M^n(\phi)\rangle_t=\lambda\sigma_n^2\int_0^t\langle\phi^2,\mu^n_u\rangle du \quad \text{for every } t\ge 0.
$$

(ii) Since $\langle 1, \mu_t^n - \mu_0^n \rangle = M_t^n(1)$ is a zero-mean martingale, by Doob's maximal inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le s\le t} \langle 1, \mu_s^n\rangle^2\Big] &\le 2\mathbb{E}\Big[\sup_{0\le s\le t} M_s^n(1)^2\Big] + 2\langle 1, \mu_0^n\rangle^2 \\ &\le 8\mathbb{E}\left[M_t^n(1)^2\right] + 2\langle 1, \mu_0^n\rangle^2 \\ &\le 8\lambda\sigma_n^2 t\, \langle 1, \mu_0^n\rangle + 2\langle 1, \mu_0^n\rangle^2.\end{aligned}$$

Note that $M_t^n(1) = \sum_{\xi\in\Re} \frac{D^\xi - 1}{\theta^n} \mathbb{1}_{\{\zeta(\xi)\le t\}}$ is a purely discontinuous martingale and $\{D^\xi - 1; \xi \in \Re\}$ are i.i.d random variables with zero mean and are independent of $\{C^\xi; \xi \in \Re\}$. Thus

$$\begin{aligned}\mathbb{E}\Big[\big(M_t^n(1)\big)^4\Big] &= \mathbb{E}\Big[\sum_{\xi,\eta\in\Re,\xi\neq\eta} \Big(\frac{(D^\xi-1)^2}{\theta^{2n}} \mathbb{1}_{\{\zeta(\xi)\le t\}} \frac{(D^\eta-1)^2}{\theta^{2n}} \mathbb{1}_{\{\zeta(\eta)\le t\}}\Big)\Big] \\ &\qquad + \mathbb{E}\Big[\sum_{\xi\in\Re} \frac{(D^\xi-1)^4}{\theta^{4n}} \mathbb{1}_{\{\zeta(\xi)\le t\}}\Big] \\ &= \frac{\sigma_n^4}{\theta^{4n}} \mathbb{E}\Big[\sum_{\xi,\eta\in\Re,\xi\neq\eta} \mathbb{1}_{\{\zeta(\xi)\le t\}}\mathbb{1}_{\{\zeta(\eta)\le t\}}\Big] + \frac{m_c^n}{\theta^{4n}} \mathbb{E}\Big[\sum_{\xi\in\Re} \mathbb{1}_{\{\zeta(\xi)\le t\}}\Big] \\ &= \frac{\sigma_n^4}{\theta^{4n}} \sum_{\xi,\eta\in\Re,\xi\neq\eta} \mathbb{E}\big[\mathbb{1}_{\{\zeta(\xi)\le t\}}\mathbb{1}_{\{\zeta(\eta)\le t\}}\big] + \frac{\lambda m_c^n}{\theta^{2n}} \mathbb{E}\Big[\int_0^t \langle 1, \mu_v^n\rangle dv\Big].\end{aligned}$$

For $\xi, \eta \in \Re$ with $\xi \neq \eta$, C^ξ and C^η are independent and so

$$\begin{aligned}&\mathbb{E}\big[\mathbb{1}_{\{\zeta(\xi)\le t\}}\mathbb{1}_{\{\zeta(\eta)\le t\}}\big] \\ &\quad = \mathbb{E}\Big[\mathbb{1}_{[0,t]}(\beta(\xi) + C^\xi)\, \mathbb{1}_{[0,t]}(\beta(\eta) + C^\eta)\Big] \\ &\quad = \lambda^2\theta^{2n}\mathbb{E}\Big[\int_0^\infty\int_0^\infty \mathbb{1}_{[0,t]}(\beta(\xi)+u)\mathbb{1}_{[0,t]}(\beta(\eta)+v)e^{-\lambda\theta^n u}e^{-\lambda\theta^n v}du\,dv\Big] \\ &\quad \le \lambda^2\theta^{2n}\mathbb{E}\Big[\int_0^\infty\int_0^\infty \mathbb{1}_{[0,t]}(\beta(\xi)+u)\mathbb{1}_{[0,t]}(\beta(\eta)+v) \\ &\qquad\qquad \times \mathbb{1}_{\{x^\xi_{\beta(\xi)+u}\neq\partial\}}\mathbb{1}_{\{x^\eta_{\beta(\eta)+u}\neq\partial\}}du\,dv\Big] \\ &\quad = \lambda^2\theta^{2n}\mathbb{E}\Big[\Big(\int_{\beta(\xi)}^\infty \mathbb{1}_{[0,t]}(r)\mathbb{1}_{\{x^\xi_r\neq\partial\}}dr\Big)\Big(\int_{\beta(\eta)}^\infty \mathbb{1}_{[0,t]}(s)\mathbb{1}_{\{x^\eta_s\neq\partial\}}ds\Big)\Big] \\ &\quad = \lambda^2\theta^{2n}\mathbb{E}\Big[\Big(\int_0^t \mathbb{1}_{[\beta(\xi),\,\zeta(\xi))}(r)dr\Big)\Big(\int_0^t \mathbb{1}_{[\beta(\eta),\,\zeta(\eta))}(s)ds\Big)\Big].\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{\xi,\eta\in\Re,\xi\neq\eta} \mathbb{E}\big[\mathbb{1}_{\{\zeta(\xi)\le t\}}\mathbb{1}_{\{\zeta(\eta)\le t\}}\big] \\
&\qquad \le \sum_{\xi,\eta\in\Re} \lambda^2\theta^{2n}\mathbb{E}\Big[\Big(\int_0^t \mathbb{1}_{[\beta(\xi),\,\zeta(\xi))}(r)dr\Big)\Big(\int_0^t \mathbb{1}_{[\beta(\eta),\,\zeta(\eta))}(s)ds\Big)\Big] \\
&\qquad = \lambda^2\theta^{2n}\mathbb{E}\left[\Big(\sum_{\xi\in\Re}\int_0^t \mathbb{1}_{[\beta(\xi),\,\zeta(\xi))}(r)dr\Big)^2\right] \\
&\qquad = \lambda^2\theta^{4n}\mathbb{E}\Big[\Big(\int_0^t \langle 1,\mu_r^n\rangle dr\Big)^2\Big].
\end{aligned}$$

It then follows

$$\begin{aligned}
\mathbb{E}\Big[\big(M_t^n(1)\big)^4\Big] &\le \frac{\sigma_n^4}{\theta^{4n}}\lambda^2\theta^{4n}\mathbb{E}\Big[\Big(\int_0^t \langle 1,\mu_r^n\rangle dr\Big)^2\Big] + \frac{\lambda m_c^n}{\theta^{2n}}\mathbb{E}\Big[\int_0^t \langle 1,\ \mu_v^n\rangle dv\Big] \\
&= \lambda^2\sigma_n^4 t^2\mathbb{E}\Big[\sup_{r\in[0,t]}\langle 1,\mu_r^n\rangle^2\Big] + \frac{\lambda m_c^n}{\theta^{2n}}\langle 1,\mu_0^n\rangle t \\
&\le \lambda^2\sigma_n^4 t^2\left(8\lambda\sigma_n^2 t\,\langle 1,\mu_0^n\rangle + 2\langle 1,\mu_0^n\rangle^2\right) + \frac{\lambda m_c^n}{\theta^{2n}}\langle 1,\mu_0^n\rangle t \\
&= 8\lambda^3\sigma_n^6 t^3\langle 1,\mu_0^n\rangle + 2\lambda^2\sigma_n^4 t^2\langle 1,\mu_0^n\rangle^2 + \frac{\lambda m_c^n}{\theta^{2n}} t\langle 1,\mu_0^n\rangle. \qquad (4.5)
\end{aligned}$$

By Doob's maximal inequality,

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le s\le t}\langle 1,\mu_s^n\rangle^4\Big] &\le \mathbb{E}\Big[\sup_{0\le s\le t}\big(\langle 1,\mu_s^n-\mu_0^n\rangle + \langle 1,\mu_0^n\rangle\big)^4\Big] \\
&\le 8\mathbb{E}\Big[\sup_{0\le s\le t}|M_s^n(1)|^4\Big] + 8\langle 1,\mu_0^n\rangle^4 \\
&\le 8(4/3)^4\,\mathbb{E}\Big[\big(M_t^n(1)\big)^4\Big] + 8\langle 1,\mu_0^n\rangle^4 \\
&\le \kappa\Big(\lambda^3\sigma_n^6 t^3\langle 1,\mu_0^n\rangle + \lambda^2\sigma_n^4 t^2\langle 1,\mu_0^n\rangle^2 \\
&\qquad\qquad + \frac{\lambda m_c^n}{\theta^{2n}} t\langle 1,\mu_0^n\rangle + \langle 1,\mu_0^n\rangle^4\Big).
\end{aligned}$$

(iii) We first prove the tightness of $\{\mu^n\}$ in $\mathbb{D}([0,\infty), M_F(\hat{\mathbb{R}}))$, where $\hat{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$. Note that (ii) above implies the compact containment property for $\{\mu^n\}$. By Theorems 4.5.4 and 4.6.1 in Dawson [7], we then only need to prove that, for any given $\varepsilon > 0$, $\eta > 0$, $T > 0$, $\phi \in C_b^2(\mathbb{R})$ and

any stopping time τ_n bounded by T, there exist $\delta > 0$ and $n_0 \geq 1$ such that $\sup_{n\geq n_0} \sup_{t\in[0,\delta]} \mathbb{P}\{|\mu^n_{\tau_n+t}(\phi) - \mu^n_{\tau_n}(\phi)| > \varepsilon\} \leq \eta$.

