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On States of Total Weighted Occupation Times of a Class of Infinitely Divisible Superprocesses on a Bounded Domain

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Abstract In this paper, general conditions of state classification for the total weighted occupation times of a class of infinitely divisible superprocesses on a bounded domain D in $\mathbb{R}^d$ are given. As an application, some sufficient and necessary conditions are found for the total weighted occupation times of some special superprocesses on D to be absolutely continuous or singular with respect to the Lebesgue measure on D.

Keywords Absolute continuity · Singularity · Total weighted occupation time · Super-stable process · Super-geometric stable process

Mathematics Subject Classifications (2000) Primary 60J80 · 35J65 · 60J50 · Secondary 60J25

1 Introduction

1.1 Model and Preliminaries

In order to give required terms, let us begin by describing the general model of a class of infinitely divisible superprocesses. Let $\xi := \{\xi_s, \Pi_x, s \geq 0, x \in \mathbb{R}^d\}$ be a right-continuous, time-homogeneous Feller process in $\mathbb{R}^d$, where Π_x is the distribution law

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of the process ξ satisfying $\Pi_x(\xi_0 = x) = 1$. ξ will serve as the random motion process of each underlying particle. Let

$$\psi(x, z) = a(x)z + b(x)z^2 + \int_0^\infty \left(e^{-uz} - 1 + uz\right) n(x, du), \quad x \in \mathbb{R}^d, z \geq 0 \tag{1.1}$$

be a branching mechanism, where n is a kernel from $\mathbb{R}^d$ into $(0, \infty)$ and $a(x) \geq 0$, $b(x) \geq 0$ and $\int_0^\infty (u \wedge u^2) n(x, du)$ are bounded Borel functions on $\mathbb{R}^d$. These assumptions imply that there exists a constant $C > 0$ such that

$$\psi(x, z) \leq C\left(z + z^2\right), \quad z \geq 0. \tag{1.2}$$

Now let us introduce some basic notation. Let $(E, \mathcal{B}(E))$ be an arbitrary Borel measurable space. Then, we use $M(E)$ to denote the set of all finite measures on $\mathcal{B}(E)$ endowed with the topology of weak convergence. The expression $\langle f, \mu\rangle$ stands for the integral of f with respect to the measure μ. We write $f \in \mathcal{B}(E)$ if f is a $\mathcal{B}(E)$-measurable function. Similarly $f \in \mathcal{B}^+(E)$ ($\mathcal{B}_b(E)$) means that f is a nonnegative (bounded) $\mathcal{B}(E)$-measurable function, respectively. We set $\mathcal{B}_b^+(E) = \mathcal{B}^+(E) \cap \mathcal{B}_b(E)$. In this paper, we mainly consider the case that $E = \mathbb{R}^d$. Let $\mathcal{T}$ denote the set of all first exit times of $\{\xi_t\}$ from open sets in $\mathbb{R}^d$. Define the σ- algebras $\mathcal{F}_r := \sigma(\xi_s, s \leq r)$ and $\mathcal{F}_\infty := \vee\{\mathcal{F}_r, r \geq 0\}$. For a given measure μ on $(E, \mathcal{B}(E))$, we use $\aleph_\mu$ to denote the support of μ. All above-mentioned notation will keep same meaning throughout the remaining part of the paper.

According to Dynkin [13] there exists a Markov process $X = \{X_t, P_\mu\}$ in $M(\mathbb{R}^d)$ satisfying the following conditions:

(a) If f is a bounded continuous function on $\mathbb{R}^d$, then $\langle f, X_t\rangle$ is right continuous in t on $\mathbb{R}^+ := [0, \infty)$.
(b) For every $\mu \in M(\mathbb{R}^d)$ and for every $f \in \mathcal{B}_b^+(\mathbb{R}^d)$,

$$P_\mu \exp\langle -f, X_t\rangle = \exp\langle -v_t, \mu\rangle, \tag{1.3}$$

where v is the unique solution to the integral equation

$$v_t(x) + \Pi_x \int_0^t \psi(\xi_s, v_{t-s}(\xi_s)) ds = \Pi_x f(\xi_t). \tag{1.4}$$

Moreover, for every $\tau \in \mathcal{T}$, there correspond random measures X_τ and Y_τ on $\mathbb{R}^d$ such that for any $\mu \in M(\mathbb{R}^d)$ and any $f, g \in \mathcal{B}_b^+(\mathbb{R}^d)$,

$$P_\mu \exp\{-\langle f, X_\tau\rangle - \langle g, Y_\tau\rangle\} = \exp\langle -u, \mu\rangle, \tag{1.5}$$

where u is the unique solution to the integral equation

$$u(x) + \Pi_x \int_0^\tau \psi(\xi_s, u(\xi_s)) ds = \Pi_x \left[f(\xi_\tau) + \int_0^\tau g(\xi_s) ds \right]. \tag{1.6}$$

$X = \{X_t, X_\tau, Y_\tau; P_\mu\}$ is called a superprocess with the underlying motion process $\{\xi_t\}$ and the branching mechanism ψ (this is the enhanced model. For more details, the reader is referred to Dawson [9] and Dynkin [13, 15, 16]). If ξ is a Brownian motion, X is called a super-Brownian motion; if ξ is a diffusion process, X is called a superdiffusion; if ξ is a symmetric α-stable process, X is called a super-α-stable process, and so on. More generally, one could replace the ds in the left hand side of Eqs. 1.4 and 1.6 by a continuous additive functional $A(ds)$ of ξ. Note that in this case one needs to assume $A(ds)$ satisfying certain conditions to ensure the existence of a superprocess. Thus, $A(ds)$ is called a branching rate if there exists $X = \{X_t, X_\tau, Y_\tau; P_\mu\}$ which satisfies the log-Laplace equations (1.5) and (1.6) (see Dynkin [14], Dawson and Fleischmann [10]).

Throughout this paper τ_D denotes the first exit time of ξ from an open subset D of $\mathbb{R}^d$, i.e., $\tau_D = \inf\{t > 0 : \xi_t \notin D\}$. X_{τ_D} is called the exit measure from D, and Y_{τ_D} is called the total weighted occupation time of X on D. It is obvious that the support of Y_{τ_D} is contained in $\overline{D}$, the closure of D. A subset $D \subset \mathbb{R}^d$ is called a domain if it is a connected, nonempty, open subset of $\mathbb{R}^d$. In the sequel, unless otherwise stated, D will denote a bounded, open subset of $\mathbb{R}^d$.

1.2 Main Results and Motivation

There are considerable discussions of the states of infinitely divisible superprocesses on a bounded domain D in $\mathbb{R}^d$. However, so far as we know there is no discussion on the states of Y_{τ_D}, the total weighted occupation time of an infinitely divisible superprocess on a bounded domain D in $\mathbb{R}^d$.

This paper will be devoted to the investigation of the states of the total weighted occupation times of a class of infinitely divisible superprocesses on a bounded domain D in $\mathbb{R}^d$. We will identify a general condition under which the total weighted occupation time of an infinitely divisible superprocess with general spatial motion ξ and general branching mechanism ψ defined as before on a bounded domain D in $\mathbb{R}^d$ is absolutely continuous with respect to the Lebesgue measure on D. For some infinitely divisible superprocesses with certain special motion processes and certain special branching mechanisms we will identify conditions that are both necessary and sufficient for their total weighted occupation times to be absolutely continuous with respect to the Lebesgue measure on D. We also found conditions under which the total weighted occupation times of infinitely divisible superprocesses on a bounded domain D in $\mathbb{R}^d$ are supported by a Lebesgue null set. Now let us describe the main results, and ideas of the paper.

Throughout the paper we assume that for any bounded open subset D of $\mathbb{R}^d$, there is a Borel function $G_D(x, y)$ defined on $D \times D$ such that

$$\Pi_x \int_0^{\tau_D} f(\xi_s) ds = \int_D G_D(x, y) f(y) dy$$

for all measurable $f \geq 0$. G_D is called the Green function for ξ on D. In other words, we assume that the Green function for ξ on D exists. We also assume that the branching rate functional is ds. Since $a(x)$ does not affect the absolute continuity, in this paper we assume that $a(x) \equiv 0$ in Eq. 1.1. For convenience, we introduce the following notation. For $f \in \mathcal{B}_b^+(D)$ and $\nu \in M(D)$, we define

$$G_D f(x) := \Pi_x \int_0^{\tau_D} f(\xi_s) ds = \int_D G_D(x, y) f(y) dy,$$

$$G_D \nu(x) := \int_D G_D(x, y) \nu(dy).$$

Obviously, if $\nu(dy) = f(y)dy$, $G_D f = G_D \nu$. If ξ is transient, we will use $G(x, y)$ to denote the Green function for ξ on $\mathbb{R}^d$, and $G_D(x, y)$ to denote the Green function for ξ on D. Throughout this paper we will also assume that the Green function for ξ on D satisfies the following assumption.

Basic Assumption (A):

(1) *$G_D(\cdot,\cdot)$ is continuous on $(D\times D)\setminus\{(x,x), x\in D\}$ and $G_D(x,y)>0$ for all $x,y\in D$.*
(2) *There exists a continuous, nonnegative function $g(x)$ on $(0,\infty)$ satisfying the integrable condition*

$$\int_{B(0,1)} g(|x|)dx<\infty$$

such that $G_D(x,y)\le g(|x-y|)$, where $|\cdot|$ is the Euclidean norm and $B(0,1):=\{x\in\mathbb{R}^d:|x|<1\}$.
(3) *For any $\delta>0$, there exists $C_\delta>0$ such that $G_D(x,y)\le C_\delta$ whenever $|x-y|\ge\delta$.*

When ξ is a transient Lévy process in $\mathbb{R}^d$, $G(0,\cdot)$ can serve as the function g in the above assumption.

We write $\mu\in M_c(D)$ if $\mu\in M(D)$ and μ has a compact support in D, and $\mu\in M_0(D)$ if $\mu\in M(D)$ and the support of μ consists of only a finite number of points. For $\nu\in M(D)$, $G_D\nu$ is integrable on each compact subset of D. Thus, $G_D\nu$ is finite almost everywhere and superharmonic for ξ on D (see Definition 2.1). Let

$$N_\nu:=\left\{x\in\mathbb{R}^d, G_D\nu(x)=\infty\right\}. \tag{1.7}$$

N_ν is then a closed subset of $\mathbb{R}^d$ with Lebesgue measure zero. Clearly for $\nu\in M_0(D)$, $N_\nu\subset\aleph_\nu$. For any $y\in\mathbb{R}^d$, let δ_y denote the Dirac measure at y. For $y_1,\dots,y_m\in D$ and $\nu=\sum_{i=1}^m\lambda_i\delta_{y_i}$, set

$$\nu_n(dy)=f_n(y)dy, \tag{1.8}$$

where

$$f_n(y)=\sum_{i=1}^m\lambda_i f_n^{y_i}(y), \tag{1.9}$$

$$f_n^{y_i}(y)=\begin{cases}\dfrac{1}{V(B(y_i,1/n))}, & y\in B(y_i,1/n),\\ 0, & y\notin B(y_i,1/n),\end{cases} \tag{1.10}$$

and $V(B(y_i,1/n))$ is the volume of the open ball $B(y_i,1/n):=\{x\in\mathbb{R}^d:|x-y_i|<1/n\}$. Clearly ν_n converges weakly to ν as $n\to\infty$.

Now consider the following integral equation:

$$u(x)+\Pi_x\int_0^{\tau_D}\psi(\xi_s,u(\xi_s))ds=G_D\nu(x),\quad x\in D\setminus N_\nu, \tag{1.11}$$

where $\nu\in M(D)$.