For this, first observe that by (4.3),

$$\begin{aligned}
&\mathbb{P}(|\mu^n_{\tau_n+t}(\phi) - \mu^n_{\tau_n}(\phi)| > \varepsilon) \\
&\quad\leq \frac{1}{\varepsilon^2}\,\mathbb{E}\Big[\big(\mu^n_{\tau_n+t}(\phi) - \mu^n_{\tau_n}(\phi)\big)^2\Big] \\
&\quad\leq \frac{6}{\varepsilon^2}\mathbb{E}\Big[\frac{1}{\theta^n}\big(U^n_{\tau_n+t}(\phi) - U^n_{\tau_n}(\phi)\big)^2 + \frac{1}{\theta^n}\big(H^n_{\tau_n+t}(\phi) - H^n_{\tau_n+t}(\phi)\big)^2 \\
&\qquad\qquad + \big(X^n_{\tau_n+t}(\phi) - X^n_{\tau_n}(\phi)\big)^2 + \big(Y^n_{\tau_n+t}(\phi) - Y^n_{\tau_n}(\phi)\big)^2 \\
&\qquad\qquad + \frac{1}{\theta^n}\big(J^n_{\tau_n+t}(\phi) - J^n_{\tau_n+t}(\phi)\big)^2 + \big(M^n_{\tau_n+t}(\phi) - M^n_{\tau_n}(\phi)\big)^2\Big].
\end{aligned}$$

Note that by the independence of $\{B^\xi; \xi \in \Re\}$,

$$\begin{aligned}
\mathbb{E}\Big[\big(U^n_{\tau_n+t}(\phi) - U^n_{\tau_n}(\phi)\big)^2\Big] &= \frac{1}{\theta^n}\sum_{\xi\in\Re}\mathbb{E}\Big[\int_{\tau_n}^{\tau_n+t} h^\xi_s(\phi' c)^2(x^\xi_s)ds\Big] \\
&= \mathbb{E}\Big[\int_{\tau_n}^{\tau_n+t}\big\langle(\phi' c)^2, \mu^n_s\big\rangle ds\Big] \\
&\leq t\,\mathbb{E}\Big[\sup_{s\leq T+t}\big\langle(\phi' c)^2, \mu^n_s\big\rangle\Big] \\
&\leq t\,\|\phi' c\|^2_\infty\,\mathbb{E}\Big[\sup_{s\leq T+t}\langle 1, \mu^n_s\rangle\Big].
\end{aligned}$$

By the independence of $\{\widehat{S}^\xi; \xi \in \Re\}$ and the Lévy system formula (2.1),

$$\begin{aligned}
&\mathbb{E}\Big[\big(H^n_{\tau_n+t}(\phi) - H^n_{\tau_n}(\phi)\big)^2\Big] \\
&\quad= \frac{1}{\theta^n}\sum_{\xi\in\Re}\mathbb{E}\Big[\sum_{\tau_n<s\leq\tau_n+t} h^\xi_s(\phi' b)^2(x^\xi_{s-})(\triangle\widehat{S}_s)^2\Big] \\
&\quad= \frac{1}{\theta^n}\sum_{\xi\in\Re}\mathbb{E}\left[\int_{\tau_n}^{\tau_n+t} h^\xi_s(\phi' b)^2(x^\xi_s)\left(\int_{\{|w|\leq 1\}}\frac{c_\alpha|w|^2}{|w|^{1+\alpha}}dw\right)ds\right] \\
&\quad\leq \frac{c_\alpha}{(2-\alpha)}\mathbb{E}\Big[\int_{\tau_n}^{\tau_n+t}\big\langle(\phi' b)^2, \mu^n_s\big\rangle ds\Big] \\
&\quad\leq \frac{c_\alpha\, t}{(2-\alpha)}\,\mathbb{E}\Big[\sup_{s\leq T+t}\big\langle(\phi' b)^2, \mu^n_s\big\rangle\Big] \\
&\quad\leq \frac{c_\alpha\, t}{(2-\alpha)}\,\|\phi' b\|^2_\infty\,\mathbb{E}\Big[\sup_{s\leq T+t}\langle 1, \mu^n_s\rangle\Big],
\end{aligned}$$

while

$$\begin{aligned}
&\mathbb{E}\Big[\big(X^n_{\tau_n+t}(\phi)-X^n_{\tau_n}(\phi)\big)^2\Big]\\
&\quad=\mathbb{E}\Big[\int_{\tau_n}^{\tau_n+t}\int_{\mathbb{R}}\langle h(y-\cdot)\phi'(\cdot),\mu^n_s\rangle^2 dy ds\Big]\\
&\quad=\mathbb{E}\left[\int_{\tau_n}^{\tau_n+t}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\rho(w-z)\phi'(w)\phi'(z)\mu^n_s(dw)\mu^n_s(dz)\right)ds\right]\\
&\quad\le\|\rho\|_\infty\|\phi'\|^2_\infty\mathbb{E}\Big[\int_{\tau_n}^{\tau_n+t}\langle 1,\mu^n_s\rangle^2 ds\Big]\le t\,\|h\|_2^2\|\phi'\|^2\,\mathbb{E}\Big[\sup_{s\le T+t}\langle 1,\mu^n_s\rangle^2\Big].
\end{aligned}$$

In view of Lemma 3.1,

$$\begin{aligned}
&\mathbb{E}\Big[\big(Y^n_{\tau_n+t}(\phi)-Y^n_{\tau_n}(\phi)\big)^2\Big]\\
&\quad\le\left(\frac{1}{2}\|a\phi''\|_\infty+\||b|^\alpha\Delta^{\alpha/2}\phi\|_\infty\right)\mathbb{E}\Big[\int_{\tau_n}^{\tau_n+t}\langle 1,\mu^n_s\rangle^2 ds\Big]\\
&\quad\le t\,\left(\frac{1}{2}\|a\phi''\|_\infty+\||b|^\alpha\Delta^{\alpha/2}\phi\|_\infty\right)\mathbb{E}\Big[\sup_{s\le T+t}\langle 1,\mu^n_s\rangle^2\Big].
\end{aligned}$$

By the independence of $\{S^\xi;\xi\in\Re\}$, we have by a similar calculation as that for (3.2) that

$$\begin{aligned}
&\mathbb{E}\Big[\big(J^n_{\tau_n+t}(\phi)-J^n_{\tau_n}(\phi)\big)^2\Big]\\
&\quad=\frac{1}{\theta^n}\sum_{\xi\in\Re}\mathbb{E}\Big[\big(M^{\phi,\xi}_{\tau_n+t}-M^{\phi,\xi}_{\tau_n}\big)^2\Big]\\
&\quad\le\frac{1}{\theta^n}\sum_{\xi\in\Re}c\,\big(\|b^2\phi''\|_\infty+\|\phi\|_\infty\big)\,\mathbb{E}\Big[\int_{\tau_n}^{\tau_n+t}\mathbb{1}_{\{x^\xi_s\neq\partial\}}ds\Big]\\
&\quad=c\,\big(\|b^2\phi''\|_\infty+\|\phi\|_\infty\big)\,\mathbb{E}\Big[\int_{\tau_n}^{\tau_n+t}\langle 1,\mu^n_s\rangle ds\Big]\\
&\quad=\big(\|b^2\phi''\|_\infty+\|\phi\|_\infty\big)\,c\,t\,\mathbb{E}\Big[\sup_{s\le T+t}\langle 1,\mu^n_s\rangle\Big].
\end{aligned}$$

Finally, we have by part (i) of Lemma 4.2 that

$$\begin{aligned}
\mathbb{E}\Big[\big(M^n_{\tau_n+t}(\phi)-M^n_{\tau_n}(\phi)\big)^2\Big]&=\lambda\sigma_n^2\mathbb{E}\Big[\int_{\tau_n}^{\tau_n+t}\langle\phi^2,\mu^n_s\rangle ds\Big]\\
&\le\lambda\sigma_n^2\|\phi\|^2_\infty\,t\,\mathbb{E}\Big[\sup_{s\le T+t}\langle 1,\mu^n_s\rangle\Big].
\end{aligned}$$

Therefore by part (ii) of Lemma 4.2 and Lemma 3.4 of Wang [33], we conclude that for every $\varepsilon > 0$ there is a constant $c > 0$ such that

$$\sup_{n\geq 1} \sup_{t\in[0,\delta]} \mathbb{P}\left(|\mu^n_{\tau_n+t}(\phi) - \mu^n_{\tau_n}(\phi)| > \varepsilon\right) \leq c\delta \quad \text{for every } \delta > 0,$$

which proves (iii). To complete the proof of the lemma, it remains to prove that the limit process does not hit the cemetery. This follows from the same arguments as those in the proof of Theorem 4.1 in Dawson et al. [10]. □

Let $\mathcal{S}(\mathbb{R})$ be the space of Schwartz test functions, i.e., the space of infinitely differentiable functions which, together with all their derivatives, are rapidly decreasing at infinity. Let $\mathcal{S}'(\mathbb{R})$ denote the Schwartz space of tempered distributions, the dual space of $\mathcal{S}(\mathbb{R})$.

Theorem 4.3. *Based on the above notations, we have following conclusions:*

(i) $(\mu^n, U^n, Y^n, M^n, H^n, J^n)$ *is tight on*

$$\mathbb{D}([0,\infty), M_F(\mathbb{R})) \times \mathbb{D}([0,\infty), (\mathcal{S}'(\mathbb{R}))^5).$$

(ii) (A Skorohod representation) *Suppose that the joint distribution of*

$$(\mu^{n_m}, U^{n_m}, Y^{n_m}, M^{n_m}, H^{n_m}, J^{n_m}, W)$$

converges weakly to the joint distribution of

$$(\mu^0, U^0, Y^0, M^0, H^0, J^0, W).$$

Then there exist a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ *and sequences*

$$(\widetilde{\mu}^{n_m}, \widetilde{U}^{n_m}, \widetilde{Y}^{n_m}, \widetilde{M}^{n_m}, \widetilde{H}^{n_m}, \widetilde{J}^{n_m}, \widetilde{W}^{n_m})$$

defined on it with values in $\mathbb{D}([0,\infty), M_F(\mathbb{R})) \times \mathbb{D}([0,\infty), (\mathcal{S}'(\mathbb{R}))^5)$ *such that*

$$\begin{aligned}\mathbb{P} \circ (\mu^{n_m}, U^{n_m}, Y^{n_m}, M^{n_m}, H^{n_m}, J^{n_m}, W)^{-1} \\ = \widetilde{\mathbb{P}} \circ (\widetilde{\mu}^{n_m}, \widetilde{U}^{n_m}, \widetilde{Y}^{n_m}, \widetilde{M}^{n_m}, \widetilde{H}^{n_m}, \widetilde{J}^{n_m}, \widetilde{W}^{n_m})^{-1}\end{aligned}$$

holds and, $\widetilde{\mathbb{P}}$*-almost surely on* $\mathbb{D}([0,\infty), M_F(\mathbb{R})) \times \mathbb{D}([0,\infty), (\mathcal{S}'(\mathbb{R}))^6)$,

$$\begin{aligned}&\left(\widetilde{\mu}^{n_m}, \widetilde{U}^{n_m}, \widetilde{Y}^{n_m}, \widetilde{M}^{n_m}, \widetilde{H}^{n_m}, \widetilde{J}^{n_m}, \widetilde{W}^{n_m}\right) \\ &\qquad \to \left(\widetilde{\mu}^0, \widetilde{U}^0, \widetilde{Y}^0, \widetilde{M}^0, \widetilde{H}^0, \widetilde{J}^0, \widetilde{W}^0\right)\end{aligned}$$

as $m \to \infty$.

(iii) *There exists a dense subset $D \subset [0,\infty)$ such that $[0,\infty) \setminus D$ is at most countable and for each $t \in D$ and each $\phi \in \mathcal{S}(\mathbb{R})$, as an $\mathbb{R}^7$-valued process*

$$\begin{aligned}(\widetilde{\mu}_t^{n_m}(\phi), \widetilde{U}_t^{n_m}(\phi), \widetilde{Y}_t^{n_m}(\phi), \widetilde{M}_t^{n_m}(\phi), \widetilde{H}_t^{n_m}(\phi), \widetilde{J}_t^{n_m}(\phi), \widetilde{W}_t^{n_m}(\phi))\\ \to (\widetilde{\mu}_t^{0}(\phi), \widetilde{U}_t^{0}(\phi), \widetilde{Y}_t^{0}(\phi), \widetilde{M}_t^{0}(\phi), \widetilde{H}_t^{0}(\phi), \widetilde{J}_t^{0}(\phi), \widetilde{W}_t^{0}(\phi))\end{aligned}$$

in $L^2(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ as $m \to \infty$. Furthermore, let $\widetilde{\mathcal{F}}_t^0$ be the σ-algebra generated by $\widetilde{\mu}_s^0(\phi)$, $\widetilde{U}_s^0(\phi)$, $\widetilde{Y}_s^0(\phi)$, $\widetilde{M}_s^0(\phi)$, $\widetilde{H}_s^0(\phi)$, $\widetilde{J}_s^0(\phi)$, $\widetilde{W}_s^0(\phi)$ with ϕ running through $\mathcal{S}(\mathbb{R})$ and $s \le t$. Then $\widetilde{M}_t^0(\phi)$ is a continuous, square-integrable $\widetilde{\mathcal{F}}_t^0$-martingale with quadratic variation process

$$\langle \widetilde{M}_t^0(\phi)\rangle = \lambda\sigma^2 \int_0^t \langle \phi^2, \widetilde{\mu}_u^0\rangle du.$$

(iv) *$\widetilde{W}^0(dy, ds)$ and $\widetilde{W}^{n_m}(dy, ds)$ are Brownian sheets. The continuous square-integrable martingale*

$$\widetilde{X}_t^{n_m}(\phi) := \int_0^t \int_{\mathbb{R}} \left\langle h(y-\cdot)\phi'(\cdot), \widetilde{\mu}_s^{n_m}\right\rangle \widetilde{W}^{n_m}(dy, ds)$$

converges to

$$\widetilde{X}_t^{0}(\phi) := \int_0^t \int_{\mathbb{R}} \left\langle h(y-\cdot)\phi'(\cdot), \widetilde{\mu}_s^{0}\right\rangle \widetilde{W}^{0}(dy, ds)$$

in $L^2(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$.

(v) *$\widetilde{\mu}^0 = \{\widetilde{\mu}_t^0; t \ge 0\}$ is a solution to the $(\mathcal{L}, \delta_{\mu_0})$-martingale problem and $\widetilde{\mu}^0$ satisfies*

$$\begin{aligned}\widetilde{\mu}_t^0(\phi) - \widetilde{\mu}_0^0(\phi) = \ & \widetilde{X}_t^0(\phi) + \int_0^t \int_{\mathbb{R}} \phi(x) \widetilde{M}^0(dx, ds)\\ & + \int_0^t \left\langle \tfrac{1}{2} a\phi'' + |b|^\alpha \Delta^{\alpha/2}\phi, \widetilde{\mu}_s^0\right\rangle ds.\end{aligned} \tag{4.6}$$

Proof. (i) The tightness of $\{mu^n; n \ge 1\}$ in $\mathbb{D}([0,\infty), M_F(\mathbb{R}))$ has been established in Lemma 4.2 (iii). So by a theorem of Mitoma [27], we only need to prove that for any $\phi \in \mathcal{S}(\mathbb{R})$ the sequence of laws of

$$(U^n(\phi), Y^n(\phi), M^n(\phi), H^n(\phi), J^n(\phi))$$

is tight in $\mathbb{D}([0,\infty), \mathbb{R}^5)$. This is equivalent to proving that each component and the sum of each pair of components are individually tight in $\mathbb{D}([0,\infty), \mathbb{R})$. Since

the same idea works for each sequence, we only give the proof for $\{M^n(\phi)\}$. By Lemma 4.2, we have

$$\mathbb{P}\left(M_t^n(\phi) > k\right) \le \frac{\lambda\sigma_n^2}{k^2}\mathbb{E}\left[\int_0^t \langle \phi^2, \mu_u^n\rangle du\right] \le \frac{\lambda\sigma_n^2\|\phi\|_\infty^2 t}{k^2}\langle 1, \mu_0^n\rangle,$$

which yields the compact containment condition. Now we use Kurtz's tightness criterion (cf. Ethier–Kurtz [17, Theorem 8.6 on p. 137]) to prove the tightness of $\{M^n(\phi)\}$.

Let $\lambda_n^T(\delta) := \delta\lambda\sigma_n^2\|\phi\|_\infty^2 \sup_{0\le u\le T}\langle 1, \mu_u^n\rangle$. Then for any $0 \le t+\delta \le T$,

$$\begin{aligned}\mathbb{E}\big[|M_{t+\delta}^n(\phi) - M_t^n(\phi)|^2 \mid \mathcal{F}_t^n\big] &= \mathbb{E}\left[\lambda\sigma_n^2\int_t^{t+\delta}\langle\phi^2, \mu_u^n\rangle du \,\middle|\, \mathcal{F}_t^n\right]\\ &\le \mathbb{E}\big[\lambda_n^T(\delta) \mid \mathcal{F}_t^n\big].\end{aligned}$$

By Lemma 4.2, $\lim_{\delta\to 0}\sup_n \mathbb{E}[\lambda_n^T(\delta)] = 0$ holds, so $\{M^n(\phi); n \ge 1\}$ is tight.