Theorem 1.1 *Let ψ be the branching mechanism defined by Eq. 1.1 with $a(x) \equiv 0$ and suppose that Basic Assumption (A) holds. If for each $y_0 \in D$, there exists a $\delta(y_0) > 0$ depending on y_0 such that*

$$\int_{B(0,\delta(y_0))} \sup_{z \in B(y_0,\delta(y_0))} \psi(y+z, g(|y|))dy < \infty \tag{1.12}$$

holds, then we have following conclusions:

(1) *For any fixed $\mu \in M_c(D)$, there exists a nonnegative random measurable function y_D defined on D such that*

$$P_\mu\{Y_{\tau_D}(dy) = y_D(y)dy\} = 1.$$

(2) *For each finite collection of points, $y_1, \ldots, y_k$, in $D \setminus N_\mu$, the following Laplace functional*

$$P_\mu \exp\left[-\langle f, Y_{\tau_D}\rangle - \sum_{i=1}^{k} \lambda_i y_D(y_i)\right] = \exp\langle -u, \mu\rangle \tag{1.13}$$

holds for any $\lambda_1, \ldots, \lambda_k \geq 0$, where $\mu \in M_c(D)$, and u is the unique nonnegative solution of Eq. 1.11 with $\nu(dy) = f(y)dy + \sum_{i=1}^{k} \lambda_i \delta_{y_i}(dy)$.

Remark 1 For some superdiffusions, Eq. 1.12 is both necessary and sufficient for Y_{τ_D} to be absolutely continuous (see Corollary 3.3). For a general right continuous Feller process ξ, so far we are unable to identify the necessary and sufficient condition for Y_{τ_D} to be absolutely continuous with respect to the Lebesgue measure on D.

We have following second main result.

Theorem 1.2 *Assume that Basic Assumption (A) holds. Let $u_n^y(x)$ be the solution to Eq. 1.11 corresponding to the measure $f_n^y(z)dz$ with f_n^y defined by Eq. 1.10. Suppose that $\mu \in M_c(D)$, ℓ is the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and for $\mu \otimes \ell$- almost all (x, y), $u_n^y(x)$ converges to zero as $n \to \infty$. Then, P_μ-almost surely, $Y_{\tau_D}(\cdot \cap D_0)$ is singular with respect to the Lebesgue measure on D_0, where $D_0 = D \setminus \aleph_\mu$.*

Now let us outline the rough ideas for approaching these problems. Essentially we will use Lemma 1.15 of Dawson et al. [12] (see also Lemma 3.4.2.2 of Dawson [9]) to prove that Y_{τ_D} is absolutely continuous with respect to the Lebesgue measure on D. In order to give the rough ideas, we cite this lemma as follows.

Lemma 1.3 (Lemma 1.15 of Dawson et al. [12]) *Let ν be a random element in $M(\mathbb{R}^d)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the following two conditions:*

(1) *There exists a Borel subset $N \in \mathcal{B}(\mathbb{R}^d)$ with Lebesgue measure zero such that for each $x \in \mathbb{R}^d \setminus N$ there is a sequence of real numbers $\{\epsilon_{x,n}\}$ such that $\epsilon_{x,n} \to 0$ as $n \to \infty$ and $\nu(B(x, \epsilon_{x,n}))/V(B(x, \epsilon_{x,n}))$ converges in law to a nonnegative, real valued random variable $\eta(x)$ as $n \to \infty$, where $V(B(x, \epsilon_{x,n}))$ is the volume of the open ball $B(x, \epsilon_{x,n})$.*

(2) *$e(x) := \mathbb{E}\eta(x)$ is locally integrable and satisfies*

$$\mathbb{E}\langle \nu, \phi \rangle = \int e(z)\phi(z)dz \quad \text{for all } \phi \in \mathcal{B}_b^+\left(\mathbb{R}^d\right). \tag{1.14}$$

Then, on $(\Omega, \mathcal{F}, \mathbb{P})$ there exists a random $f \in \mathcal{B}^+(\mathbb{R}^d)$ such that $\mathbb{P}(\nu(dz) = f(z)dz) = 1$, and for each $z \in \mathbb{R}^d \setminus N$ the random variables $f(z)$ and $\eta(z)$ are identically distributed. In particular, ν is absolutely continuous with respect to the Lebesgue measure almost surely $(\mathbb{P})$.

Thus, the proof of Theorem 1.1 is subject to the verification of the conditions (1) and (2) of Lemma 1.3. To verify condition (1), based on the log-Laplace functional technique, the question is transformed into the proof of the existence, uniqueness, and convergence of the solutions to Eq. 1.11 with removable singularities. The proof of this part is motivated by Dawson and Fleischmann [10], Dynkin and Kuznetsov [17], Klenke [22], and Kuznetsov [23]. The condition (2) is mainly subject to the proof of the existence of the Green function for the corresponding superprocess. We know that the Green function $G_D(x, y)$ for ξ is the same as the occupation time density for the underlying process on D. Thus, condition (2) follows from log-Laplace functional and Basic Assumption (A). The proof of Theorem 1.2 is based on the following lemma. Let ℓ be the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Lemma 1.4 *Let ν be a random element in $M(\mathbb{R}^d)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and suppose that there is a Borel subset $N \in \mathcal{B}(\mathbb{R}^d)$ with Lebesgue measure zero such that for each $x \in \mathbb{R}^d \setminus N$ there is a sequence of real numbers $\{\epsilon_{x,n}\}$ such that $\epsilon_{x,n} \to 0$ as $n \to \infty$ and*

$$\nu(B(x, \epsilon_{x,n}))/V(B(x, \epsilon_{x,n})) \mapsto 0 \qquad \text{in law as} \quad n \to \infty.$$

Then ν is singular with respect to the Lebesgue measure ℓ almost surely $(\mathbb{P})$.

Proof For each $\omega \in \Omega$, we denote the Lebesgue decomposition of ν with respect to the Lebesgue measure ℓ by $\nu = \nu_{ac}(\omega) + \nu_s(\omega)$, where for each $\omega \in \Omega$, $\nu_{ac} << \ell$ (absolutely continuous part) and $\nu_s \perp \ell$ (singular part). By Lemma 3.4.2.1 of Dawson [9] the absolutely continuous part and the singular part are both measurable maps of $(\Omega, \mathcal{F})$ into $(M(\mathbb{R}^d), \mathcal{B}(M(\mathbb{R}^d)))$. For each ω by the Lebesgue density theorem the limit

$$\lim_{n\to\infty} \frac{\nu(B(x, \epsilon_n))}{V(B(x, \epsilon_n))} = \lim_{n\to\infty} \frac{\nu_{ac}(B(x, \epsilon_n))}{V(B(x, \epsilon_n))} = \eta_{ac}(\omega, x) \tag{1.15}$$

exists for each $x \in \mathbb{R}^d \setminus \hat{N}(\omega)$, where $\hat{N}(\omega)$ is a Borel subset of $\mathbb{R}^d$ with Lebesgue measure zero and $\eta_{ac}(\omega, z)$ is a version of the Radon–Nikodym derivative of ν_{ac} with respect to the Lebesgue measure ℓ. It is easy to verify that Eq. 1.15 holds almost surely with respect to the product measure $\mathbb{P} \otimes \ell$. Thus, the lemma follows from the assumption that $\eta_{ac} = 0$ almost everywhere on $\Omega \times \mathbb{R}^d$ with respect to the product measure $\mathbb{P} \otimes \ell$. □

1.3 Organization and Related Results

The content of the paper is organized as follows. Section 1 provides an introduction of the paper. In Section 2 we discuss the fundamental solutions to Eq. 1.11. The fundamental solution to the integral equation (1.11) plays an important role in the investigation of the absolute continuity of Y_{τ_D}. Theorem 2.3 builds up the approximation or convergence property of the fundamental solutions to Eq. 1.11. Then based on Theorem 2.3 we give a proof of Theorem 1.1. Theorem 1.2 will then follow from Lemma 1.4.

In Section 3 we restrict our attention to the results relative to superdiffusions. We prove that if X is a superdiffusion in $M(\mathbb{R}^d)$ with $d > 2$ and with a branching mechanism $\psi(x, z)$ which does not depend on the space variable x and satisfies both condition $\triangle_2$ and condition ∇_2 (see Section 3), then Eq. 1.12 is both necessary and sufficient for Y_{τ_D} to be absolutely continuous. Then, we will establish that the following four statements are equivalent:

(a) Any single point of $\mathbb{R}^d$ is polar, that is, any single point of $\mathbb{R}^d$ is not hit, with probability one, by the range of $(\mathcal{L}, \psi)$-superdiffusion, where $\mathcal{L}$ denotes the generator of diffusion ξ (See [13] and [23], for more details).
(b) If u is a nonnegative solution to the equation $\mathcal{L}u = \psi(u)$ in $\mathbb{R}^d \setminus \{x\}$, then $u = 0$ (in other words, every single point set is removable), where $\mathcal{L}$ denotes the generator of diffusion ξ.
(c) $\mathrm{Cap}_\psi(\{x\}) = 0$ for every $x \in \mathbb{R}^d$, where $\mathrm{Cap}_\psi(\{x\})$ is the Orlicz capacity defined by Eq. 3.3.
(d) For every bounded smooth domain D, P_μ-almost surely $Y_{\tau_D}(\cdot \cap D_0)$ is singular with respect to the Lebesgue measure on D, where $D_0 = D \setminus \aleph_\mu$.

Actually the equivalence of the first three statements is given by Dynkin [13] and Kuznetsov [23]. One of the contributions of the present paper is to add the new equivalent statement (d).

Finally, the last section is devoted to the applications of Theorem 1.1 to super-α-stable processes and super-geometric stable processes.

Throughout this article C will always denote a constant which may change value from line to line.

Here we briefly review some results on the absolute continuity of X_t and X_{τ_D}.

The absolute continuity of X_t has been discussed by many authors. It is well-known that if the spatial motion ξ is an α-stable process $(1 < \alpha \leq 2)$ and the branching mechanism $\psi(z) = z^{1+\beta}$ $(0 < \beta \leq 1)$, then X_t is absolutely continuous if and only if $d < \alpha/\beta$ (see Dawson and Hochberg [11] for the case that X_t is a super-Brownian motion with the branching mechanism $\psi(z) = z^2$, and Fleischmann [18] for general α and β). In the case that X_t is a super-Brownian motion with the branching mechanism $\psi(z) = z^2$ and a general branching rate $A(dt)$, Dawson and Fleischmann [10] gave a sufficient condition on $A(dt)$ for X_t to be absolutely continuous. Klenke [22] generalized the result for a broader class of infinitely divisible superprocesses. Similar results were independently obtained by Ren [27].

For a super-Brownian motion on a bounded smooth domain D, the absolute continuity of the exit measure X_{τ_D} has been studied by several authors. When $\psi(x, z) = z^{1+\beta}, 0 < \beta \leq 1$, the states of the exit measures X_τ were studied by Sheu [28, 29]. Sheu [29] stated that X_{τ_D} is absolutely continuous if and only if

$d < 1 + 2/\beta$. In Ren [26], X_{τ_D} with a general branching mechanism ψ and a general branching rate function $A(dt)$ was studied and a sufficient condition on $A(dt)$ and $\psi(z)$ was given for X_{τ_D} to be absolutely continuous.

2 Proofs of Main Results

Let ξ be the right continuous, time-homogeneous Feller process discussed in the previous section.

Definition 2.1 Let D be a bounded, open subset of $\mathbb{R}^d$, and f a locally integrable function defined on $\mathbb{R}^d$ taking values in $(-\infty, \infty]$.

(1) f is a harmonic function on D with respect to ξ if f is continuous on D and for each $x \in D$ and each ball $B(x, r)$ with $\overline{B(x, r)} \subset D$,

$$f(x) = \Pi_x[f(\xi(\tau_{B(x,r)}))].$$

(2) f is a superharmonic function on D with respect to ξ if f is lower semicontinuous on D and for each $x \in D$ and each ball $B(x, r)$ with $\overline{B(x, r)} \subset D$,

$$f(x) \geq \Pi_x[f(\xi(\tau_{B(x,r)}))].$$

For $\nu \in M_c(D)$, $G_D\nu$ is a harmonic function with respect to ξ on $D \setminus \aleph_\nu$. For a given bounded subset $A \subset \mathbb{R}^d$, we use $\partial A := \overline{A} \cap \overline{A^c}$ to denote the boundary of A, where $A^c := \mathbb{R}^d \setminus A$.