(ii) If we choose any countable dense subset $\{g_i\}_{i\in\mathbb{N}}$ of $\mathcal{S}(\mathbb{R})$ and any enumeration $\{t_j\}_{j\in\mathbb{N}}$ of all rational numbers, then Theorem 1.7 of Jakubowski [21] shows that the countable family $\{f_{ij}; i, j \in \mathbb{N}\}$ are continuous functions (with respect to Skorohod topology on $\mathbb{D}([0,\infty), \mathcal{S}'(\mathbb{R}))$) and separate points, where

$$f_{ij} : x \in \mathbb{D}([0,\infty), \mathcal{S}'(\mathbb{R})) \to f_{ij}(x) := \arctan\langle g_i, x(t_j)\rangle \in [-\pi, \pi].$$

This proves that the space $\mathbb{D}([0,\infty), \mathcal{S}'(\mathbb{R}))$, thus the space $\mathbb{D}([0,\infty), (\mathcal{S}'(\mathbb{R}))^7)$ as well, satisfies the basic assumption for a version of the Skorohod Representation Theorem due to Jakubowski [22].

(iii) For each $t \in D$ and each $\phi \in \mathcal{S}(\mathbb{R})$, from Lemma 4.2 we obtain the uniform integrability of

$$\left\{\widetilde{\mu}_t^{n_m}(\phi)^2,\ \widetilde{U}_t^{n_m}(\phi)^2,\ \widetilde{Y}_t^{n_m}(\phi)^2,\ \widetilde{M}_t^{n_m}(\phi)^2,\ \widetilde{H}_t^{n_m}(\phi)^2,\ \widetilde{J}_t^{n_m}(\phi)^2,\ \widetilde{W}_t^{n_m}(\phi)^2\right\}.$$

So (ii) implies their convergence in $L^2(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ as $m \to \infty$. For each $\phi \in \mathcal{S}(\mathbb{R})$, $G_i \in C_b(\mathbb{R}^7)$ and any $0 < t_1 \le \cdots \le t_n = s < t$ with $t_i, t \in D$, $i = 1, \ldots, n$, let

$$\begin{aligned}f^{n_m}(t_1, \ldots, t_n) := \prod_{i=1}^n G_i\Big(&\widetilde{\mu}_{t_i}^{n_m}(\phi), \widetilde{U}_{t_i}^{n_m}(\phi), \widetilde{Y}_{t_i}^{n_m}(\phi), \widetilde{M}_{t_i}^{n_m}(\phi),\\ &\widetilde{H}_{t_i}^{n_m}(\phi), \widetilde{J}_{t_i}^{n_m}(\phi), \widetilde{W}_{t_i}^{n_m}(\phi)\Big).\end{aligned}$$

Then we have

$$\widetilde{\mathbb{E}}\left[(\widetilde{M}_t^{n_m}(\phi) - \widetilde{M}_s^{n_m}(\phi))f^{n_m}(t_1, \ldots, t_n)\right] = 0 \tag{4.7}$$

and

$$\widetilde{\mathbb{E}}\Big[\Big(\widetilde{M}_t^{n_m}(\phi)^2 - \lambda\sigma_{n_m}^2\int_0^t\langle\phi^2,\widetilde{\mu}_u^{n_m}\rangle du - \widetilde{M}_s^{n_m}(\phi)^2 + \lambda\sigma_{n_m}^2\int_0^s\langle\phi^2,\widetilde{\mu}_u^{n_m}\rangle du\Big)f^{n_m}(t_1,\dots,t_n)\Big] = 0. \tag{4.8}$$

By the above convergence in $L^2(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$, this implies that the processes $\widetilde{M}_t^0$ and $\widetilde{M}_t^0(\phi)^2 - \lambda\sigma^2\int_0^t\langle\phi^2,\widetilde{\mu}_u^0\rangle du$ are $\widetilde{\mathcal{F}}_t^0$-martingales. Let $K = \sup_{x\in\mathbb{R}}\phi^2(x)$. Based on (4.5), we can get

$$\begin{aligned}
&\widetilde{\mathbb{E}}\big[(\widetilde{M}_t^n(\phi) - \widetilde{M}_s^n(\phi))^4\big]\\
&\quad= \mathbb{E}\big[(M_t^n(\phi) - M_s^n(\phi))^4\big]\\
&\quad= \mathbb{E}\Big[\sum_{\xi,\eta\in\mathfrak{R},\xi\neq\eta}\Big(\frac{(D^\xi-1)^2}{\theta^{2n}}\phi^2(x^\xi_{\zeta(\xi)-})\mathbb{1}_{\{s<\zeta(\xi)\le t\}}\frac{(D^\eta-1)^2}{\theta^{2n}}\\
&\qquad\qquad\times\phi^2(x^\eta_{\zeta(\eta)-})\mathbb{1}_{\{s<\zeta(\eta)\le t\}}\Big)\Big]\\
&\qquad\quad+\mathbb{E}\Big[\sum_{\xi\in\mathfrak{R}}\Big(\frac{(D^\xi-1)^4}{\theta^{4n}}\phi^4(x^\xi_{\zeta(\xi)-})\mathbb{1}_{\{s<\zeta(\xi)\le t\}}\Big)\Big]\\
&\quad\le K^2\mathbb{E}\Big[\sum_{\xi,\eta\in\mathfrak{R},\xi\neq\eta}\Big(\frac{(D^\xi-1)^2}{\theta^{2n}}\mathbb{1}_{\{s<\zeta(\xi)\le t\}}\frac{(D^\eta-1)^2}{\theta^{2n}}\mathbb{1}_{\{s<\zeta(\eta)\le t\}}\Big)\Big]\\
&\qquad\quad+K^2\mathbb{E}\Big[\sum_{\xi\in\mathfrak{R}}\Big(\frac{(D^\xi-1)^4}{\theta^{4n}}\mathbb{1}_{\{s<\zeta(\xi)\le t\}}\Big)\Big]\\
&\quad\le K^2\Big(8\lambda^3\sigma_n^6(t-s)^3\langle 1,\mu_0^n\rangle + 2\lambda^2\sigma_n^4(t-s)^2\langle 1,\mu_0^n\rangle^2\\
&\qquad\quad+\frac{\lambda m_c^n}{\theta^{2n}}(t-s)\langle 1,\mu_0^n\rangle\Big).
\end{aligned}$$

In particular, for any $m\ge 1$ we have

$$\begin{aligned}
\widetilde{\mathbb{E}}\left[(\widetilde{M}_t^{n_m}(\phi) - \widetilde{M}_s^{n_m}(\phi))^4\right] \le K^2\Big(&8\lambda^3\sigma_{n_m}^6(t-s)^3\langle 1,\mu_0^{n_m}\rangle\\
&+2\lambda^2\sigma_{n_m}^4(t-s)^2\langle 1,\mu_0^{n_m}\rangle^2\\
&+\frac{\lambda m_c^{n_m}}{\theta^{2n_m}}(t-s)\langle 1,\mu_0^{n_m}\rangle\Big).
\end{aligned} \tag{4.9}$$

Letting $m \to \infty$, we get

$$\widetilde{\mathbb{E}}\left[(\widetilde{M}_t^0(\phi) - \widetilde{M}_s^0(\phi))^4\right] \le K^2\Big(8\lambda^3\sigma^6(t-s)^3\langle 1, \mu_0\rangle + 2\lambda^2\sigma^4(t-s)^2\langle 1, \mu_0\rangle^2\Big). \tag{4.10}$$

Thus $\widetilde{M}_t^0$ has a continuous modification according to Kolmogorov's continuity criterion and

$$\langle \widetilde{M}_t^0(\phi)\rangle = \lambda\sigma^2 \int_0^t \langle \phi^2, \widetilde{\mu}_u^0\rangle du.$$

(iv) Since W, $\widetilde{W}^0$, $\widetilde{W}^{n_m}$ have the same distribution, $\widetilde{W}^0$ and $\widetilde{W}^{n_m}$ are Brownian sheets. The conclusion follows from (ii) and Theorem 2.1 of Cho [5].