Lemma 2.1 *Let D_2 and D_1 be two bounded open subsets of $\mathbb{R}^d$ satisfying $\overline{D}_1 \subset D_2$. Assume that h is a nonnegative harmonic function on D_2 and that f is a nonnegative function on $D_1^c := \mathbb{R}^d \setminus D_1$ satisfying $f \leq h$ on D_1^c. Let u_1 and u_2 be solutions to the integral equations*

$$u_1(x) + \Pi_x \int_0^{\tau_{D_1}} \psi(\xi_s, u_1(\xi_s))ds = h(x), x \in D_1, \tag{2.1}$$

and

$$u_2(x) + \Pi_x \int_0^{\tau_{D_1}} \psi(\xi_s, u_2(\xi_s))ds = h(x) - \Pi_x f(\xi_{\tau_{D_1}}), x \in D_1, \tag{2.2}$$

respectively. Then $u_1(x) \geq u_2(x)$ for every $x \in D_1$.

Proof Since h is harmonic on D_2 and $\overline{D}_1 \subset D_2$, we have

$$h(x) = \Pi_x h(\xi_{\tau_{D_1}}), \quad x \in D_1.$$

Then

$$h(x) - \Pi_x f(\xi_{\tau_{D_1}}) = \Pi_x(h - f)(\xi_{\tau_{D_1}}), \quad x \in D_1.$$

By Eqs. 1.5 and 1.6, we have

$$u_1(x) = -\log P_{\delta_x} \exp\{-\langle h, X_{\tau_{D_1}}\rangle\}, \quad x \in D_1,$$

and

$$u_2(x) = -\log P_{\delta_x} \exp\{-\langle h - f, X_{\tau_{D_1}}\rangle\}, \quad x \in D_1.$$

Then we have $u_1(x) \geq u_2(x)$ for every $x \in D_1$. □

Lemma 2.2 *Suppose that Basic Assumption (A) and Eq. 1.12 hold. Let D be a bounded domain in $\mathbb{R}^d$. For each $\nu \in M_0(D)$, let ν_n be defined by Eqs. 1.8 and 1.9. Then, there exists a sequence of bounded smooth open subsets $\{D_k\}$ of $D \setminus N_\nu$ satisfying $D_k \uparrow D \setminus N_\nu$ such that for each $x \in D \setminus N_\nu$,*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \Pi_x \int_{\tau_{D_k}}^{\tau_D} \psi(\xi_s, G_D\nu_n(\xi_s))ds = 0. \tag{2.3}$$

Proof For $\nu \in M_0(D)$, we first claim that the assumption (1.12) implies

$$\int_D G_D(x, y)\psi(y, G_D\nu(y))dy = \Pi_x \int_0^{\tau_D} \psi(\xi_s, G_D\nu(\xi_s))ds < \infty, \quad x \in D \setminus N_\nu. \tag{2.4}$$

In fact, without loss of generality, we may assume that $\nu = \delta_{y_0}$. Then for $x \neq y_0$ and $\delta \in (0, |x - y_0|/2)$, by Basic Assumption (A), we have

$$\begin{aligned}
&\int_D G_D(x, y)\psi(y, G_D\nu(y))dy \\
&\quad = \int_{D\cap B(x,\delta)} G_D(x, y)\psi(y, G_D(y, y_0))dy + \int_{D\setminus B(x,\delta)} G_D(x, y)\psi(y, G_D(y, y_0))dy \\
&\quad \leq C_\delta \int_{B(x,\delta)} G_D(x, y)dy + C_\delta \int_D \psi(y, G_D(y, y_0))dy \\
&\quad \leq C_\delta \int_{B(0,\delta)} g(|y|)dy + C_\delta \int_{D\cap B(y_0,\delta)} \psi(y, G_D(y, y_0))dy + C_\delta \int_{D\setminus B(y_0,\delta)} \psi(y, G_D(y, y_0))dy \\
&\quad \leq C_\delta + C_\delta \int_{B(0,\delta)} \psi(y + y_0, g(|y|))dy \\
&\quad < \infty,
\end{aligned}$$

where in the last inequality we have used Eq. 1.12, and C_δ is a positive constant which may change value from line to line.

Let ν_n and f_n be defined by Eqs. 1.8 and 1.9 and let $\hat{D}_k$ be a bounded open subset of D with smooth boundary such that $\hat{D}_k \uparrow D$. Define $D_k := \hat{D}_k \setminus \bigcup_{i=1}^m \overline{B\left(y_i, \frac{1}{k}\right)}$. Then, we know $D_k \uparrow D \setminus N_\nu$. For any given $x \in D \setminus N_\nu$, without loss of generality we may assume that $x \in D_k$ for $k \geq 1$. Let τ_k denote the first exit time of ξ from D_k. Note that

$$\begin{aligned}
&\Pi_x \int_{\tau_k}^{\tau_D} \psi(\xi_s, G_D\nu_n(\xi_s))ds \\
&\quad = \Pi_x \int_0^{\tau_D} \psi(\xi_s, G_D\nu_n(\xi_s))ds - \Pi_x \int_0^{\tau_k} \psi(\xi_s, G_D\nu_n(\xi_s))ds \\
&\quad = \int_D G_D(x, y)\psi(y, G_D\nu_n(y))dy - \int_{D_k} G_{D_k}(x, y)\psi(y, G_D\nu_n(y))dy.
\end{aligned} \tag{2.5}$$

By Eq. 2.5, we obtain

$$
\begin{aligned}
&\left|\Pi_x \int_{\tau_k}^{\tau_D} \psi(\xi_s, G_D\nu_n(\xi_s))ds\right| \\
&\quad \leq \left| \int_D G_D(x,y)\psi(y, G_D\nu_n(y))dy - \int_{D_k} G_D(x,y)\psi(y, G_D\nu_n(y))dy \right| \\
&\quad + \left| \int_{D_k} G_D(x,y)\psi(y, G_D\nu_n(y))dy - \int_{D_k} G_{D_k}(x,y)\psi(y, G_D\nu_n(y))dy \right|. \qquad (2.6)
\end{aligned}
$$

To apply the dominated convergence theorem, note that $G_D(x, y)$ and $G_{D_k}(x, y)$ are locally integrable with respect to y, and that $G_D\nu_n(x) \to \sum_{i=1}^m \lambda_i G_D(x, y_i)$ pointwise in D_k. Since for large n, $G_D\nu_n(x) = \int_D G_D(x, y) f_n(y)dy$ is uniformly bounded in D_k and D is a bounded domain in $\mathbb{R}^d$, by the dominated convergence theorem we have

$$
\lim_{n\to\infty} \int_{D_k} G_D(x,y)\psi(y, G_D\nu_n(y))dy = \int_{D_k} G_D(x,y)\psi(y, G_D\nu(y))dy < \infty, \qquad (2.7)
$$

and

$$
\lim_{n\to\infty} \int_{D_k} G_{D_k}(x,y)\psi(y, G_D\nu_n(y))dy = \int_{D_k} G_{D_k}(x,y)\psi(y, G_D\nu(y))dy < \infty. \qquad (2.8)
$$

Then, for any fixed $k_0 \in \mathbb{N}$ and any $x \in D_{k_0}$, by Eq. 2.7, Eq. 2.8, the monotone property of D_k, and the nonnegativity of the integrands, we have

$$
\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}\left(\int_{D_k} G_D(x,y)\psi(y, G_D\nu_n(y))dy - \int_{D_k} G_{D_k}(x,y)\psi(y, G_D\nu_n(y))dy \right) \\
&\quad = \lim_{k\to\infty}\left(\int_{D_k} G_D(x,y)\psi(y, G_D\nu(y))dy - \int_{D_k} G_{D_k}(x,y)\psi(y, G_D\nu(y))dy \right) \\
&\quad = \int_D G_D(x,y)\psi(y, G_D\nu(y))dy - \lim_{k\to\infty} \Pi_x \int_0^{\tau_{D_k}} \psi(\xi_s, G_D\nu(\xi_s))ds \\
&\quad = \Pi_x \int_0^{\tau_D} \psi(\xi_s, G_D\nu(\xi_s))ds - \Pi_x \int_0^{\tau_{D\setminus N_\nu}} \psi(\xi_s, G_D\nu(\xi_s))ds \\
&\quad = 0, \qquad (2.9)
\end{aligned}
$$

where the last equality follows from the fact that N_ν is a polar subset of D, $\tau_{D\setminus N_\nu} = \tau_D \wedge \tau_{N_\nu}$ and $\tau_{N_\nu} = \infty \quad a.s.(\Pi_x)$ (see [4] p. 282–283, Proposition 5.1 and Proposition 5.2 or [5]).

Thus, to prove Eq. 2.3, based on Eqs. 2.6 and 2.9 we only need to prove that

$$
\limsup_{k\to\infty}\limsup_{n\to\infty}\left| \int_D G_D(x,y)\psi(y, G_D\nu_n(y))dy - \int_{D_k} G_D(x,y)\psi(y, G_D\nu_n(y))dy \right| = 0. \qquad (2.10)
$$

To prove Eq. 2.10, it suffices to prove

$$
\limsup_{k\to\infty}\left\{ \limsup_{n\to\infty} \int_{D\setminus D_k} G_D(x,y)\psi(y, G_D\nu_n(y))dy \right\} = 0. \qquad (2.11)
$$

Since $\nu \in M_0(D)$, without loss of generality, we may simply assume $y_0 \in D$ and $\nu = \delta_{y_0}$. Thus $D_k = \hat{D}_k \setminus \overline{B(y_0, 1/k)}$ and we have

$$\begin{aligned}
&\int_{D\setminus D_k} G_D(x, y)\psi(y, G_D\nu_n(y))dy \\
&\quad = \int_{D\setminus \hat{D}_k} G_D(x, y)\psi(y, G_D\nu_n(y))dy + \int_{\overline{B(y_0,1/k)}} G_D(x, y)\psi(y, G_D\nu_n(y))dy.
\end{aligned} \tag{2.12}$$

For k and n sufficiently large, there exists a constant $C > 0$ such that

$$G_D(y, z) \le C, \quad y \in D \setminus \hat{D}_k, z \in B(y_0, 1/n).$$

So by Eq. 1.2, for large k and n, there exists a constant $C_1 \ge 0$ such that

$$\int_{D\setminus \hat{D}_k} G_D(x, y)\psi(y, G_D\nu_n(y))dy \le C_1 \int_{D\setminus \hat{D}_k} G_D(x, y)dy.$$

Then

$$\limsup_{k\to\infty} \left\{ \limsup_{n\to\infty} \int_{D\setminus \hat{D}_k} G_D(x, y)\psi(y, G_D\nu_n(y))dy \right\} = 0. \tag{2.13}$$

Now we are going to prove that

$$\limsup_{k\to\infty} \left\{ \limsup_{n\to\infty} \int_{\overline{B(y_0,1/k)}} G_D(x, y)\psi(y, G_D\nu_n(y))dy \right\} = 0. \tag{2.14}$$

Note that

$$G_D\nu_n(y) = \int_{B(y_0,1/n)} G_D(y, z) f_n^{y_0}(z)dz$$

and $\int_{B(y_0,1/n)} f_n^{y_0}(z)dz = 1$, where $f_n^{y_0}$ is defined by Eq. 1.10. Since for every fixed y, $\psi(y, z)$ is a convex function in z, by Jensen's inequality we have

$$\psi(y, G_D\nu_n(y)) \le \int_{B(y_0,1/n)} \psi(y, G_D(y, z)) f_n^{y_0}(z)dz.$$

Then we have

$$\begin{aligned}
&\int_{\overline{B(y_0,1/k)}} G_D(x, y)\psi(y, G_D\nu_n(y))dy \\
&\quad \le \int_{B(y_0,1/n)} f_n^{y_0}(z)dz \int_{\overline{B(y_0,1/k)}} G_D(x, y)\psi(y, G_D(y, z))dy \\
&\quad \le \sup_{z\in B(y_0,1/n)} \int_{\overline{B(y_0,1/k)}} G_D(x, y)\psi(y, G_D(y, z))dy.
\end{aligned} \tag{2.15}$$

Since $x \ne y_0$, there exists a constant $C(x) > 0$ depending on x such that for large k,

$$G_D(x, y) \le C(x), \quad y \in \overline{B(y_0, 1/k)}.$$

Thus,

$$\begin{aligned}
&\sup_{z\in B(y_0,1/n)}\int_{\overline{B(y_0,1/k)}} G_D(x,y)\psi(y,G_D(y,z))dy\\
&\quad\le C(x)\sup_{z\in B(y_0,1/n)}\int_{\overline{B(y_0,1/k)}}\psi(y,g(|y-z|))dy\\
&\quad\le C(x)\int_{\overline{B(0,1/k+1/n)}}\sup_{z\in B(y_0,1/n)}\psi(y+z,g(|y|))dy. \qquad (2.16)
\end{aligned}$$

The assumption (1.12) implies that there exist $n,k>0$ such that

$$\int_{\overline{B(0,1/k+1/n)}}\sup_{z\in B(y_0,1/n)}\psi(y+z,g(|y|))dy<\infty. \qquad (2.17)$$

Based on Eqs. 2.16 and 2.17 we obtain

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\sup_{z\in B(y_0,1/n)}\int_{\overline{B(y_0,1/k)}} G_D(x,y)\psi(y,G_D(y,z))dy=0. \qquad (2.18)$$

Consequently, Eqs. 2.18 and 2.15 imply Eq. 2.14. Then, Eq. 2.11 follows from Eqs. 2.12–2.14. This completes the proof of Lemma 2.2. □

Remark 2 If in Eq. 1.1 b is not zero almost everywhere with respect to ℓ, then condition (1.12) is equivalent to either one of the following two conditions:

(a) For each $y_0\in D$, there exists a $\delta(y_0)>0$ depending on y_0 such that

$$\int_{B(0,\delta(y_0))}\psi(y+y_0,g(|y|))dy<\infty; \qquad (2.19)$$

(b) There exists a $\delta>0$ such that

$$\int_{B(0,\delta)} g^2(|y|)dy<\infty. \qquad (2.20)$$

It is obvious that Eq. 1.12 implies Eq. 2.19. Now suppose that Eq. 2.19 holds. Then there exist a $y_0\in D$ and a $\delta>0$ such that in the ball $B(y_0,\delta)$, b has a positive essential infimum

$$m:=\sup_{N,\ell(N)=0}\left\{\inf_{y\in B(y_0,\delta)\setminus N,\ell(N)=0} b(y)\right\}>0.$$

By Eq. 1.1, we have

$$\int_{B(y_0,\delta)} mg^2(|y-y_0|)dy\le\int_{B(y_0,\delta)}\psi(y,g(|y-y_0|))dy<\infty,$$

which implies that Eq. 2.20 holds. Next suppose that Eq. 2.20 holds. Based on Eq. 1.2, we have

$$\begin{aligned}\int_{B(0,\delta)} \sup_{z\in B(y_0,\delta)} \psi(y+z, g(|y|))dy \\ \le C\int_{\overline{B(0,\delta)}} [g(|y|)+g^2(|y|)]dy < \infty. \quad (2.21)\end{aligned}$$

This means that Eq. 1.12 holds.

Let $\{f, f_n, n = 1, 2, \cdots\}$ be a sequence of real valued functions defined on a topological space E. We say that f_n boundedly and pointwise converges to f if $\{f_n\}$ are uniformly bounded and for each $x \in E$, $\lim_{n\to\infty} f_n(x) = f(x)$; we denote this by $f_n \xrightarrow{bp} f$. In the following, we will often use $U[\nu]$ to denote the solution to Eq. 1.11 in order to clearly indicate that it depends on ν or other parameters.

Theorem 2.3 (Fundamental Solutions) *Let D be a bounded domain and ψ be the function defined by Eq. 1.1 with $a(x) \equiv 0$ that satisfies Eq. 1.12. Let $\nu \in M_0(D)$ and let ν_n and f_n be defined by Eqs. 1.8 and 1.9, respectively. Suppose that Basic Assumption* (A) *holds. Then, we have following conclusions:*

(1) (*Existence and Uniqueness*). *The Eq. 1.11 has a unique solution which is a measurable, nonnegative function defined on D.*
(2) (*Initial Data Continuity*). *The solution to Eq. 1.11, denoted by $U[\nu]$, is continuous in ν in the following sense:*

$$U[\nu_n](\cdot) \xrightarrow{bp} U[\nu](\cdot) \text{ as } n \to \infty \quad (2.22)$$

on each compact subset $K \subset D \setminus N_\nu$.
(3) (*Derivative with Respect to Parameter $\lambda \ge 0$*).

$$\lambda^{-1}U[\lambda\nu](\cdot) \xrightarrow{bp} G_D\nu(\cdot) \text{ as } \lambda \to 0 \quad (2.23)$$

on each compact subset $K \subset D \setminus N_\nu$.

Proof Let $\{D_k\}$ be the sequence defined in Lemma 2.2. For simplicity, we write τ and τ_k for τ_D and τ_{D_k}, respectively.

(1) *Existence*: Let u_m be the solution to the integral equation:

$$u_m(x) + \Pi_x \int_0^{\tau_m} \psi(\xi_s, u_m(\xi_s))ds = G_D\nu(x), \quad x \in D \setminus N_\nu. \quad (2.24)$$

Note that Eq. 2.24 has a unique, nonnegative solution since $G_D\nu(x)$ is bounded on D_m (see Dynkin [14]). For $x \in D_m^c := (D \setminus N_\nu) \setminus D_m$, we have $\tau_m = 0$ and Eq. 2.24 yields $u_m(x) = G_D\nu(x)$. Then, for $x \in D_m^c$, we have $u_m(x) = G_D\nu(x) \ge u_{m+1}(x)$. For $x \in D_m$, we have

$$u_{m+1}(x) + \Pi_x \int_0^{\tau_m} \psi(\xi_s, u_{m+1}(\xi_s))ds = G_D\nu(x) - \Pi_x \int_{\tau_m}^{\tau_{m+1}} \psi(\xi_s, u_{m+1}(\xi_s))ds. \quad (2.25)$$

By the strong Markov property of ξ, for $x \in D_m$, we have

$$\Pi_x \int_{\tau_m}^{\tau_{m+1}} \psi(\xi_s, u_{m+1}(\xi_s))ds = \Pi_x f(\xi_{\tau_m}),$$

where

$$f(z) := \Pi_z \int_0^{\tau_{m+1}} \psi(\xi_s, u_{m+1}(\xi_s))ds \le G_D\nu(z), \quad z \in D_m^c.$$

Then, we can rewrite Eq. 2.25 as

$$u_{m+1}(x) + \Pi_x \int_0^{\tau_m} \psi(\xi_s, u_{m+1}(\xi_s))ds = G_D\nu(x) - \Pi_x f(\xi_{\tau_m}). \tag{2.26}$$

Since $G_D\nu$ is a harmonic function on $D \setminus N_\nu \supset D_m$ with respect to ξ, using Lemma 2.1, we see that $u_m(x) \ge u_{m+1}(x)$ for every $x \in D_m$. By the monotonicity of u_m and D_k, for each $x \in D \setminus N_\nu$, there exists a D_m such that $x \in D_m$. Thus, we can define $u(x) = \lim_{m\to\infty} u_m(x)$.

Now we are going to prove that for each $x \in D \setminus N_\nu$,

$$\begin{aligned}\lim_{n\to\infty} \int_D G_D(x, y)\psi(y, G\nu_n(y))dy &= \int_D G_D(x, y)\psi(y, G_D\nu(y))dy \\ &= \Pi_x \int_0^\tau \psi(\xi_s, G_D\nu(\xi_s))ds < \infty. \end{aligned} \tag{2.27}$$

Note that

$$\begin{aligned}&\int_D G_D(x, y)\psi(y, G\nu_n(y))dy \\ &\quad = \int_{D_k} G_D(x, y)\psi(y, G_D\nu_n(y))dy + \int_{D\setminus D_k} G_D(x, y)\psi(y, G_D\nu_n(y))dy. \end{aligned} \tag{2.28}$$

By an argument similar to the proof of Lemma 2.2, we can prove that

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_{D_k} G_D(x, y)\psi(y, G_D\nu_n(y))dy = \int_D G_D(x, y)\psi(y, G_D\nu(y))dy < \infty.$$

Letting $n \to \infty$ first, and then $k \to \infty$ in Eq. 2.28, we see that to prove Eq. 2.27 it suffices to prove that

$$\limsup_{k\to\infty} \left\{ \limsup_{n\to\infty} \int_{D\setminus D_k} G_D(x, y)\psi(y, G\nu_n(y))dy \right\} = 0. \tag{2.29}$$

However, this is just Eq. 2.11 which is already proved in Lemma 2.2, thus Eq. 2.27 follows. It is obvious that $u_m(x) \le G_D\nu(x)$ and $\psi(x, z)$ is monotonically increasing in z for $z \ge 0$. Then, based on Eq. 2.27 it follows from the dominated convergence theorem that

$$u(x) + \Pi_x \int_0^\tau \psi(\xi_s, u(\xi_s))ds = G_D\nu(x), \quad x \in D \setminus N_\nu. \tag{2.30}$$

Therefore, u is a solution to Eq. 1.11.

Uniqueness Let v be another solution to Eq. 1.11. Then

$$v(x) + \Pi_x \int_0^\tau \psi(\xi_s, v(\xi_s))ds = G_D\nu(x), \quad x \in D \setminus N_\nu. \tag{2.31}$$

For $x \in D_m$ we have

$$v(x) + \Pi_x \int_0^{\tau_m} \psi(\xi_s, v(\xi_s))ds = G_D\nu(x) - \Pi_x \int_{\tau_m}^\tau \psi(\xi_s, v(\xi_s))ds. \tag{2.32}$$

By the strong Markov property of ξ, we have

$$\Pi_x \int_{\tau_m}^\tau \psi(\xi_s, v(\xi_s))ds = \Pi_x \tilde{f}(\xi_{\tau_{D_m}}), \quad x \in D_m,$$

where

$$\tilde{f}(x) := \Pi_x \int_0^\tau \psi(\xi_s, v(\xi_s))ds, \quad x \in D_m^c.$$

Since $G_D\nu$ is a harmonic function on $D \setminus N_\nu$ with respect to ξ, using Lemma 2.1, we see that

$$u_m(x) \geq v(x), \quad x \in D_m.$$

Thus, on the one hand we get

$$u(x) \geq v(x), \quad x \in D \setminus N_\nu, \tag{2.33}$$

but on the other hand, Eq. 2.33 gives

$$\begin{aligned} u(x) &= G_D\nu(x) - \Pi_x \int_0^\tau \psi(\xi_s, u(\xi_s))ds \\ &\leq G_D\nu(x) - \Pi_x \int_0^\tau \psi(\xi_s, v(\xi_s))ds = v(x), \quad x \in D \setminus N_\nu. \end{aligned}$$

This proves the uniqueness.

(2) For a fixed compact subset K of $D \setminus N_\nu$, there exists an integer n_0 such that, for $n \geq n_0$, $f_n = 0$ in a neighborhood of K. Thus we have the following inequality:

$$\begin{aligned} 0 \leq U[\nu_n](x) &\leq \int_D G_D(x, y) f_n(y)\mathrm{d}y \\ &\leq C \int_D f_n(y)dy = C\nu_n(D) \leq C \end{aligned} \tag{2.34}$$

for all $x \in K, n \geq n_0$.
Let $U_m[\nu_n](\cdot)$ be the solution to Eq. 2.24 with ν replaced by ν_n. Then

$$U[\nu_n](x) \leq U_m[\nu_n](x), \quad x \in D \setminus N_\nu, m \leq n. \tag{2.35}$$

Note that

$$U_m[\nu_n](x) = \begin{cases} -\log P_{\delta_x} \exp\langle -G_D\nu_n, Y_{\tau_m}\rangle, & x \in D_m, \\ G_D\nu_n(x), & x \in (D \setminus N_\nu) \setminus D_m. \end{cases}$$

Since $G_D \nu_n$ is uniformly bounded on $\overline{D}_m$ and $\lim_{n\to\infty} G_D \nu_n(x) = G_D \nu(x)$ for every $x \in D \setminus N_\nu$, by the dominated convergence theorem, for any $x \in D \setminus N_\nu$,

$$\lim_{n\to\infty} U_m[\nu_n](x) = U_m[\nu](x). \tag{2.36}$$

By the monotonicity of $U_m[\nu]$ we have

$$\lim_{m\to\infty} U_m[\nu](x) = U[\nu](x), \tag{2.37}$$

where $U_m[\nu](x)$ is the solution to Eq. 2.24. It follows from Eqs. 2.35–2.37 that

$$\limsup_{n\to\infty} U[\nu_n](x) \le U[\nu](x), \quad x \in D \setminus N_\nu. \tag{2.38}$$

On the other hand, since $G_D \nu_n \ge U_m[\nu_n] \ge U[\nu_n]$, we have, for $x \in D \setminus N_\nu$,

$$\begin{aligned} U[\nu_n](x) &\ge G_D \nu_n(x) - \Pi_x \int_0^\tau \psi(\xi_s, U_m[\nu_n](\xi_s))ds \\ &= U_m[\nu_n](x) - \Pi_x \int_{\tau_m}^\tau \psi(\xi_s, U_m[\nu_n](\xi_s))ds \\ &\ge U_m[\nu_n](x) - \Pi_x \int_{\tau_m}^\tau \psi(\xi_s, G_D \nu_n(\xi_s))ds. \end{aligned} \tag{2.39}$$

Letting $n \to \infty$ in the above inequality and using Eq. 2.36, we have for any m,

$$\liminf_{n\to\infty} U[\nu_n](x) \ge U_m[\nu](x) - \limsup_{n\to\infty} \Pi_x \int_{\tau_m}^\tau \psi(\xi_s, G_D \nu_n(\xi_s))ds, \quad x \in D \setminus N_\nu.$$

Then letting $m \to \infty$ and using Eq. 2.3, we obtain

$$\liminf_{n\to\infty} U[\nu_n](x) \ge \lim_{m\to\infty} U_m[\nu](x) = U[\nu](x). \tag{2.40}$$

Combining Eqs. 2.38 and 2.40 we obtain

$$\lim_{n\to\infty} U[\nu_n](x) = U[\nu](x), \quad x \in D \setminus N_\nu.$$

Thus statement (2) is proved.