(v) Since $\widetilde{U}_t^{n_m}(\phi)^2$, $\widetilde{H}_t^{n_m}(\phi)^2$ and $\widetilde{J}_t^{n_m}(\phi)^2$ are uniformly integrable, we have $\widetilde{\mathbb{P}}$-a.s. and in $L^2(\widetilde{\mathbb{P}})$

$$\lim_{m\to\infty} \frac{1}{\sqrt{\theta^{n_m}}}\widetilde{U}_t^{n_m}(\phi) = \lim_{m\to\infty} \frac{1}{\sqrt{\theta^{n_m}}}\widetilde{H}_t^{n_m}(\phi) = \lim_{m\to\infty} \frac{1}{\sqrt{\theta^{n_m}}}\widetilde{J}_t^{n_m}(\phi) = 0.$$

By passing $n \to \infty$ along the subsequence $\{n_m; m \ge 1\}$ in (4.3), we have

$$\widetilde{\mu}_t^0(\phi) - \widetilde{\mu}_0^0(\phi) = \widetilde{X}_t^0(\phi) + \widetilde{Y}_t^0(\phi) + \widetilde{M}_t^0(\phi) \quad \text{for every } \phi \in \mathcal{S}(\mathbb{R}) \text{ and } t \ge 0.$$

As

$$\widetilde{Y}_t^n(\phi) = \int_0^t \left\langle \tfrac{1}{2}a\phi'' + |b|^\alpha \Delta^{\alpha/2}\phi, \widetilde{\mu}_s^n\right\rangle ds$$

and

$$\widetilde{M}_t^n(\phi) = \int_0^t\int_{\mathbb{R}} \phi(x)\widetilde{M}^n(dx, dy),$$

we see from (ii) above that

$$\widetilde{Y}_t^0(\phi) = \int_0^t \left\langle \tfrac{1}{2}a\phi'' + |b|^\alpha \Delta^{\alpha/2}\phi, \widetilde{\mu}_s^0\right\rangle ds$$

and

$$\widetilde{M}_t^0(\phi) = \int_0^t\int_{\mathbb{R}} \phi(x)\widetilde{M}^0(dx, dy).$$

So, $\widetilde{\mu}_t^0$ satisfies (4.6).

By Itô's formula, we see that $\{\widetilde{\mu}_t^0; t \ge 0\}$ is a solution to the martingale problem for $(\mathcal{L}, \delta_{\mu_0})$ with sample pathes in $\mathbb{D}([0,\infty), M_F(\mathbb{R}))$. □

Theorem 4.3 (v) tells us that $\widetilde{\mu}_0 = \{\widetilde{\mu}_t^0; t \ge 0\}$ is a solution to the martingale problem for $(\mathcal{L}, \delta_{\mu_0})$. For uniqueness, we will use a duality argument due to Dawson–Kurtz [9]. For $f \in \bigcup_{m=1}^\infty C_b(\mathbb{R}^m)$, we write $N(f)$ for m if $f \in C_b(\mathbb{R}^m)$ and define

$$F_\mu(f) := F_f(\mu) := \int_{\mathbb{R}^m} f(x_1, \dots, x_m)\mu(dx_1)\dots\mu(dx_m) \quad \text{for } \mu \in M_F(\mathbb{R}).$$

Such a function F_f is called a monomial function on space $M_F(\mathbb{R})$. Note that for such a monomial function F_f,

$$\frac{\partial F_f(\mu)}{\partial \mu(x)} = \sum_{j=1}^{N(f)} \int_{\mathbb{R}^{N(f)-1}} f(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_{N(f)}) \prod_{l=1, l\neq j}^{N(f)} \mu(dx_l)$$

and

$$\frac{\partial^2 F_f(\mu)}{\partial \mu(y)\partial \mu(x)} = \sum_{j,k=1,\, j\neq k}^{N(f)} \int_{\mathbb{R}^{N(f)-2}} f(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_{k-1}, y, \\ x_{k+1}, \ldots, x_{N(f)}) \prod_{l=1, l\neq j,k}^{N(f)} \mu(dx_l).$$

For $f \in C_b^2(\mathbb{R}^m)$, $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, we define

$$\begin{aligned} \mathcal{L}^{(m)} f(x) := & \sum_{k=1}^{m} \left(\tfrac{1}{2} a(x_k) \frac{\partial^2}{\partial x_k^2} + |b(x_k)|^\alpha \Delta_{x_k}^{\alpha/2} \right) f(x) \\ & + \frac{1}{2} \sum_{j,k=1,\, j\neq k}^{m} \rho(x_k - x_j) \frac{\partial^2}{\partial x_k \partial x_j} f(x) \qquad (4.11) \\ = & \sum_{j,k=1}^{m} \Gamma_{jk} \frac{\partial^2}{\partial x_k \partial x_j} f(x) + \left(\sum_{k=1}^{m} |b(x_k)|^\alpha \Delta_{x_k}^{\alpha/2} \right) f(x), \end{aligned}$$

where Γ_{jk} is defined (1.9). Then by (1.2), we have for monomial function F_f on $M_F(\mathbb{R})$ with $N(f) = m$,

$$\begin{aligned} \mathcal{L} F_f(\mu) &= \mathcal{A} F_f(\mu) + \mathcal{B} F_f(\mu) \\ &= F_{\mathcal{L}^{(m)} f}(\mu) + \frac{\lambda\sigma^2}{2} \sum_{j,k=1,\, j\neq k}^{m} F_{\Phi_{jk} f}(\mu) \\ &= F_{\mathcal{L}^{(m)} f}(\mu) + \frac{\lambda\sigma^2}{2} \sum_{j,k=1,\, j\neq k}^{m} \big(F_{\Phi_{jk} f}(\mu) - F_f(\mu) \big) \\ &\quad + \frac{\lambda\sigma^2}{2} m(m-1) F_f(\mu) \\ &= F_\mu(\mathcal{L}^{(m)} f) + \frac{\lambda\sigma^2}{2} \sum_{j,k=1,\, j\neq k}^{m} \big(F_\mu(\Phi_{jk} f) - F_\mu(f) \big) \\ &\quad + \frac{\lambda\sigma^2}{2} m(m-1) F_\mu(f) \\ &=: \mathcal{L}^* F_\mu(f) + \frac{1}{2} \lambda\sigma^2 m(m-1) F_\mu(f). \end{aligned}$$

Here for $j < k$, $\Phi_{jk} f$ is a function on $\mathbb{R}^{m-1}$ defined by

$$\Phi_{jk} f(y) := \Phi_{kj}(y) := f(y_1, \dots, y_j, \dots, y_{k-1}, y_j, y_k, \dots, y_{m-1})$$

for $y = (y_1, \dots, y_{m-1}) \in \mathbb{R}^{m-1}$.

In order to use Dawson–Kurtz's duality method, we establish $C^{2+\gamma}$-regularity on the transition semigroup of $\mathcal{L}^{(m)}$ for some $\gamma \in (0,1)$. Following [23], we introduce the Hölder spaces on $\mathbb{R}^m$ and $[0,T] \times \mathbb{R}^m$ as follows. Given $\gamma \in (0,1)$, define for $f \in C(\mathbb{R}^m)$

$$[f]_\gamma := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\gamma}$$

and for $\phi \in C([0,T] \times \mathbb{R}^m)$

$$\lfloor \phi \rfloor_{(s),\gamma;T} := \sup_{s,t \in [0,T],\, x \in \mathbb{R}^m} \frac{|\phi(s,x) - \phi(t,x)|}{|s-t|^\gamma},$$

$$\lfloor \phi \rfloor_{(x),\gamma;T} := \sup_{s \in [0,T],\, x,y \in \mathbb{R}^m} \frac{|\phi(s,x) - \phi(s,y)|}{|x-y|^\gamma}.$$

For $k = (k_1, \dots, k_m)$ with non-negative integer components, let $|k| := \sum_{i=1}^m k_i$ and $\partial^k := \frac{\partial^{|k|}}{\partial^{k_1} x_1 \dots \partial^{k_m} x_m}$. For $j = 0,1,2$ and $\gamma \in (0,1)$, let $C_b^{j+\gamma}(\mathbb{R}^m)$ denote the Banach space with the norm

$$\|f\|_{j+\gamma} := \sum_{|k| \le j} \sup_{(s,x) \in [0,T] \times \mathbb{R}^m} |\partial^k f(s,x)| + \sum_{|k|=j} [\partial^k f]_\gamma,$$

and $C_b^{j+(\gamma/2),2j+\gamma}([0,T] \times \mathbb{R}^m)$ the Banach space with the norm

$$\begin{aligned}\|\phi\|_{2j+\gamma,T} := &\sum_{2l+|k| \le 2j} \sup_{(s,x) \in [0,T \times \mathbb{R}^m} |\partial_s^l \partial_x^k \phi(s,x)| \\ &+ \sum_{0 < 2j+\gamma-2l-|k| < 2} \lfloor \partial_s^l \partial_x^k \phi \rfloor_{(s),(2j+\gamma-2l-|k|)/2;T} \\ &+ \sum_{2l+|k|=2j} \lfloor \partial_s^l \partial_x^k \phi \rfloor_{(x),\gamma;T}.\end{aligned}$$

We also define the Banach spaces $C_b^{0,2}([0,T] \times \mathbb{R}^m)$ and $C_b^{1,2}([0,T] \times \mathbb{R}^m)$ of bounded $C^{0,2}$- and $C^{1,2}$-smooth functions on $[0,T] \times \mathbb{R}^m$, respectively, with norms

$$\|\phi\|_{0,2;T} := \sum_{|k| \le 2} \sup_{(s,x) \in [0,T] \times \mathbb{R}^m} |\partial_x^k \phi(s,x)|$$

and

$$\|\phi\|_{1,2;T} := \sum_{2l+|k|\le 2} \sup_{(s,x)\in[0,T]\times\mathbb{R}^m} |\partial_s^l \partial_x^k \phi(s,x)|.$$

For any given integer $m \ge 1$, let $Z_t^{(m)} = (z_t^{\mathbb{1}}, \dots, z_t^m)$ be the process of m particles given by (3.1). For the next theorem, the law of $Z^{(m)}$ with initial starting point from $x \in \mathbb{R}^m$ will be denoted as $\mathbb{P}_x$, and the mathematical expectation under $\mathbb{P}_x$ will be denoted as $\mathbb{E}_x$.