(3) Note that $U[\lambda\nu](x) \le \lambda G_D \nu$ for $\lambda \ge 0$.

Thus, we have

$$\limsup_{\lambda\downarrow 0} \lambda^{-1} \Pi_x \int_0^\tau \psi(\xi_s, U(\lambda\nu)(\xi_s))ds \le \limsup_{\lambda\downarrow 0} \Pi_x \int_0^\tau \lambda^{-1} \psi(\xi_s, \lambda G_D \nu(\xi_s))ds. \tag{2.41}$$

Since for every $x \in \mathbb{R}^d$, $\psi(x, 0) = 0$, and $\psi(x, u)$ is a convex function of u, $\lambda^{-1}\psi(x, \lambda z)$ is increasing in λ for each $z \ge 0$. Then, we have

$$I_{0\le s\le\tau} \lambda^{-1} \psi(\xi_s, \lambda G_D \nu(\xi_s)) \le I_{0\le s\le\tau} \psi(\xi_s,\ G_D \nu(\xi_s)) \text{ for } \lambda < 1.$$

We have proved in Eq. 2.27 that the assumption (1.12) implies

$$\Pi_x \int_0^\infty I_{0\le s\le\tau} \psi(\xi_s, G_D \nu(\xi_s))ds = \Pi_x \int_0^\tau \psi(\xi_s, G_D \nu(\xi_s))ds < \infty. \tag{2.42}$$

Note that

$$G_D\nu(\xi_s) < \infty \quad \Pi_x - \text{a.s.},$$

and for each $0 \leq z < \infty$, by Eq. 1.1, we have $\lim_{\lambda\downarrow 0} \lambda^{-1}\psi(\xi_s, \lambda z) = z\frac{\partial}{\partial\lambda}\psi(\xi_s, 0+) = 0$. Hence we have

$$\lim_{\lambda\downarrow 0} \lambda^{-1}\psi(\xi_s, \lambda G_D\nu(\xi_s)) = 0 \quad \Pi_x - \text{a.s.}.$$

So, we can apply the dominated convergence theorem to the righthand side of Eq. 2.41 to obtain

$$\limsup_{\lambda\downarrow 0} \lambda^{-1}\Pi_x \int_0^\tau \psi(\xi_s, U(\lambda\nu)(\xi_s))ds = 0.$$

By Eq. 1.11 this clearly implies Eq. 2.23 and this completes the proof of Theorem 2.3 □

Proof of Theorem 1.1 To prove Theorem 1.1 we only need to check that Y_{τ_D} fulfills the two assumptions of Lemma 1.3.

We first check that for almost all $y \in D$, $\eta(y) := \lim_{n\to\infty}\langle f_n^y, Y_{\tau_D}\rangle$ in distribution exists with f_n^y defined by Eq. 1.10, and $P_\mu\eta(y) = \langle G(\cdot, y), \mu\rangle$. Recall that the Lebesgue measure of N_μ is zero. For $y \in D \setminus N_\mu$,

$$P_\mu \exp\langle -\lambda f_n^y, Y_{\tau_D}\rangle = \exp\langle -U[\lambda f_n^y(z)dz], \mu\rangle, \quad \lambda > 0.$$

Theorem 2.3 (2) states that $U[\lambda f_n^y(z)dz] \xrightarrow{bp} U[\lambda\delta_y]$ on each compact subset K of $D \setminus \{y\}$. Hence there exists a nonnegative random measurable function $\eta(y)$ such that

$$P_\mu \exp(-\lambda\eta(y)) = \exp\langle -U[\lambda\delta_y], \mu\rangle. \tag{2.43}$$

For every fixed $y \in D \setminus N_\mu$, we have $U[\lambda\delta_y]/\lambda \leq G_D(\cdot, y)$. Since for every fixed $y \in D \setminus N_\mu$, $\langle G_D(\cdot, y), \mu\rangle < \infty$, and $U[\lambda\delta_y]/\lambda \to G_D(\cdot, y)$ on $D \setminus \{y\}$ as $\lambda \to 0$ (see Theorem 2.3 (3)), by the dominated convergence theorem, we have

$$\lim_{\lambda\to 0}\langle U[\lambda\delta_y]/\lambda, \mu\rangle = \langle G_D(\cdot, y), \mu\rangle.$$

Then, Eq. 2.43 implies $P_\mu\eta(y) = \langle G_D(\cdot, y), \mu\rangle$.

Next we prove that $P_\mu\langle f, Y_{\tau_D}\rangle = \int_D f(y)P_\mu(\eta(y))dy$ for any $f \in \mathcal{B}^+(D)$. First for any $\lambda \geq 0$ and $f \in \mathcal{B}^+(D)$, we have

$$P_\mu \exp\langle -\lambda f, Y_{\tau_D}\rangle = \exp\langle -U[\lambda f], \mu\rangle, \tag{2.44}$$

where $U[\lambda f]$ is the solution to

$$u(x) + \Pi_x \int_0^{\tau_D} \psi(\xi_s, u(\xi_s))ds = \lambda G_D f(x). \tag{2.45}$$

Taking derivative with respect to λ in Eq. 2.44 and using Theorem 2.3(3), we obtain

$$\begin{aligned} P_\mu\langle f, Y_{\tau_D}\rangle &= \langle G_D f, \mu\rangle \\ &= \left\langle \int_D f(y)G_D(\cdot, y)dy, \mu\right\rangle \\ &= \int_D f(y)P_\mu\eta(y)dy. \end{aligned} \tag{2.46}$$

This proves (1). Now let us consider the proof of (2). Let $f_n^{y_i}(y)$ be defined by Eq. 1.10 and $f_n(y) := \sum_{i=1}^k \lambda_i f_n^{y_i}(y)$. For $\nu_n(dy) := f(y)dy + f_n(y)dy$,

$$P_\mu \exp[-\langle f + f_n, Y_{\tau_D}\rangle] = \exp\langle -u_n, \mu\rangle \tag{2.47}$$

holds for any $\lambda_1, \dots, \lambda_k \geq 0$, where $\mu \in M_c(D)$, and u_n is the unique nonnegative solution to Eq. 1.11 with $\nu(dy)$ replaced by $\nu_n(dy)$. Then, (2) follows from a limit argument similar to that of (1). This completes the proof of Theorem 1.1 . □

Proof of Theorem 1.2 Recall $D_0 = D \setminus \aleph\mu$ and note that

$$\langle U[\lambda f_n^y(z)dz], \mu\rangle = -\log[P_\mu \exp(-\langle \lambda f_n^y(\cdot), Y_{\tau_D}\rangle)].$$

Let

$$v(y, \lambda) := -\lim_{n\to\infty} \log[P_\mu \exp(-\langle \lambda f_n^y(\cdot), Y_{\tau_D}\rangle)].$$

The assumption implies that for any given $\lambda \geq 0$ and for $\mu \otimes \ell$-almost all (x, y),

$$\lim_{n\to\infty} U[\lambda f_n^y(z)dz](x) = 0.$$

Since $v(y, \lambda) = \lim_{n\to\infty}\langle U[\lambda f_n^y(z)dz], \mu\rangle$ and for any $y \in D_0$, by Eq. 1.11 there exists a positive integer n_0 such that for all $n \geq n_0$, $U[\lambda f_n^y(z)dz]$ is uniformly bounded on $\aleph_\mu$, thus we have

$$v(y, \lambda) = 0, \quad \forall \lambda \geq 0.$$

Then we have for all $y \in D_0$,

$$\lim_{n\to\infty} P_\mu \exp\left(-\lambda\langle f_n^y(\cdot), Y_{\tau_D}\rangle\right) = \exp(-v(y, \lambda)), \quad \forall \lambda \geq 0.$$

It is obvious that $v(y, \lambda)$ is continuous at $\lambda = 0$. Thus, for each $y \in D_0$ there exists a nonnegative random variable $\eta(y)$ such that $\eta(y) = \lim_{n\to\infty}\langle f_n^y(\cdot), Y_{\tau_D}\rangle = 0$ in distribution with respect to the probability P_μ and

$$v(y, \lambda) = -\log P_\mu \exp(-\lambda\eta(y)). \tag{2.48}$$

Then by Lemma 1.4, $Y_{\tau_D}(\cdot \cap D_0)$ is singular with respect to the Lebesgue measure on D_0. □

3 Application to Super-diffusions

In the rest of this paper we assume that the branching mechanism does not depend on the space variable x, i.e., the branching mechanism is given by

$$\psi(z) = b z^2 + \int_{(0,\infty)} \left(e^{-uz} - 1 + uz\right) n(du), \tag{3.1}$$

where $b \geq 0$ is a constant and n is a positive measure on $(0, \infty)$ such that

$$\int_{(0,\infty)} \left(u \wedge u^2\right) n(du) < \infty.$$

In Appendix we will give some concrete examples of branching mechanisms.

In this section we assume that the spatial motion ξ is a diffusion in $\mathbb{R}^d$ $(d > 2)$ corresponding to the uniformly elliptic operator

$$\mathcal{L}u = \sum_{i,j} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b^i(x) \frac{\partial u}{\partial x_i}.$$

We assume that coefficients a^{ij} and b^i are bounded and $a^{ij} \in C^{2,\lambda}(\mathbb{R}^d)$, $b^i \in C^{1,\lambda}(\mathbb{R}^d)$, where $C^{k,\lambda}(\mathbb{R}^d)$ is the Hölder space defined as a class of functions whose k^{th} order partial derivatives are locally Hölder continuous with exponent λ on $\mathbb{R}^d$ (see [23] for more details). Suppose that a branching mechanism ψ is defined by Eq. 3.1. In this section we will assume that ψ satisfies following conditions:

($\triangle_2$): $\psi(2x) \leq K\psi(x)$ for some $K > 0$ and all $x > 0$;
(∇_2): $\psi(x) \leq \psi(K_0 x)/(2K_0)$ for some $K_0 > 1$ and all $x > 0$.

Kuznetsov [23] proved that the conditions ($\triangle_2$) and (∇_2) imply that for every $a > 0$,

$$\int_a^\infty \frac{dt}{\left[\int_0^t \psi(s)ds\right]^{1/2}} < \infty$$

(see [23], Lemma 2.4). Then, Section 2.7 of Dynkin and Kuznetsov [17] gave the following conclusions:

(1) There is an upper bound for every bounded positive solution to $\mathcal{L}u = \psi(u)$ in D. This upper bound, denoted by $\bar{u}(x)$, is called an absolute barrier on D. The boundary condition for the absolute barrier is defined as follows. For every $y \in \partial D$,

$$\bar{u}(x) \to \infty \qquad \text{as } x \to y, x \in D\,. \tag{3.2}$$

(2) If $\{u_n\}$ is a sequence of solutions to $\mathcal{L}u = \psi(u)$ in D, and the limit, $v(x) = \lim_{n\to\infty} u_n(x)$, exists everywhere in D, then v is also a solution to $\mathcal{L}u = \psi(u)$ in D.
(3) If B is a relatively open subset of ∂D such that all points of B are regular for D and if $u_n = f$ in B with f being a continuous function on B, then $v = f$ in B.

Here the reader is referred to [17] and [23] for some new terms and more details.

Let $\Gamma \subset \mathbb{R}^d$ be a compact set and $d > 2$. The Orlicz capacity of Γ is defined by

$$\mathrm{Cap}_\psi(\Gamma) = \sup\left\{\nu(\Gamma) : \nu \in M\left(\mathbb{R}^d\right), \nu\left(\mathbb{R}^d \setminus \Gamma\right) = 0, \int \psi(G\nu(x))dx \le 1\right\}. \tag{3.3}$$

The condition $\mathrm{Cap}_\psi(\Gamma) = 0$ is equivalent to

$$\sup\left\{\nu(\Gamma) : \nu \in M\left(\mathbb{R}^d\right), \nu\left(\mathbb{R}^d \setminus \Gamma\right) = 0, \int_B \psi\left(\int_\Gamma \frac{\nu(dy)}{|x-y|^{d-2}}\right) dx \le 1\right\} = 0, \tag{3.4}$$

where B is an open ball containing Γ (see [1] and [2]).

Theorem 3.1 *Suppose that the underlying dimension $d > 2$ and X is the superdiffusion with the initial value μ, where the underlying particle motion process ξ is governed by the uniformly elliptic operator $\mathcal{L}$ and the branching mechanism ψ is defined by Eq. 3.1 and satisfies both $(\triangle_2)$ and (∇_2). Then the following statements are equivalent:*

(a) *Any single point of $\mathbb{R}^d$ is polar.*
(b) *If $u \ge 0$ and $\mathcal{L}u = \psi(u)$ in $\mathbb{R}^d \setminus \{x\}$, then $u = 0$ (in other words, any single point set is removable).*
(c) *$Cap_\psi(\{x\}) = 0$ for every $x \in \mathbb{R}^d$.*
(d) *For every bounded smooth domain D, P_μ-almost surely $Y_{\tau_D}(\cdot \cap D_0)$ is singular with respect to the Lebesgue measure on D_0, where $D_0 = D \setminus \aleph_\mu$.*

Remark 3 By Eq. 3.4, $\mathrm{Cap}_\psi(\{x\}) = 0$ for every $x \in \mathbb{R}^d$ if and only if $\mathrm{Cap}_\psi(\{0\}) = 0$.

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2 *Suppose that the branching mechanism ψ defined by Eq. 3.1 satisfies both $(\triangle_2)$ and (∇_2), D is a bounded smooth domain in $\mathbb{R}^d$ and for almost all $y \in D$, the following equation has no nontrivial, nonnegative solution:*

$$\begin{cases} \mathcal{L}u = \psi(u), & in\ D \setminus \{y\}, \\ u = 0, & on\ \partial D. \end{cases} \tag{3.5}$$

Then, for any given $\mu \in M_c(D)$, P_μ-almost surely $Y_{\tau_D}(\cdot \cap D_0)$ is singular with respect to the Lebesgue measure on D_0, where $D_0 = D \setminus \aleph_\mu$.

Proof By the above assumption, there is a Lebesgue null set A such that for every $y \in D \setminus A$, there is no nontrivial solution to Eq. 3.5. By Theorem 1.2, we only need to prove that for each fixed $y \in D \setminus A$,

$$\lim_{m\to\infty} U\left[f_m^y(z)dz\right](x) = 0, \quad x \in D \setminus \{y\}. \tag{3.6}$$

To simplify notation, $U[f_m^y(z)dz](x)$ is denoted by u_m, i.e., u_m is the unique solution to the integral equation

$$u_m(x) + \Pi_x \int_0^{\tau_D} \psi(u_m)(\xi_s)ds = G_D f_m^y(x), \quad x \in D.$$

Then we have, for $x \in D \setminus \overline{B(y, 1/n)}$,

$$u_m(x) + \Pi_x \int_0^{\tau_n} \psi(u_m)(\xi_s)ds = G_D f_m^y(x) - \Pi_x \int_{\tau_n}^{\tau_D} \psi(u_m)(\xi_s)ds, \tag{3.7}$$

where τ_n denotes the first exit time of ξ from $D \setminus \overline{B(y, 1/n)}$. Using the strong Markov property of ξ, we can rewrite Eq. 3.7 as

$$u_m(x) + \Pi_x \int_0^{\tau_n} \psi(u_m)(\xi_s)ds = G_D f_m^y(x) - \Pi_x \widetilde{f}(\xi_{\tau_n}), \tag{3.8}$$

where

$$\widetilde{f}(x) = \Pi_x \int_0^{\tau_D} \psi(u_m)(\xi_s)ds \le G_D f_m^y(x), \quad x \in \partial\Big(D \setminus \overline{B(y, 1/n)}\Big).$$

Let v_n be the unique solution to the integral equation

$$v_n(x) + \Pi_x \int_0^{\tau_n} \psi(v_n)(\xi_s)ds = G_D(x, y), \quad x \in D \setminus \overline{B(y, 1/n)}. \tag{3.9}$$

Note that for $x \in D \setminus \overline{B(y, 1/n)}$,

$$v_{n+1}(x) + \Pi_x \int_0^{\tau_n} \psi(v_{n+1})(\xi_s)ds = G_D(x, y) - \Pi_x \int_{\tau_n}^{\tau_{n+1}} \psi(v_{n+1})(\xi_s)ds. \tag{3.10}$$

As in the existence proof of Theorem 2.3 we can prove that

$$v_{n+1}(x) \le v_n(x), \qquad x \in D \setminus \overline{B(y, 1/n)}.$$

Then, we define $v(x) = \lim_{n\uparrow\infty} v_n(x)$ for $x \in D \setminus \{y\}$.

Let $v_{n,m}$ be the solution to the following integral equation

$$v_{n,m}(x) + \Pi_x \int_0^{\tau_n} \psi(v_{n,m})(\xi_s)ds = G_D f_m^y(x), \quad x \in D \setminus \overline{B(y, 1/n)}. \tag{3.11}$$

Since $G_D f_m^y(x)$ is harmonic on $D \setminus \overline{B(y, 1/m)}$ with respect to ξ, we have for $m > n$,

$$G_D f_m^y(x) = \Pi_x \left[G_D f_m^y(\xi_{\tau_n})\right], \quad x \in D \setminus \overline{B(y, 1/n)}.$$

It follows from Eqs. 1.5 and 1.6 that we have

$$v_{n,m}(x) = -\log P_{\delta_x} \exp\langle -G_D f_m^y, X_{\tau_n}\rangle, \quad x \in D \setminus \overline{B(y, 1/n)}. \tag{3.12}$$

Note that for every fixed n, $\{G_D f_m^y, m > n\}$ is uniformly bounded on $\partial B(y, 1/n)$ and has boundary value 0 on ∂D, and for $x \in \partial B(y, 1/n)$, $\lim_{m\to\infty} G_D f_m^y(x) = G_D(x, y)$. By the dominated convergence theorem, we see that $\lim_{m\to\infty} v_{n,m}(x)$ exists for each $x \in D \setminus \overline{B(y, 1/n)}$. If we define

$$v_n(x) = \lim_{m\to\infty} v_{n,m}(x), \quad \text{for } x \in D \setminus \overline{B(y, 1/n)},$$

then v_n satisfies Eq. 3.9. Based on Eqs. 3.8 and 3.11, by Lemma 2.1 we have

$$u_m \le v_{n,m} \quad \text{in } D \setminus \overline{B(y, 1/n)}. \tag{3.13}$$

Letting $m \to \infty$ first, then $n \to \infty$, we have

$$\limsup_{m\to\infty} u_m \le v, \quad \text{in } D \setminus \{y\}.$$

Since v_n is a solution to $\mathcal{L}u = \Psi(u)$ in $D \setminus \overline{B(y, 1/n)}$ with boundary value 0 on ∂D, the limit, v, is a solution to Eq. 3.5 (see [17, 23]). By the assumption that there is no nontrivial solution to Eq. 3.5, we have $v \equiv 0$ in $D \setminus \{y\}$, and hence

$$\lim_{m\to\infty} u_m = 0, \quad \text{in } D \setminus \{y\}.$$

This shows Eq. 3.6 and completes the proof of Lemma 3.2. □

Proof of Theorem 3.1 The equivalence of (a) and (b) was established by Dynkin [13] for the particular branching mechanism $\psi(z) = z^{1+\beta}$ with $0 < \beta \leq 1$. His proof depends crucially on the existence of the absolute barrier to $\mathcal{L}u = u^{1+\beta}$ in every domain D. For a general ψ satisfying $(\triangle_2)$ and (∇_2), the absolute barrier to $\mathcal{L}u = \psi(u)$ in D does exist by Proposition 3.3 of Kuznetsov [23]. Thus Dynkin's proof works for all general ψ which satisfies both $(\triangle_2)$ and (∇_2).

The equivalence of (b) and (c) was proved by Kuznetsov [23].

Now we prove the equivalence of (c) and (d). First, we prove that (d) implies (c). On the contrary, suppose $\mathrm{Cap}_\psi(\{z\}) > 0$ for some $z \in \mathbb{R}^d$. Hence there exists an open ball B such that $0 \in B$ and

$$\int_B \psi\left(|y|^{2-d}\right) dy < \infty.$$

By Theorem 2.5 of ([24] p. 136), the Green function $G_D(x, y)$ satisfies Basic Assumption (A) (1). It follows from Hueber and Sieveking [20] that there exists a constant $C > 0$ such that

$$G_D(x, y) \leq C|x - y|^{2-d},$$

which implies that G_D satisfies Basic Assumption (A) (2) and (3) with $g(s) = Cs^{2-d}$.

Therefore, there exists a $\delta > 0$ such that $B(0, \delta) \subset B$, hence

$$\int_{B(0,\delta)} \psi(g(|y|))dy = \int_{B(0,\delta)} \psi\left(C|y|^{2-d}\right) dy.$$

Choose a positive integer m_C such that $C \leq 2^{m_C}$. Since $\psi(z)$ is increasing in z and satisfies $(\triangle_2)$, we obtain

$$\psi(Cz) \leq \psi\left(2^{m_C} z\right) \leq K^{m_C}\psi(z), \quad \forall z > 0.$$

Therefore,

$$\begin{aligned} \int_{B(0,\delta)} \psi(g(|y|))dy &\leq C \int_{B(0,\delta)} \psi\left(|y|^{2-d}\right) dy \\ &\leq C \int_B \psi\left(|y|^{2-d}\right) dy < \infty. \end{aligned} \tag{3.14}$$

This is exactly the condition (1.12), since ψ does not depend on any space variable. Then by Theorem 1.1, $Y_{\tau_D}(\cdot \cap D_0)$ is absolutely continuous with respect to the Lebesgue measure on D. This gives a direct contradiction to the assumption (d).

Conversely, assume that (c) holds. We prove that (d) is true. Suppose $\mathrm{Cap}_\psi(\{0\}) = 0$. By the equivalence of (b) and (c), for every $y \in D$, there is no nontrivial solution to Eq. 3.5. Hence, by Lemma 3.2, $Y_{\tau_D}(\cdot \cap D_0)$ is singular with respect to the Lebesgue measure on D_0. □

As an application of Theorem 3.1, we have the following corollary.

Corollary 3.3 *Suppose that the underlying dimension $d > 2$, D is a bounded smooth domain in $\mathbb{R}^d$ and the branching mechanism ψ defined by Eq. 3.1 satisfies both $(\triangle_2)$ and (∇_2). Then, Eq. 1.12 holds if and only if Y_{τ_D} is absolutely continuous with respect to the Lebesgue measure on D. In particular, if $\psi(z) = z^{1+\beta}$ and $(0 < \beta \leq 1)$, Y_{τ_D} is absolutely continuous if and only if $d < 2 + 2/\beta$.*

Proof This follows from the proof of Theorem 3.1. □

Corollary 3.4 *Suppose that the branching mechanism ψ defined by Eq. 3.1 satisfies both $(\triangle_2)$ and (∇_2) and the underlying dimension $d > 2$ and D is a bounded smooth domain in $\mathbb{R}^d$.*

(1) *If the branching mechanism ψ satisfies*

$$\psi(z) \leq Cz^{1+\beta_1}(\ln(z))^{\beta_2}, \quad 0 \leq \beta_1 \leq 1, \beta_2 \geq 0 \tag{3.15}$$

for sufficiently large z and $d < 2 + 2/\beta_1$ $(\frac{1}{0} = \infty)$, then Y_{τ_D} is absolutely continuous with respect to the Lebesgue measure on D.