Theorem 4.4. *Assume that the diffusion matrix* $(\Gamma_{ij})_{1\le i,j\le m}$ *defined by* (1.9) *is uniformly elliptic and bounded. Assume also that each* Γ_{ij} *and* $|b|^\alpha$ *are bounded and* γ*-Hölder continuous for some* $\gamma \in (0, 1 \wedge (2-\alpha))$. *Let* $\{P_t^m; t \ge 0\}$ *be the transition semigroup for* $Z^{(m)}$, *that is,*

$$P_t^m f(x) := \mathbb{E}_x\left[f(Z_t^{(m)})\right] \quad \textit{for } t \ge 0 \textit{ and } f \in \mathcal{B}_b(\mathbb{R}^m).$$

Then for every $f \in C_b^{2+\gamma}(\mathbb{R}^m)$, $P_t^m f(x)$ *as a function of* (t,x) *is* $C_b^{1+(\gamma/2),\,2+\gamma}$ *on* $[0,T]\times\mathbb{R}^m$ *for every* $T > 0$. *In particular,* $C_b^{2+\gamma}(\mathbb{R}^m)$ *is invariant under* P_t^m *for every* $t > 0$ *and* $m \ge 1$.

The above theorem gives sufficient regularity for the semigroup $P_t^{(m)}$ needed to apply Dawson–Kurtz's duality method for the well-posedness of the martingale problem. To keep the flow of the argument for the uniqueness of the martingale problem, we postpone its proof into the next section.

Fix an arbitrary $\gamma \in (0, 1 \wedge \alpha \wedge (2-\alpha))$. Let $\mathcal{S} = \bigcup_{k=0}^\infty C_b^{2+\gamma}(\mathbb{R}^k)$ (disjoint union) with $C_b^{2+\gamma}(\mathbb{R}^0) := \mathbb{R}$. We see from the proof of Theorem 4.4 that $\mathcal{L}^{(m)}$ coincides on $C_b^{2+\gamma}(\mathbb{R}^m)$ with the infinitesimal generator of the strong Markov process $Z^{(m)}$ for the motion m particles given by (3.1). Thus $\mathcal{L}^*$ has the structure of the infinitesimal generator of an $\mathcal{S}$-valued strong Markov process Y, whose dynamics involve two mechanisms as follows.

(a) Jump mechanism: Let $\{M_t; t \ge 0\}$ be a non-negative integer-valued cádlág Markov process with $M_0 = 0$ and transition intensities $\{q_{i,j}\}$ such that

$$q_{i,i-1} = -q_{i,i} = \frac{\lambda\sigma^2}{2} i(i-1) \quad \text{and} \quad q_{i,j} = 0 \quad \text{for all other pairs } (i,j).$$

Thus $\{M_t; t \ge 0\}$ is just the well-known Kingman's coalescent process. Let $\tau_0 = 0$, $\tau_{M_0+1} = \infty$ and $\{\tau_k; 1 \le k \le M_0\}$ be the sequence of jump times of $\{M_t; t \ge 0\}$. Let $\{S_k; 1 \le k \le M_0\}$ be a sequence of random operators which are conditionally independent given $\{M_t; t \ge 0\}$ and satisfy

$$\mathbb{P}\{S_k = \Phi_{i,j} \mid M(\tau_k-) = l\} = \frac{1}{l(l-1)}, \quad 1 \le i \ne j \le l.$$

(b) Spatial jump-diffusion semigroup: Let $\boldsymbol{B}$ denote the topological union of $\{C_b(\mathbb{R}^m); m = 1, 2, \dots\}$ endowed with supremum norm on each $C_b(\mathbb{R}^m)$. Then

$$Y_t = P_{t-\tau_k}^{M_{\tau_k}} S_k P_{\tau_k-\tau_{k-1}}^{M_{\tau_{k-1}}} S_{k-1} \dots P_{\tau_2-\tau_1}^{M_{\tau_1}} S_1 P_{\tau_1}^{M_0} Y_0,$$
$$\tau_k \le t < \tau_{k+1}, \quad 0 \le k \le M_0,$$

defines a Markov process $Y := \{Y_t; t \ge 0\}$. By Theorem 4.4, the process Y takes values in $\mathcal{S} \subset \boldsymbol{B}$. Clearly, $\{(M_t, Y_t); t \ge 0\}$ is also a Markov process.

The duality relationship can be described as follows. Take any $\gamma \in (0, 1)$. Let us denote by $\mathcal{D}(\mathcal{L})$ the set of all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ with $f \in C_b^{2+\gamma}(\mathbb{R}^m)$. If $\{X_t; t \ge 0\}$ is a solution of the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem with $X_0 = \mu_0$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then, by the Feynman–Kac formula (see [9]), we have

$$\mathbb{E}\left[\langle f, X_t^m \rangle\right] = \mathbb{E}_{m,f}\left[\langle Y_t, \mu_0^{M_t} \rangle \exp\left(\frac{\lambda\sigma^2}{2}\int_0^t M_s(M_s - 1)ds\right)\right] \tag{4.12}$$

for any $t \ge 0$, $f \in C_b^{2+\gamma}(\mathbb{R}^m)$ and integer $m \ge 1$, where the right hand side is the expectation taking on the probability space for which the dual process is defined with giving $M_0 = m$ and $Y_0 = f \in C_b^{2+\gamma}(\mathbb{R}^m)$. From this, we see that the marginal distribution of X is uniquely determined and hence the law of X is unique (see, e.g., [17, Theorem 4.4.2] or [9, Theorem 2.1]). This proves the uniqueness of the martingale problem for $\mathcal{L}$.

We summarize these results in the following theorem.

Theorem 4.5. *Assume that $b, c \in \mathrm{Lip}_b(\mathbb{R})$ and $h \in L^2(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$. For the uniqueness for the martingale problem below, assume further that c is bounded from below by a strictly positive constant and $h \in L^1(\mathbb{R})$. For any $\mu \in M_F(\mathbb{R})$, the $(\mathcal{L}, \delta_\mu)$-martingale problem has a unique solution μ_t with sample pathes in $C([0, \infty), M_F(\mathbb{R}))$, which is a diffusion process and satisfies*

$$\begin{aligned}\mu_t(\phi) - \mu_0(\phi) = X_t(\phi) &+ \int_0^t\int_{\mathbb{R}} \phi(x)M(dx, ds) \\ &+ \int_0^t \langle \tfrac{1}{2}a\phi'' + |b|^\alpha \Delta^{\alpha/2}\phi, \mu_s \rangle ds\end{aligned} \tag{4.13}$$

for every $t > 0$ and $\phi \in C_b^2(\mathbb{R})$, where W is a Brownian sheet,

$$X_t(\phi) := \int_0^t\int_{\mathbb{R}} \langle h(y - \cdot)\phi'(\cdot), \mu_s \rangle W(dy, ds), \quad t \ge 0,$$

and M is a square-integrable martingale measure with

$$\langle M(\phi)\rangle_t = \lambda\sigma^2 \int_0^t \langle \phi^2, \mu_u\rangle du \quad \textit{for every } t > 0 \textit{ and } \phi \in C_b^2(\mathbb{R}).$$

Here

$$M_t(\phi) := \int_0^t \int_{\mathbb{R}} \phi(y) M(ds, dy)$$

is a square-integrable, continuous $\{\mathcal{F}_t\}$-martingale, where

$$\mathcal{F}_t := \sigma\{\mu_s(f), M_s(f), X_s(f), f \in \mathcal{B}_b(\mathbb{R}), s \le t\}.$$

Moreover $X_t(\phi)$, $M_t(\phi)$ are orthogonal square-integrable $\{\mathcal{F}_t\}$-martingales for every $\phi \in \mathcal{S}(\mathbb{R})$.