(2) *If the branching mechanism ψ satisfies both $\triangle_2$ and ∇_2, and*

$$\psi(z) \geq Cz^{1+\beta_1}/(\ln(z))^{\beta_2}, \quad 0 < \beta_1 \leq 1, 0 \leq \beta_2 < 1 \tag{3.16}$$

for sufficiently large z and $d \geq 2 + 2/\beta_1$, then P_μ-almost surely $Y_{\tau_D}(\cdot \cap D_0)$ is singular with respect to the Lebesgue measure on D.

Proof It is easy to see that the condition (3.15) implies that

$$\int_B \psi\left(C|y|^{2-d}\right) dy < \infty,$$

where B is an open ball such that $0 \in B$. Then by Theorem 1.1, Y_{τ_D} is absolutely continuous.

On the other hand, Eq. 3.16 implies that

$$\int_B \psi\left(C|y|^{2-d}\right) dy = \infty$$

for any open ball B such that $0 \in B$. This gives that $\mathrm{Cap}_\Psi(\{0\}) = 0$. Since $d \geq 2 + 2/\beta_1$ implies $d > 2$, by Corollary 3.3, this proves that P_μ-almost surely $Y_{\tau_D}(\cdot \cap D_0)$ is singular with respect to the Lebesgue measure on D. □

4 Application to Super-stable Processes and Super-geometric Stable Processes

In this section we assume that D is a bounded $C^{1,1}$ domain and that the branching mechanism is given by Eq. 3.1. However, in this section the branching mechanism ψ is not assumed to satisfy the conditions $(\triangle_2)$ and (∇_2).

Theorem 4.1 *Let D be a bounded domain in $\mathbb{R}^d$. Suppose that the spatial motion process ξ is a symmetric α-stable process and there exists a constant $C > 0$ such that the branching mechanism satisfies*

$$\psi(z) \leq Cz^{1+\beta_1}(\ln(z))^{\beta_2}, \quad 0 \leq \beta_1 \leq 1, \beta_2 \geq 0 \tag{4.1}$$

for sufficiently large z. If $d < \alpha + \alpha/\beta_1$ (where by convention $\frac{1}{0} = \infty$), then Y_{τ_D} is absolutely continuous and Eq. 1.13 holds.

Proof We will only prove this result for $\alpha < 2$, the proof for the case $\alpha = 2$ is similar. It is easy to see that G_D satisfies Basic Assumption (A). From [6] and [7], we have the following estimate:

$$G_D(x, y) \leq \begin{cases} C|\ln|x-y||, & d = \alpha = 1; \\ C|x-y|^{\alpha-d}, & \alpha \neq 1 \text{ or } d > \alpha. \end{cases} \tag{4.2}$$

For $r > 0$, define

$$g(r) = \begin{cases} C|\ln r|, & d = \alpha = 1; \\ Cr^{\alpha-d}, & \alpha \neq 1 \text{ or } d > \alpha. \end{cases} \tag{4.3}$$

Then, we only need to check whether g satisfies Eq. 1.12 or

$$\int_{B(0,\delta)} (g(|y|))^{1+\beta_1}(\ln(g(|y|)))^{\beta_2} dy < \infty. \tag{4.4}$$

If $d = \alpha = 1$ (in this case, $d < \alpha + \alpha/\beta_1$ for every $0 \leq \beta_1 \leq 1$), we have

$$\begin{aligned} &\int_{B(0,\delta)} (g(|y|))^{1+\beta_1}(\ln(g(|y|)))^{\beta_2} dy \\ &\quad \leq C\int_0^\delta |\ln r|^{1+\beta_1}(\ln(|\ln r|))^{\beta_2} dr \\ &\quad = C\int_{1/\delta}^\infty |\ln r|^{1+\beta_1}(\ln(|\ln r|))^{\beta_2} r^{-2} dr \\ &\quad < \infty. \end{aligned} \tag{4.5}$$

If $d = 1 < \alpha (< \alpha + \alpha/\beta_1)$, then g is bounded and

$$\int_D (g(|y|))^{1+\beta_1}(\ln(g(|y|)))^{\beta_2} dy < \infty. \tag{4.6}$$

If $\alpha < d < \alpha + \alpha/\beta_1$, then $(1 + \beta_1)(\alpha - d) + d - 1 > -1$. Thus,

$$\begin{aligned} &\int_D (g(|y|))^{1+\beta_1}(\ln(g(|y|)))^{\beta_2} dy \\ &\quad \leq C\int_0^{r_0} r^{(1+\beta_1)(\alpha-d)+d-1}|\ln(r)|^{\beta_2} dr \\ &\quad < \infty, \end{aligned} \tag{4.7}$$

where r_0 is a positive constant. □

A Lévy process $\xi = (\xi_t, \Pi_x)$ is called a symmetric, geometric, strictly α-stable process in $\mathbb{R}^d$ with $d \geq 3$ if its characteristic exponent is given by $\Psi(z) = \log(1 + |z|^\alpha)$, $z \in \mathbb{R}^d$ ($\alpha \in (0, 2]$). We will simply call it a geometric α-stable process (see Šikić, Song and Vondraček [30]).

Theorem 4.2 *Suppose that the spatial motion ξ is a geometric α-stable process in $\mathbb{R}^d$ with $d \geq 3$ and D is a bounded domain in $\mathbb{R}^d$. Assume that there exist a $C > 0$, and a $\beta \in (0, 1)$ such that the branching mechanism satisfies*

$$\psi(x, z) = \psi(z) \leq Cz(\ln z)^\beta \tag{4.8}$$

for sufficiently large z. Then, Y_{τ_D} is absolutely continuous with respect to the Lebesgue measure on D and Eq. 1.13 holds.

Proof We use Theorem 1.1 to prove this theorem. By Theorem 3.2 of [30], for any $\alpha \in (0, 2]$,

$$G(x) \sim \frac{\Gamma(d/2)}{2\alpha\pi^{d/2}|x|^d \log^2 \frac{1}{|x|}}, \qquad |x| \to 0. \tag{4.9}$$

It is easy to check that $G(x)$ is locally integrable. Since $\psi(x, z) = \psi(z) \leq Cz(\ln z)^\beta$ for sufficiently large z and $G_D(x, y) \leq G(x, y) = G(x - y)$, Eq. 1.12 is equivalent to

$$\int_{B(0,\delta)} G(y)(\ln(G(y)))^\beta dy < \infty \qquad \text{for some } \delta > 0. \tag{4.10}$$

From Eq. 4.9, we see that for $\delta \in (0, 1)$,

$$\begin{aligned}
\int_{B(0,\delta)} G(y)\,(\ln(G(y)))^\beta\,dy &\leq C\int_0^\delta (\ln r)^{-2} r^{-1}(\ln(r^{-d}(\ln r)^{-2}))^\beta dr \\
&= C\int_0^\delta (\ln r)^{-2}(-d\,\ln r - 2\ln(-\ln r))^\beta d(\ln r) \\
&= d^\beta C\int_{-\infty}^{\ln\delta} s^{-2}\left(-s - \frac{2}{d}\ln(-s)\right)^\beta ds \\
&= d^\beta C\int_{-\ln\delta}^{\infty} s^{-2}\left(s - \frac{2}{d}\ln s\right)^\beta ds \\
&\leq d^\beta C\int_{-\ln\delta}^{\infty} s^{-2+\beta} ds < \infty,
\end{aligned}$$

where the constant C depends only on the dimension d. This gives Eq. 4.10. □

For $z \in (0, 1)$, set

$$\Phi^{(1)}(z) = \log(1 + z), \tag{4.11}$$

and

$$\Phi^{(n)}(z) = \Phi\left(\Phi^{(n-1)}(z)\right), \quad n > 1. \tag{4.12}$$

For convenience, we introduce the following notation:

$$l_n(z) = \log\log\cdots\log z, \quad (\,n \text{ iterations})$$

and

$$L_n(z) = l_1(z)l_2(z)\cdots l_n(z).$$

Let $B = (B_t, t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ $(d \geq 3)$, and let $S^{(n)} = (S_t^{(n)}, t \geq 0)$ be a subordinator with Laplace exponent $\Phi^{(n)}$. If B and $S^{(n)}$ are independent, we define a subordinate process $\xi^{(n)} = \left(\xi_t^{(n)}, t \geq 0\right)$ as follows:

$$\xi_t^{(n)} = B\left(S_t^{(n)}\right), \quad t \geq 0. \tag{4.13}$$

Theorem 4.3 *Suppose that the spatial motion process is defined by the above $\xi^{(n)}$ for $n \geq 1$ and D is a bounded domain. Assume that there exist two real numbers $C > 0$ and β satisfying $0 < \beta < 1$ such that the branching mechanism satisfies*

$$\psi(x, z) = \psi(z) \leq Cz(l_n(z))^\beta$$

for sufficiently large z. Then, Y_{τ_D} is absolutely continuous with respect to the Lebesgue measure on D and Eq. 1.13 holds.

Proof Let $G^{(n)}$ denote the Green function of $\xi^{(n)}$. By Theorem 4.7 of [30], for any $\alpha \in (0, 2]$,

$$G^{(n)}(x) \sim \frac{\Gamma(d/2)}{2\alpha\pi^{d/2}|x|^d L_{n-1}\left(1/|x|^2\right) l_n^2\left(1/|x|^2\right)}, \quad |x| \to 0. \tag{4.14}$$

It is easy to check that $G^{(n)}(x)$ is locally integrable. Since $\psi(x, z) = \psi(z) \leq Cz(l_n(z))^\beta$ for sufficiently large z, and $G_D^{(n)}(x, y) \leq G^{(n)}(x, y) = G^{(n)}(x - y)$, Eq. 1.12 is equivalent to

$$\int_{B(0,\delta)} G^{(n)}(y)\left(l_n(G^{(n)}(y))\right)^\beta dy < \infty \quad \text{for some } \delta > 0. \tag{4.15}$$

By Eq. 4.14, for $\delta \in (0, 1)$ we have

$$\begin{aligned}
&\int_{B(0,\delta)} G^{(n)}(y) \left(l_n(G^{(n)}(y))\right)^{\beta} dy \\
&\quad \le C \int_0^{\delta} L_{n-1}^{-1}\left(1/r^2\right) l_n^{-2}\left(1/r^2\right) r^{-1} l_n^{\beta}\left[r^{-d} L_{n-1}^{-1}\left(1/r^2\right) l_n^{-2}\left(1/r^2\right)\right] dr \\
&\quad = C \int_{1/\delta}^{\infty} L_{n-1}^{-1}\left(r^2\right) l_n^{-2}\left(r^2\right) r^{-1} l_n^{\beta}\left[r^{d} L_{n-1}^{-1}\left(r^2\right) l_n^{-2}\left(r^2\right)\right] dr \\
&\quad = C \int_{1/\delta}^{\infty} l_n^{-2}\left(r^2\right) l_n^{\beta}\left[r^{d} L_{n-1}^{-1}\left(r^2\right) l_n^{-2}\left(r^2\right)\right] d\left(l_n\left(r^2\right)\right) \\
&\quad \le C \int_{1/\delta}^{\infty} l_n^{-2}(r^2) l_n^{\beta}\left(r^{d}\right) d\left(l_n\left(r^2\right)\right) \quad \left(L_{n-1}^{-1}\left(r^2\right) l_n^{-2}\left(r^2\right) \text{ is bounded in } (1/\delta, \infty)\right) \\
&\quad = C \int_{1/\delta}^{\infty} l_n^{-2}\left(r^2\right) l_{n-1}^{\beta}\left(\frac{d}{2}\log\left(r^2\right)\right) d\left(l_n\left(r^2\right)\right) \\
&\quad \le C \int_{1/\delta}^{\infty} l_n^{-2}\left(r^2\right) l_{n-1}^{\beta}\left(\log\left(r^2\right)\right) d\left(l_n\left(r^2\right)\right) \\
&\quad = C \int_{l_n(1/\delta^2)}^{\infty} s^{-2+\beta} ds < \infty,
\end{aligned}$$

where C is a positive constant which only depends on the dimension d and which may change its value from line to line. This gives Eq. 4.15 and completes the proof of Theorem 4.3. □

Fleischmann and Sturm [19] constructed a measure-valued Markov process X with an α-stable process as the underlying motion process and

$$\psi(x, z) = z \log z \tag{4.16}$$

as the branching mechanism. The above branching mechanism has a special feature: The mean of the total mass $X_t(\mathbb{R}^d)$ is infinite. This process can be approximated by a sequence of supercritical super α-stable processes X^{β}. We also call it a super α-stable process (unusual). X shows interesting new properties compared with the properties of a usual superprocess. For instance, X_t is absolutely continuous with respect to the Lebesgue measure on D for any dimension $d \geq 1$. The following theorem states that the total weighted occupation time Y_{τ_D} is also absolutely continuous with respect to the Lebesgue measure on D for any dimension $d \geq 1$.

Theorem 4.4 *Suppose that the spatial motion ξ is a symmetric α-stable process in $\mathbb{R}^d$, D is a bounded domain in $\mathbb{R}^d$, and the branching mechanism is given by $\psi(x, z) = z \log z$. Then Y_{τ_D} is absolutely continuous with respect to the Lebesgue measure on D.*

Proof In the following we will prove Theorem 4.4 by checking whether Eq. 1.12 is satisfied according to different cases.

Case 1. First, for $d > \alpha$ and $\alpha \in (0,1) \cup (1,2)$, by [6] and [7] we have

$$G_D(x,y) \le C \frac{1}{|x-y|^{d-\alpha}}. \tag{4.17}$$

For every fixed $x \in D$, $G_D(x,\cdot)$ is locally integrable. Thus we only need to check whether G satisfies Eq. 1.12 which is equivalent to

$$\int_{B(0,1)} G_D(0,y) \log G_D(0,y) dy < \infty. \tag{4.18}$$

Note that Eq. 4.18 is equivalent to

$$-\int_0^1 r^{\alpha-1} \log r \, dr < \infty, \tag{4.19}$$

and Eq. 4.19 holds since $\alpha > 0$. Thus, Case 1 is proved.

Case 2. For $d = \alpha = 1$ or $d = \alpha = 2$, Eq. 1.12 follows from the inequality

$$G_D(x,y) \le C|\ln|x-y||, \tag{4.20}$$

where Eq. 4.20 comes from Eq. 4.2 for the case $d = \alpha = 1$ and [25] for the case $d = \alpha = 2$.

Case 3. For $d = 1, \alpha = 2$, Eq. 1.12 follows from the fact that $G_D(x,y) \le C$ for every $x, y \in D$, where C is a nonnegative constant (see [25], Section 3.5).

This completes the proof of Theorem 4.4. □

Appendix: Examples of Branching Mechanisms

In this appendix we provide a number of examples of branching mechanisms that are of the form needed in Sections 3 or 4.

We begin with some connections between (sub)critical branching mechanisms and Bernstein functions which were given in Bertoin, Roynette and Yor [3], where (sub)critical means critical or subcritical. For every triple (a, b, Λ) with $a, b \ge 0$ and Λ a positive measure on $(0,\infty)$ such that

$$\int_{(0,\infty)} (x \wedge 1)\Lambda(dx) < \infty, \tag{5.1}$$

there corresponds a Bernstein function $\Phi_{a,b,\Lambda}$ defined by

$$\Phi_{a,b,\Lambda}(z) := a + bz + \int_{(0,\infty)} \left(1 - e^{-zx}\right) \Lambda(dx), \quad z \ge 0. \tag{5.2}$$

We call (a, b, Λ) the characteristic of $\Phi_{a,b,\Lambda}$. Then, we have the following connections between (sub)critical branching mechanisms and Bernstein functions (see [3]).

Proposition 5.1 *Suppose* $\lim_{z\to 0} \psi(z) = 0$, *then the following statements are equivalent:*

(1) *ψ is a (sub)critical branching mechanism.*
(2) *$\psi'(z)$ is a Bernstein function, where ψ' is the derivative of ψ.*

Proposition 5.2 *Suppose $\psi(z) = z\Phi(z)$, then the following statements are equivalent:*

(1) *ψ is a (sub)critical branching mechanism and* $\lim_{z\to 0}\psi(z) = 0$.
(2) *Φ is a Bernstein function and $\Phi = \Phi_{a,b,\Lambda}$ for some $a, b \geq 0$, and $\Lambda(dx) = g(x)dx$ with $g \geq 0$ decreasing and $\int_0^\infty (x \wedge 1)g(x)dx < \infty$.*

In the following we collect some Bernstein functions. Then, by Propositions 5.1 and 5.2, we can obtain plenty of branching mechanisms.

$$\Phi(z) = z^\beta, \quad \beta \in (0, 1],$$
$$\Phi(z) = (z+1)^\beta - 1, \quad \beta \in (0, 1],$$
$$\Phi(z) = \ln(z+1),$$
$$\Phi(z) = \frac{z}{z+1},$$
$$\Phi(z) = \sqrt{z}\arctan\frac{1}{\sqrt{z}},$$
$$\Phi(z) = \ln\left(1 + z^\beta\right), \quad \beta \in (0, 1],$$
$$\Phi(z) = (\ln(1+z))^\beta, \quad \beta \in (0, 1],$$
$$\Phi(z) = z^{\beta_1}(\ln(1+z))^{\beta_2}, \quad \beta_1, \beta_2 \geq 0, \beta_1 + \beta_2 \in (0, 1],$$
$$\Phi(z) = z^{\beta_1}\left(\ln\left(1 + \frac{1}{z}\right)\right)^{-\beta_2}, \quad \beta_1, \beta_2 \geq 0, \beta_1 + \beta_2 \in (0, 1],$$
$$\Phi(z) = z^{\beta_1+\beta_2}(\ln(1+z))^{-\beta_2}, \quad \beta_1, \beta_2 \geq 0, \beta_1 + \beta_2 \in (0, 1],$$
$$\Phi(z) = \frac{z-1}{\ln z},$$
$$\Phi(z) = \frac{z}{\ln(1+z)},$$
$$\Phi(z) = \frac{z\ln z - z + 1}{(\ln z)^2}.$$

In fact, the above Bernstein functions are all complete Bernstein functions. For the definition of complete Bernstein function, see Jacob [21]. The first five Bernstein functions are well known and they are taken from [8, 31] and [21]. The next five Bernstein functions can be constructed from the first five by using the properties of complete Bernstein functions listed in Section 2 of [31]. The last few were obtained from a private communication with R. Song.

We can check that some of the above branching mechanisms satisfy both $(\triangle_2)$ and (∇_2). In the following, we give two examples to show how to check whether a branching mechanism satisfies both $(\triangle_2)$ and (∇_2). We leave remaining examples to the interested reader.

Example 1 Let $\psi(x) = x[(x+1)^\beta - 1]$, $\beta \in (0, 1]$, and $x > 0$. Then, ψ satisfies both $(\triangle_2)$ and (∇_2).

Proof Define

$$g_{K_0}(x) := \frac{\psi(K_0 x)}{K_0\psi(x)}, \qquad x > 0.$$

First, we verify that $\psi(x)$ satisfies (∇_2).

It is easy to see that (∇_2) is equivalent to

$$g_{K_0}(x) \geq 2, \qquad x > 0. \tag{5.3}$$

Note that

$$\begin{aligned}\frac{\psi(K_0 x)}{K_0\psi(x)} &= \frac{(K_0 x+1)^\beta - 1}{(x+1)^\beta - 1}\\ &\sim \frac{K_0\beta(K_0x+1)^{\beta-1}}{\beta(x+1)^{\beta-1}} (\text{ by l'Hôspital's rule as } x\to\infty \text{ or as } x\to 0+)\\ &= K_0\left(\frac{K_0x+1}{x+1}\right)^{\beta-1}\\ &\to \begin{cases} K_0^\beta, & \text{as } x\to\infty,\\ K_0, & \text{as } x\to 0+. \end{cases}\end{aligned}$$

In the following, if we choose $K_0^\beta > 2$ and if we can prove that for any given $K_0 > 1$, g_{K_0} is a decreasing function on $(0,\infty)$. Then, Eq. 5.3 is proved and consequently, (∇_2) is satisfied.

Note that

$$\frac{\partial}{\partial x}\left\{\frac{(1+K_0x)^\beta - 1}{(1+x)^\beta - 1}\right\} = \frac{\beta h_{K_0}(x)}{[(1+x)^\beta - 1]^2}, \tag{5.4}$$

where $h_{K_0}(x) := (1+K_0x)^{\beta-1}[(1+x)^\beta - 1] - (1+x)^{\beta-1}[(1+K_0x)^\beta - 1]$. To prove that g_{K_0} is a decreasing function on $(0,\infty)$, it is enough to prove that $h_{K_0}(x) < 0$ on $(0,\infty)$. For $x\in(0,\infty)$, since $K_0 > 1$,

$$(1+K_0x)^\beta > (1+x)^\beta \tag{5.5}$$

is obvious. Note that Eq. 5.5 implies that

$$(1+K_0x)^\beta(1+x)^\beta - (1+x)^\beta > (1+K_0x)^\beta(1+x)^\beta - (1+K_0x)^\beta \tag{5.6}$$

or

$$\left[(1+K_0x)^\beta - 1\right](1+x)^\beta > (1+K_0x)^\beta\left[(1+x)^\beta - 1\right]. \tag{5.7}$$

We can rewrite Eq. 5.7 as

$$\frac{\left[(1+K_0x)^\beta - 1\right]}{\left[(1+x)^\beta - 1\right]} > \frac{(1+K_0x)^\beta}{(1+x)^\beta}. \tag{5.8}$$

Since

$$\frac{(1+K_0x)^\beta}{(1+x)^\beta} > \frac{(1+K_0x)^{\beta-1}}{(1+x)^{\beta-1}},$$

thus, it follows from Eq. 5.8 that

$$\frac{\left[(1+K_0x)^\beta - 1\right]}{\left[(1+x)^\beta - 1\right]} > \frac{(1+K_0x)^{\beta-1}}{(1+x)^{\beta-1}}.$$

This implies that $h_{K_0} < 0$ on $(0,\infty)$ and (∇_2) holds.

Now let us turn to the proof of $(\triangle_2)$.

Define

$$f(x) := \frac{\psi(x)}{\psi(2x)}, \qquad x \in (0,\infty).$$

It is easy to see that $(\triangle_2)$ is equivalent to

$$f(x) \geq \frac{1}{K}, \qquad \text{for some } K > 0, x \in (0,\infty). \tag{5.9}$$

By l'Hôspital's rule, we can prove that

$$\lim_{x\to\infty} f(x) = \frac{1}{2^{1+\beta}}$$

and

$$\lim_{x\to 0+} f(x) = \frac{1}{4}.$$

By an argument similar to the proof of (∇_2), we can prove that f is increasing on $(0,\infty)$. Thus, $(\triangle_2)$ holds if we choose $K > 4$. □

Example 2 Let $\psi(x) = x\ln(1+x)$, $x \in (0,\infty)$. Then ψ does not satisfy condition (∇_2).

Proof It is easy to see that (∇_2) is equivalent to

$$\frac{\psi(K_0x)}{K_0\psi(x)} \geq 2, \qquad x > 0. \tag{5.10}$$

Note that

$$\begin{aligned}\frac{\psi(K_0x)}{K_0\psi(x)} &= \frac{\ln(1+K_0x)}{\ln(1+x)}\\ &\sim \frac{K_0(1+x)}{1+K_0x}(\text{ by l'Hôspital's rule as } x\to\infty \text{ or as } x\to 0+)\\ &\to \begin{cases} 1, & \text{as } x\to\infty,\\ K_0, & \text{as } x\to 0+.\end{cases}\end{aligned}$$

Therefore, it is impossible that $\frac{\psi(K_0x)}{K_0\psi(x)} > 2$ holds for all $x > 0$. This proves that (∇_2) does not hold. □

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