Proof. According to previous results, we only need to prove the continuity of μ_t as a process taking values in $M_F(\mathbb{R})$. Hence we only need to show that for any bounded continuous function f on $\mathbb{R}$, $\langle \phi, \mu_t\rangle$ is continuous in $t \ge 0$. However, this is just a simple application of Bakry–Emery's result (see [1, Proposition 2] and the last section in Wang [33]). □

5 $C^{2+\gamma}$-regularity for the pseudo-differential operator $\mathcal{L}^{(m)}$

In this section, we give the proof of the $C^{2+\gamma}$-regularity for the semigroup of the pseudo-differential operator $\mathcal{L}^{(m)}$ of (4.11).

Proof of Theorem 4.4. For $1 \le j \le m$, let e_j denote the unit vector in the positive x_j direction. Note that for $f \in C_b^2(\mathbb{R}^m)$ the infinitesimal generator $\mathcal{L}^{(m)}$ given by (1.8) can be written as

$$\mathcal{L}^{(m)} = \mathcal{G}_m + \mathcal{J}_m,$$

where

$$\mathcal{G}_m f(x) := \frac{1}{2} \sum_{i,j=1}^m \Gamma_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

and

$$\mathcal{J}_m f(x) = c_\alpha \sum_{j=1}^m |b(x_j)|^\alpha \int_{\mathbb{R}\setminus\{0\}} \Big(f(x + we_j) - f(x) - \frac{\partial}{\partial x_j} f(x) w \mathbb{1}_{\{|w|\le 1\}} \Big) \frac{1}{|w|^{1+\alpha}} dw.$$

Since Γ_{ij} is uniformly elliptic and Hölder continuous, it is known from Theorem 5.1 in [23, p. 320] that for every $T > 0$, every $f \in C^{2+\gamma}(\mathbb{R}^m)$ and every $\phi \in C_b^{(\gamma/2),\gamma}([0,T] \times \mathbb{R}^m)$, the differential equation

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} = \mathcal{G}_m v(t,x) + \phi(t,x) & \text{for } (t,x) \in (0,1] \times \mathbb{R}^m, \\ v(0,x) = f(x) & \text{for} \quad x \in \mathbb{R}^m, \end{cases} \tag{5.1}$$

has a unique solution $v \in C_b^{1+(\gamma/2),2+\gamma}([0,T] \times \mathbb{R}^m)$. Moreover, there is a constant $c_0(T) > 0$ such that

$$\|v\|_{2+\gamma,T} \le c_0(T) \left(\|f\|_{2+\gamma} + \|\phi\|_{\gamma,T} \right). \tag{5.2}$$

Let $\{\eta_s; s \ge 0, \mathbb{P}_x^0, x \in \mathbb{R}^m\}$ be the diffusion on $\mathbb{R}^m$ with infinitesimal generator $\mathcal{G}_m$, whose transition semigroup will be denoted as $\{T_t^m; t \ge 0\}$. For any $t \in (0,T]$, by using Itô's formula, we can conclude that $M_s := v(t-s,\eta_s) + \int_0^s \phi(t-r,\eta_r)dr$, $0 \le s \le t$, is a bounded $\{\mathcal{F}_s^0\}_{0\le s\le t}$-martingale under $\mathbb{P}_x^0$ for every $x \in \mathbb{R}^m$, where $\{\mathcal{F}_s^0; s \ge 0\}$ denotes the filtration generated by the diffusion process η. Thus

$$\begin{aligned} v(t,x) &= \mathbb{E}_x[M_0] = \mathbb{E}_x[M_t] = \mathbb{E}_x[f(\eta_t)] + \mathbb{E}_x\left[\int_0^t \phi(t-r,\eta_r)dr\right] \\ &= \mathbb{E}_x[f(\eta_t)] + \mathbb{E}_x\left[\int_0^t \phi(r,\eta_{t-r})dr\right]. \end{aligned}$$

In other words,

$$v(t,x) = T_t^m f(x) + \int_0^t T_{t-s}^m \phi(s,\cdot)(x)ds \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^m. \tag{5.3}$$

In fact, by [23, Chapter IV], the operator $\mathcal{G}_m$ has a fundamental solution $p(t,x,y)$ such that

$$v(t,x) = \int_{\mathbb{R}^m} p(t,x,y)f(y)dy + \int_0^t \int_{\mathbb{R}^m} p(t-s,x,y)\phi(s,y)dyds.$$

(It follows that $p(t,x,y)$ is the transition density function of $\{\eta_t; t \ge 0\}$.) Define

$$v_1(t,x) := \int_0^t \int_{\mathbb{R}^m} p(t-s,x,y)\phi(s,y)dyds$$

and

$$v_2(t,x) := \int_{\mathbb{R}^m} p(t,x,y)f(y)dy.$$

By estimates (13.1) and (14.11)–(14.12) in [23, Chapter IV], there is a constant $c_1 > 0$ such that for every $t \geq 0$,

$$\|v_1\|_{0,2;t} \leq c_1 t^{\gamma/2} \left(\lfloor \phi \rfloor_{(x),\gamma;t} + \|\phi\|_{L^\infty([0,t]\times R^m)} \right). \tag{5.4}$$

For $f \in C_b^{2+\gamma}(\mathbb{R}^m)$, define for $(t,x) \in [0,1] \times \mathbb{R}^m$

$$\mathcal{J}_0 f(t,x) = T_t^m f(x),$$
$$\mathcal{J}_k f(t,x) = \int_0^t T_{t-s}^m (\mathcal{I}_m \mathcal{J}_{k-1} f(s,\cdot))(x) ds, \quad k \geq 1.$$

Let $\phi \in C_b^{0,2}([0,1] \times \mathbb{R}^m)$ and for $1 \leq j \leq m$, write $\phi_j(t,x) := \frac{\partial \phi(t,x)}{\partial x_j}$. For each $i \leq j \leq m$, $x, y \in \mathbb{R}^m$ and $w \in (-1,1)$, there is $\theta \in [0,1]$ such that

$$\begin{aligned}
&\big|\big(\phi(t,x+we_j) - \phi(t,x) - \phi_j(t,x)w\big) \\
&\qquad - \big(\phi(t,y+we_j) - \phi(t,y) - \phi_j(t,y)w\big)\big| \\
&= \big|\big(\phi_j(t,x+\theta we_j) - \phi_j(t,y+\theta we_j)\big) - \big(\phi_j(t,x) - \phi_j(t,y)\big)\big| \, |w| \\
&\leq \min\Big\{ \sum_{|k|=2} \|\partial_x^k \phi(t,\cdot)\|_\infty |w|^2, \sum_{|k|=2} \|\partial_x^k \phi(t,\cdot)\|_\infty |x-y||w| \Big\} \\
&\leq \sum_{|k|=2} \|\partial_x^k \phi(t,\cdot)\|_\infty |x-y|^\gamma |w|^{2-\gamma}.
\end{aligned} \tag{5.5}$$

For each $i \leq j \leq m$, $x, y \in \mathbb{R}^m$ and $|w| \geq 1$, by the mean-value theorem,

$$\begin{aligned}
&\big|\big(\phi(t,x+we_j) - \phi(t,x)\big) - \big(\phi(t,y+we_j) - \phi(t,y)\big)\big| \\
&\qquad \leq \min\big\{2\|\phi_j(t,\cdot)\|_\infty |x-y|, \, 4\|\phi(t,\cdot)\|_\infty\big\}.
\end{aligned} \tag{5.6}$$

Recall that $\gamma \in (0, 1 \wedge (2-\alpha))$ and

$$\begin{aligned}
\Delta_{x_j}^{\alpha/2} \phi(t,x) = c_\alpha &\int_{\{w\in\mathbb{R}:\, 0<|w|<1\}} \frac{\phi(t,x+we_j) - \phi(t,x) - \frac{\partial}{\partial x_j}\phi(t,x)w}{|w|^{1+\alpha}} dw \\
&+ c_\alpha \int_{\{w\in\mathbb{R}:\, |w|\geq 1\}} \frac{\phi(t,x+we_j) - \phi(t,x)}{|w|^{1+\alpha}} dw.
\end{aligned}$$

Note 2 in formula, subscript of the first integral: replace colon by semi-colon?

We deduce from (5.5)–(5.6) and Lemma 3.1 that there is a constant $c_2 > 0$ so that for every $\phi \in C^{0,2}([0,1] \times \mathbb{R}^m)$, $1 \leq j \leq m$, and $t \in (0,1]$,

$$\lfloor \Delta_{x_j}^{\alpha/2} \phi \rfloor_{(x),\gamma;t} + \|\phi\|_{L^\infty([0,t]\times\mathbb{R}^m)} \leq c_2 \sum_{|k|\leq 2} \|\partial_x^k \phi\|_{L^\infty([0,t]\times\mathbb{R}^m)}.$$

Since the function $r \mapsto |b(r)|^\alpha$ is bounded, γ-Hölder continuous and

$$\mathcal{I}_m\phi(t,x) = \sum_{j=1}^m |b(x_j)|^\alpha \Delta_{x_j}^{\alpha/2}\phi(t,\cdot)(x),$$

there is a constant $c_3 > 0$ such that for $t \in (0,1]$,

$$\lfloor \mathcal{I}_m\phi \rfloor_{(x),\gamma;t} + \|\mathcal{I}_m\phi\|_{L^\infty([0,t]\times\mathbb{R}^m)} \le c_3\|\phi\|_{0,2;t}.$$

From this and (5.2)–(5.4), with $c_0 = c_0(1) > 0$, we have for $t \in [0,1]$,

$$\begin{aligned}
&\|\mathcal{J}_0 f\|_{2+\gamma,1} \le c_0\,\|f\|_{2+\gamma},\\
&\|\mathcal{J}_1 f\|_{0,2;t} \le c_1c_3t^{\gamma/2}\,\|\mathcal{J}_0 f\|_{0,2;t} \le c_0c_1c_3t^{\gamma/2}\,\|f\|_{2+\gamma},
\end{aligned}$$

and hence by induction,

$$\|\mathcal{J}_k f\|_{0,2;t} \le c_0c_1^kc_3^kt^{k\gamma/2}\|f\|_{2+\gamma} \quad \text{for } k \ge 2.$$

Now take $T_0 = \left(\frac{1}{2}\min\{c_1^{-1}c_3^{-1},\,1\}\right)^{2/\gamma}$. Then

$$\sum_{k=0}^\infty \|\mathcal{J}_k f\|_{0,2;T_0} \le \frac{c_0}{1-c_1c_3T_0^{\gamma/2}}\,\|f\|_{2+\gamma} \le 2c_0\,\|f\|_{2+\gamma}.$$

Define $u(t,x) := \sum_{k=0}^\infty \mathcal{J}_k f(t,x)$. Then $u \in C_b^{0,2}([0,T_0]\times\mathbb{R}^m)$ with

$$\|u\|_{0,2;T_0} \le 2c_0\,\|f\|_{2+\gamma}.$$

Since

$$u(t,x) = T_t^m f(x) + \int_0^t T_{t-s}^m(\mathcal{I}_m u(s,\cdot))(x)ds, \tag{5.7}$$

we have $u \in C^{1,2}([0,T_0]\times\mathbb{R}^m)$ by Theorem 12 in [19, Chapter 1].

We next show that $\mathcal{I}_m u(s,\cdot)(x) \in C_b^{\gamma/2,\gamma}([0,T_0]\times\mathbb{R}^m)$. To prove this, first observe that for each $1 \le j \le m$,

$$\begin{aligned}
\Delta_{x_j}^{\alpha/2}u(s,\cdot)(x) &= \lim_{i_0\to\infty} c_\alpha \sum_{i=1}^{i_0}\int_{\{w\in\mathbb{R}:2^{-i}<|w|\le 2^{1-i}\}} \frac{u(s,x+we_j)-u(s,x)}{|w|^{1+\alpha}}dw\\
&\quad + c_\alpha\int_{\{w\in\mathbb{R}:|w|>1\}} \frac{u(s,x+we_j)-u(s,x)}{|w|^{1+\alpha}}dw\\
&=: \lim_{i_0\to\infty} c_\alpha\sum_{i=1}^{i_0} I_i(s,x) + c_\alpha I_0(s,x).
\end{aligned}$$

For $0 \le s \le t \le T_0$ and $1 \le i \le i_0$,

$$\begin{aligned}
|I_i(s,x)| &= \left| \int_{\{w\in\mathbb{R}:2^{-i}<|w|\le 2^{1-i}\}} \Big(u(s,x+we_j) - u(s,x) \\
&\qquad\qquad - \frac{\partial}{\partial x_j} u(s,x) w \Big) \frac{1}{|w|^{1+\alpha}} dw \right| \\
&\le \sum_{|k|=2} \left(\sup_{x\in\times\mathbb{R}^m} |\partial_x^k u(s,x)| \right) \int_{2^{-i}}^{2^{(1-i)}} w^{1-\alpha} dw \\
&= \frac{2^{2-\alpha}-1}{2-\alpha} 2^{-(2-\alpha)i} \sum_{|k|=2} \left(\sup_{x\in\mathbb{R}^m} |\partial_x^k u(s,x)| \right)
\end{aligned}$$

and so

$$|I_i(s,x) - I_i(t,x)| \le 2\frac{2^{2-\alpha}-1}{2-\alpha} 2^{-(2-\alpha)i} \|u\|_{1,2;T_0}.$$

On the other hand,

$$\begin{aligned}
|I_i(s,x) - I_i(t,x)| &= \left| \int_{\{w\in\mathbb{R}:2^{-i}<|w|\le 2^{1-i}\}} \Big(u(s,x+we_j) - u(t,x+we_j) \\
&\qquad\qquad - (u(s,x) - u(t,x)) \Big) \frac{1}{|w|^{1+\alpha}} dw \right| \\
&\le 4 \left(\sup_{(s,x)\in[0,T_0]\times\mathbb{R}^m} |\partial_s u(s,x)| \right) |t-s| \int_{2^{-i}}^{2^{(1-i)}} w^{-1-\alpha} dw \\
&\le \frac{4(1-2^{-\alpha})}{\alpha} 2^{i\alpha} \|u\|_{1,2;T_0} |t-s|.
\end{aligned}$$

By the above two displays, there exists a positive constant $c_5 > 0$, independent of $i \ge 1$, such that for every $s,t \in [0,T_0]$ and $x \in \mathbb{R}^m$,

$$\begin{aligned}
|I_i(s,x) - I_i(t,x)| &\le c_5 \left(2^{-(2-\alpha)i}\right)^{1-\gamma/2} \left(2^{i\alpha}|t-s|\right)^{\gamma/2} \\
&= c_5 2^{-(2-\alpha-\gamma)i} |t-s|^{\gamma/2}.
\end{aligned}$$

By increasing the value of c_5 if necessary, clearly we also have by the mean-value theorem

$$|I_0(s,x) - I_0(t,x)| \le c_5|t-s| \le c_5|t-s|^{\gamma/2} \quad \text{for } s,t \in [0,T_0] \text{ and } x \in \mathbb{R}^m.$$

Since $\gamma < 2-\alpha$, we conclude that for each $1 \le j \le m$,

$$|\Delta_{x_j}^{\alpha/2} u(s,\cdot)(x) - \Delta_{x_j}^{\alpha/2} u(t,\cdot)(x)| \le c_6|t-s|^{\gamma/2} \quad \text{for } s,t \in [0,T_0] \text{ and } x \in \mathbb{R}^m.$$

This shows that the function $\mathcal{I}_m u(s,\cdot)(x) = \sum_{j=1}^m |b(x_j)|^\alpha \Delta_{x_j}^{\alpha/2}\phi(s,\cdot)(x)$ is in $C_b^{\gamma/2,\gamma}([0,T_0]\times\mathbb{R}^m)$.

Now we have from (5.2)–(5.3) and (5.7) that $u \in C_b^{1+\gamma/2,2+\gamma}([0,T_0]\times\mathbb{R}^m)$ and

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \mathcal{G}_m u(t,x) + \mathcal{I}_m u(t,x) = \mathcal{L}^{(m)} u(t,x) & \text{for } (t,x) \in (0,T_0]\times\mathbb{R}^m, \\ u(0,x) = f(x) & \text{for} \quad x \in \mathbb{R}^m. \end{cases}$$

For any $t \in (0,T_0]$, applying Itô's formula to $s \mapsto u(t-s, Z_s^{(m)})$ (see, e.g., the calculation for (3.4)), we see that $s \mapsto u(t-s, Z_s^{(m)})$ is a bounded martingale for $s \in [0,t]$. Thus

$$u(t,x) = \mathbb{E}_x\left[u(0,Z_t^{(m)})\right] = \mathbb{E}_x\left[f(Z_t^{(m)})\right] = P_t^m f(x) \quad \text{for } t \le T_0.$$

This together with the semigroup property of $\{P_t^m; t \ge 0\}$ proves the theorem. □

Remark. Defining $\rho(x) := \int_{\mathbb{R}} h(y-x)h(y)dy$, we have by Hölder's inequality, $\|\rho\|_\infty \le \int_{\mathbb{R}} h(y)^2 dy$. Moreover, for every $\{x_1,\dots,x_m,\xi_1,\dots,\xi_m\} \subset \mathbb{R}$,

$$\sum_{i,j=1}^m \rho(x_i - x_j)\xi_i\xi_j = \int_{\mathbb{R}} \left(\sum_{i=1}^m h(y-x_i)\xi_i\right)^2 dy \ge 0.$$

Thus if the function $a(x) = c(x)^2$ is bounded between two strictly positive constants, then $(\Gamma_{ij}(x)))_{1\le i,j\le m}$ is uniformly elliptic and bounded. The γ-Hölder condition on Γ and $|b|^\alpha$ are satisfied when $c, b \in \mathrm{Lip}_b(\mathbb{R})$ and $h \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$ for any $\gamma \in (0, 1\wedge\alpha\wedge(2-\alpha))$.

Acknowledgments. We would like to thank the anonymous referee for constructive suggestions which lead to an improvement of the paper.

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Received November 26, 2009; revised November 28, 2010.

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